Effective Goldstone dynamics on cosmological spacetimes

Marc Schneider [*]

Department of Physics and EHU Quantum Center, University of the Basque Country UPV/EHU, Barrio Sarriena s/n, Leioa 48940, Spain

(Received 1 May 2026; accepted 15 June 2026; published 6 July 2026)

We derive the Lehmann-Symanzik-Zimmermann reduction formalism for a massive spin-2 particle on Minkowski spacetime and extend the formalism to cosmological spacetimes. The reduction formalism allows for a versatile proof that the Goldstone boson equivalence theorem holds in Friedmann-Lemaître-Robertson-Walker spacetimes. For the de Sitter and the radiation-filled universe, we investigate the Goldstone dynamics and perform an analysis of the range of validity provided by the effective kinetic operator.

DOI: 10.1103/brs1-x1dz

I. INTRODUCTION

The Goldstone boson equivalence theorem (GBET) links high-energy physics to the longitudinal degrees of freedom. Its foundation can be linked to the applicability of the Lehmann-Szymanzik-Zimmermann (LSZ) reduction formalism [1], an operational method to calculate S-matrix elements by cutting external states. Intuitively, the GBET can be summarized as follows: whenever the energy scale of a scattering process E is much larger than the mass of the external state m, the corresponding S-matrix elements involving external, longitudinally polarized gauge bosons are well approximated (up to $\mathcal{O}(m/E)$-corrections) by those containing their corresponding Goldstone scalars. In other words, if we have a process with a longitudinally polarized (tensor) field, we can replace it by its corresponding scalar Goldstone partner. With this substitution, we reproduce the correct values for scattering amplitudes to high accuracy in the high-energy regime. The astonishing fact is that Goldstone bosons themselves do not represent physical degrees of freedom; they can rather be arranged to disappear from any theory by gauge transformations or intrinsic cancellations. However, despite not appearing in the external spectrum, the Goldstone bosons still influence physical observables.

An intuitive understanding of the mechanism in the GBET can be inferred from considering a massive spin-1 particle in the rest frame. There, its three polarizations are approximately comparable, but if the particle starts moving, the picture changes. In particular, the two transverse polarizations become negligible at high velocities, whereas the longitudinal polarization ε_L aligns with the momentum k in the direction of propagation: $\varepsilon^a \to \varepsilon^a_L = \frac{k^a}{m} + \mathcal{O}(\frac{m}{E})$ in the limit $|k| \to \infty$, such that a massive spin-1 particle behaves like its corresponding Goldstone scalar. Effectively, the kinetic term dominates and the scattering is perfectly well described by the degree of freedom pointing into the direction of propagation. Even though it may be a purely fiducial infrared regulator, the existence of a mass as an additional scale in the theory is presumed to be a necessity for the GBET.

Not only does the GBET simplify calculations—it undoubtedly provides one of the most remarkable insights into modern gauge theory. For the Standard Model, the GBET relates the electroweak symmetry breaking mechanism with longitudinal weak-boson scattering experiments. This is insofar significant as the GBET offers another criterion for probing the Higgs mechanism that is responsible for mass creation within the Standard Model. Moreover, the GBET is a fabulous tool to describe models involving a heavy Higgs boson, longitudinally polarized external gauge bosons, and scalar particles [2,3].

Historically, the GBET was first formulated in 1974 by Cornwall, Levin, and Tiktopoulos [4] for a tree-level process with one external longitudinal vector boson. Later, it was generalized by Chanowitz and Gaillard [5]; Lee, Quigg, and Thacker [6]; and Gounaris, Kögerler, and Neufeld [7] to an arbitrary number of external, longitudinal vector bosons. Their claim that the GBET were true for all orders in perturbation theory has been proven wrong by Yao and Yuan [8] and Bagger and Schmidt [9], who pointed out that at higher order in loop corrections, the renormalization condition for the unphysical Goldstone boson has to be taken into account.

Based on these results, there have been many investigations on the precise formulation of the GBET [10–12]

[*]Contact author: marc.schneider@ehu.eus

but so far most research has been confined to lower spin fields like in the Standard Model, or related chiral theories. However, the equivalence theorem offers advantages that go beyond mere computational convenience; it has been conjectured for spin-2 theories both on Minkowski and Friedmann-Lemaître-Robertson-Walker (FLRW) backgrounds [13–15]. The conjecture for validity of a spin-2 GBET basically rests upon the same observation that leads to the hitherto equivalence theorems.

This rather generic statement that the longitudinal term dominates at high energies should equally apply to massive spin-2 particles, since at high energies, the longitudinal polarization scales as $\varepsilon^L_{ab} \sim k_a k_b/m^2$, implying that scattering amplitudes are dominated by the scalar Stückelberg decomposition. In curved spacetimes, this statement holds locally provided that the curvature is sufficiently small compared to the energy. This ensures that the curvature operators are suppressed.

To extract the leading order contribution to scattering amplitudes, we focus our attention on scalar Goldstone bosons. In this sense, the GBET represents a natural diagnostic tool for investigating the consistency of theories involving longitudinal degrees of freedom. In particular, the authors of Ref. [13] found that Goldstone bosons in Minkowski spacetime become strongly coupled at far lower energy than naïvely expected from unitary gauge considerations due to obscured intrinsic cancellations. At low energies, a proper UV complete theory of gravity requires, at the linearized level, a transfer to the Goldstone theory. On top of that, the Goldstone boson description helps elucidating many peculiarities associated with IR deformations of gravity in an intuitive fashion: the explicit form of the Fierz-Pauli mass term [16] is uniquely determined because any deviation would cause terms with four derivatives to occur, indicating pathological degrees of freedom [17]. The influence of the Goldstone degree of freedom develops the van Dam-Veltman-Zakharov discontinuity, which is the disagreement between a massless spin-2 and the massless limit of a massive spin-2 field coupled to the stress-energy tensor. In other words, a massive spin-2 field with 5 degrees of freedom coupled to the covariantly conserved stress-energy tensor yields, in the massless limit, a massless spin-2 and a massless spin-1 field coupled to the stress-energy tensor and a spin-0 field coupled to its trace with gravitational strength; this does not occur for an initially massless spin-2. In general, relevant modifications of Einstein's gravity present an exciting possibility to probe the rigidity of general relativity on cosmological scales. Especially, proposals for massive gravity [18] or even higher spin theories could be tested effectively from prediction in the high-energy limit. A valid GBET simplifies calculations tremendously and provides a window of opportunity to test the consistency of the theory by means of the longitudinal (scalar) degree of freedom. For instance, the GBET has been shown to be valid for inflationary spacetimes [19].

The present analysis serves as an effective field theoretic statement and does not assume the existence of a UV completion of massive gravity. Throughout this article, we presume that below a characteristic cutoff scale, the theory admits a perturbative description in terms of a massive spin-2 field with well-defined helicity sectors. Furthermore, we assume the existence of asymptotic particle states such that standard scattering-theoretic methods can be employed. The derivation relies on the LSZ reduction formalism and the Stückelberg decomposition used to identify the longitudinal degrees of freedom. As a statement about an effective spin-2 field theory, the GBET remains insensitive to the details of the UV completion.

This article aims to derive the GBET on cosmological spacetimes in a versatile way. Section II will give a short introduction to the LSZ reduction formalism and the corresponding GBET for massive spin-1 fields on flat spacetime before we extend the formalism to spin-2. In Sec. III, we present the generalization of the GBET to curved spacetimes for a spin-2 field with Fierz-Pauli mass term. Section IV particularizes our findings to de Sitter and radiation-dominated universes, and we discuss the validity of the approximation. In the subsequent conclusion, we present an outlook and emphasize applications of this theorem. Our sign convention for the metric is mostly plus, and we work in natural units, where $\hbar = c = G_N = 1$.

II. MINKOWSKI BACKGROUND

Let us start with a brief introduction into the LSZ reduction formalism (cf., Refs. [1,20] for a detailed presentation). Consider a (bosonic) field which transforms as an (r, s)-rank tensor $\mathcal{F}^{b_1 \ldots b_r}_{a_1 \ldots a_s}$ under the diffeomorphism group. This tensor describes the system's degrees of freedom. In general, $\mathcal{F}$ may denote either a symmetric (bosonic) or an antisymmetric (fermionic) field. In this case, the degree of freedom also carries Lorentz indices. So far, the GBET has been thoroughly studied for rank (0,0) (scalar or simply spin-0) and rank (0,1) (vector or spin-1) bosons as well as for fermionic degrees of freedom described via spinors [20]. In the following, we restrict ourselves to bosonic degrees of freedom with the note that the fermionic generalization follows immediately.

By using the two metalabels $A = \{a_1, \ldots, a_s\}$, and $B = \{b_1, \ldots, b_r\}$, which collect all the individual indices and their contractions, we will show the derivation of the LSZ formalism schematically and then apply it to the vector boson before extending to the case of a spin-2 field described by a (0,2)-tensor. Our starting point is a scattering process with one ingoing rank-(B, A) quantum field $\mathcal{F}_{\rm in}(k, \varepsilon)$ described by the creation operator $a^\dagger_{\rm in}(k, \varepsilon)$ acting on the in-state $|\psi_{\rm in}\rangle$. We expand the field operator into positive frequency modes and their Hermitian conjugates

$$F_A^B(x) = \sum_n (a_n(k)(f_{A,n}^B(k;x))^* + a_n^\dagger(k) f_{A,n}^B(k;x)), \qquad (1)$$

where the polarization tensor $\varepsilon_{A,n}^B(k)$ has been absorbed into the basis $f_{A,n}^B(k,x) = \varepsilon_{A,n}^B(k) f(k;x)$, which is given by plane waves $f(k;x) = e^{ikx}$ in flat spacetime. Note, x denotes the spacetime location such that the exponent has to be understood as $kx \coloneqq k_a x^a$. We can now perform a shift in the momentum to change to a four-dimensional integral and write [20]

$$|\psi_{\rm in}\rangle = i \int \mathrm{d}^4 x \partial^c (F_A^B \overleftrightarrow{\partial}_c (f_B^A(k;x))^*) |\psi_{\rm in} - k\rangle$$
$$- i \lim_{t\to+\infty} \int \mathrm{d}x^4 n^c F_A^B \overleftrightarrow{\partial}_c (f_B^A(k;x))^* |\psi_{\rm in} - k\rangle, \qquad (2)$$

where n^a is the normal vector while the left and right antisymmetric derivative operator is defined as

$$f(x) \overleftrightarrow{\partial}_a g(x) = f(x)(\partial_a g(x)) - (\partial_a f(x)) g(x). \qquad (3)$$

Note, we have used asymptotic freedom for F_A^B, that is, for $t \to \mp\infty$ the fields become effectively free, i.e.,

$$F_A^B \to Z F_A^{B,\mathrm{in/out}}, \qquad (4)$$

where the normalization Z of the field $0 < Z \leq 1$ for which equality occurs in the free field case.[1] Physically, this conditions relates quantum fields to free one-particle states at the asymptotic boundaries that are elements of the free-field vacuum's Fock space.

Since, essentially, the LSZ reduction formalism amputates the external legs in scattering amplitudes, it in fact connects the measurements from scattering amplitudes, which describes how N initial particles scatter into M final particles, with the dynamical content of the theory, i.e., the degrees of freedom in the propagator. For the amplitude transferring an initial into a final state, we get

$$\mathcal{M} = \langle \psi_{\rm out} | \psi_{\rm in} \rangle = \mathrm{disc.} + i \int \mathrm{d}^4 x (f_B^A(k_1;x))^* \mathfrak{D}_{AD}^{BC}(x_1)$$
$$\times \langle \psi_{\rm out} | \mathcal{T}[F_C^D(k_1)] | \psi_{\rm in} \rangle,$$

with $\mathfrak{D}_{ABCD}$ being the differential operator that constitutes the equation of motion for F^{AB} and $\mathcal{T}$ the time ordering, which is included in case there exists more than one field. Notice, we used once again that asymptotically F_A^B represents a free one-particle state. The time ordering then ensures causality if more than one operator is involved. Here, the disconnected terms (denoted by disc.) describe processes where not all participating particles scatter.[2] Since the addition of a spectator particle should not influence the GBET, these terms are not essential and will be neglected for further considerations.

It is important to mention that the LSZ formalism requires the presence of a (fiducial) mass; this is because mass terms break gauge invariance. Thus, a formerly spurious degree of freedom becomes observable; in case of a vector boson a longitudinally polarized mode will enter the physical spectrum. By a standard argument in differential geometry, ε_A^B can be split into a tensor product of A polarization vectors and B covectors, such that $\varepsilon = (\otimes_A \varepsilon) \otimes (\otimes_B \varepsilon^*)$. Recall the Ward-Takahashi identity for amplitudes $\mathcal{M} = \varepsilon_B^A \mathcal{M}_A^B$, scaling relations for high energies unveil that the dominant contribution is described by the polarization, which is mostly aligned with the propagation direction. This allows us to replace the polarization tensor as follows: $\varepsilon_A \to \frac{1}{m^r} \prod_{i=1}^r k_{a_i}$. As a result, we get for the reduced scattering amplitude (for the sake of simplicity only for one momentum)

$$k_b \mathcal{M}^b = i \int \mathrm{d}^4 x e^{ikx} \partial_b \mathfrak{D}_A^{B(b)}(x_1) \langle \psi_{\rm out} | \mathcal{T}\{F_B^A(x_1)\} | \psi_{\rm in} \rangle, \qquad (6)$$

where we have pulled k_b under the Fourier integral such that $k_a \to i\partial_a$. We will specify the explicit form of $\mathfrak{D}$ in $\mathcal{M}^a$ in the next sections, which cover the specific examples of the tensor fields. This rest-matrix element covers the amputated scattering diagrams and will therefore be central in the derivation of the GBET.

A. The GBET for Proca theory

To illustrate the methodology, we begin with a well-known theory to study the effect as well as the apparent physical relevance of the Goldstone boson. Consider a massive vector (or a (0,1)-tensor) field $\tilde{A}$ that is described by Proca theory. This theory offers the right amount of complexity while being essential in the Standard Model for the description of the massive gauge bosons, i.e., $W^\pm$ and Z^0, of the weak interaction sector. The Lagrange density is given by

[1]The constant Z is subject to a normalization procedure with respect to the equal time commutator. It takes into account that the field content $F_A^B(x)|0\rangle$ is not fully covered by $\langle 1|F_A^B(x)|0\rangle$, while it would be by using $F_A^{B,\mathrm{in}}(x)$ instead [20].

[2]Those terms are proportional to Dirac δ-distributions; explicitly, for one noninteracting particle with initial momentum q_1 we get

$$\mathrm{disc.} = \sum_{k=1}^{M} (2\pi)^3 p_k^0 \delta^{(3)}(q_1 - p_k) \langle p_1, \ldots, p_{k-1}, p_{k+1}, \ldots | q_2, \ldots \rangle, \qquad (5)$$

where the summation extends over all possible out-states. Here, p and q denote the momenta.

$$\tilde{\mathcal{L}}_P = -\frac{1}{4}F^{ab}(\tilde{A})F_{ab}(\tilde{A}) + \frac{1}{2}m^2\tilde{A}_a\tilde{A}^a + J^a\tilde{A}_a, \quad (7)$$

with rank (0,2) field strength tensor $F = \mathrm{d}\tilde{A}$ and external current J. The vector field $\tilde{A}$ propagates three physical degrees of freedom: one longitudinal scalar degree of freedom and two transversal vector degrees of freedom. The kinetic operator for Proca theory expressed in the R_ξ-gauge [20], as follows:

$$\mathcal{D}_{ab} = (\Box - m^2)\eta_{ab} + \left(\frac{1}{\xi} - 1\right)\partial_a\partial_b, \quad (8)$$

where η is the Minkowski metric and $\Box \coloneqq \eta^{ab}\partial_a\partial_b$ the d'Alembert operator. Let us emphasize that for our methodology it will be essential to work in this gauge.

If we compare Proca theory to its massless counterpart, Maxwell theory, we see that the mass term breaks the gauge freedom explicitly, such that an additional degree of freedom arises. As pointed out in Ref. [13], gauge freedom corresponds to the existence of spurious degrees of freedom s in the theory. Those are in principle part of the theory but cannot contribute to physical observables, since $\|s\| = 0$ and hence for any operator, $\mathcal{O}$ follows $\langle s|\mathcal{O}|s\rangle = 0$ [21]. Although they are present in the theory, we might not see their influence as long as the corresponding gauge symmetry is active. Observables will stay unaffected from those degrees of freedom such that before making any prediction, we need to fix a gauge. In other words, whenever we break a gauge freedom, we activate a spurious degree of freedom. This happens through addition of a (fiducial) mass term in a massless theory. To turn it around, we can also artificially amend the gauge freedom by introducing a Stückelberg field ϕ through the transformation

$$\tilde{A}_a \to A_a + \frac{1}{m}\partial_a\phi. \quad (9)$$

As we see immediately, the existence of a mass term in $\mathcal{L}$ is essential for this formalism. The mass m is introduced because ϕ is canonically normalized. The Stückelberg field ϕ plays the role of a Goldstone boson and now Eq. (7) shows that the artificially created symmetry grants us some additional gauge freedom. In fact, we have included a helicity-1 shift symmetry $\delta A_a = \partial_a\chi$, which can be restored by a shift in the Stückelberg field $\delta\phi = -m\chi$. Using this, we implement the following R_ξ-gauge-fixing condition by adding[3]

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(\partial_a A^a + \xi m\phi)^2 \quad (10)$$

to $\mathcal{L}_P$ with arbitrary gauge parameter ξ implying $\langle\psi|\partial_a A^a + \xi m\phi|\psi\rangle = 0$; that is, for physical states, the Stückelberg Lagrange density equals the Proca Lagrange density. Although in principle, Eq. (10) may lead to pathologies, Delbourgo *et al.* [24] have shown that the theory does not feature ghost states that couple to physical states.

Before proceeding, we need to check whether this gauge condition can be consistently imposed; that is, if it can be reached by an appropriate gauge transformation. If there were no additional Stückelberg field ϕ, we could choose the Lorenz gauge $\partial_a\tilde{A}^a = 0$. However, using Eq. (9), we find the condition

$$\partial_a A^a + \frac{\Box\chi}{m} = 0 \quad \Rightarrow \quad \Box\chi = -m\partial_a A^a. \quad (11)$$

The gauge is compatible with the theory whenever there exists a solution for χ to Eq. (11) under consideration of the inhomogenous Gupta-Bleuler condition. The gauge condition implies the constraint

$$\partial_a A^a = -m\phi, \quad (12)$$

which removes the timelike component but leaves the longitudinal mode dynamical.[4] Now, χ has to fulfill just the Klein-Gordon equation, which with the gauge condition Eq. (12) becomes

$$\Box\chi = \xi m^2\phi. \quad (13)$$

Its solution is known to be given by means of the fundamental solution $G(x,x')$ as

$$\chi(x) = -m^2\xi\int_{\mathcal{M}}\mathrm{d}^4x' G(x,x')\phi(x'), \quad (14)$$

where $G(x,x')$ denotes a Green's function of the d'Alembert operator (e.g., the retarded Green's function), which exists on globally hyperbolic spacetimes. Hence, Eq. (12) defines a valid gauge fixing because every field configuration can be mapped into it.

Let us have a look at the polarization vector in this theory: ε_a is normalized such that $\|\varepsilon\| = -1$. A canonical basis can be found by choosing the propagation direction such that k_a points in the z-direction, with $E^2 - k_z^2 = m^2$. Imposing the Gupta-Bleuler quantization condition, $\varepsilon_a k^a = 0$ has to be fulfilled. The two obvious choices for the transversal polarizations are found to be $\varepsilon_a^{T_1} = (0,1,0,0)$ and $\varepsilon_a^{T_2} = (0,0,1,0)$ while the third polarization is $\varepsilon_a^L = (\frac{k_z}{m},0,0,\frac{E}{m})$. Note, ε_a^L is the longitudinal

[3]The gauge fixing conditions and the symmetries are closely related to the Becchi-Rouet-Stora-Tyutin (BRST) transformation [22], which is explicitly discussed for the case of the Standard Model in Ref. [23].

[4]In classical electrodynamics, this would correspond to an inhomogenous Lorenz gauge, which eliminates only the temporal entry in A_a but not the one parallel to the direction of propagation.

polarization. Its energy dependence is a direct consequence of the little group and constitutes the correspondence between ε_a^L and $\frac{k_a}{m}$.

In order to extract the kinetic operator of A_a from $\mathcal{L}_P + \mathcal{L}_{\mathrm{gf}}$, we employ the LSZ reduction formula as well as the condition that for high energies, $\varepsilon \to \frac{k}{m} + \mathcal{O}(\frac{m}{E})$, which projects out the longitudinal degree of freedom A^L. We use a very direct approach by applying the derivative to the kinetic operator. Then, we are able to derive the GBET for one spin-1 particle in the in-state [20], as follows:

$$
\begin{aligned}
&\varepsilon^a \mathcal{M}_a(\psi_{\mathrm{in}} + A_a(k,\varepsilon) \to \psi_{\mathrm{out}}) \\
&\quad \overset{(8)}{=} \frac{i}{m}\int \mathrm{d}^4 x e^{-ikx}\partial^a\left[(\partial^c\partial_c - m^2)\eta_{ab} + \left(\frac{1}{\xi} - 1\right)\partial_a\partial_b\right] \\
&\qquad \times \langle\psi_{\mathrm{out}}|A_b(x)|\psi_{\mathrm{in}}\rangle \\
&\quad = -\frac{i}{m}\int \mathrm{d}^4 x e^{-ikx}\left(\frac{\Box}{\xi} - m^2\right)\langle\psi_{\mathrm{out}}|\partial_a A^a(x)|\psi_{\mathrm{in}}\rangle \\
&\quad \overset{(10)}{=} -i\int \mathrm{d}^4 x e^{-ikx}(\Box - m^2)\langle\psi_{\mathrm{out}}|\phi(x)|\psi_{\mathrm{in}}\rangle \\
&\quad = -i\mathcal{M}(\psi_{\mathrm{in}} + \phi(x) \to \psi_{\mathrm{out}})\left[1 + \mathcal{O}\left(\frac{m}{E}\right)\right],
\end{aligned} \tag{15}
$$

where we used Eq. (12) in the last step to replace $\partial_a A^a(x)$ by $\xi m\phi(x)$. After canonically normalizing ϕ by ϕ/m, we find the Klein-Gordon equation for the Goldstone scalar and thus the GBET for Proca. Let us emphasize that in our methodology, it was essential to work in the R_ξ-gauge. This is because we do not intend to enforce certain conditions *a priori*. The proof of the GBET in Proca theory used Schwarz's theorem, that is, partial derivatives commute, as such the differential operator (8) commutes with ∂_a. Application of $\mathcal{D}^{ab}$ to A_a will not give zero because A_a is not an asymptotic state; instead, it solves $\mathcal{D}^{ab}A_a = J^b$ for $J^b \neq 0$. Finally, the differential operator is transformed to the Klein-Gordon operator for the scalar field, which look similar here. However, we will see examples where this is not the case, especially in curved spacetimes.

The term marking the low-energy corrections comes from the transversal contributions, which are ignored in the replacement of the polarization tensor by the longitudinal wave vector. A generalization to arbitrary numbers of spin-1 particles in the in- and out-states can be accomplished in a similar way [5]. Intuitively, from Eq. (15) follows that in the high-energy limit, the amplitude for a (0,1)-field is equivalent those of a (0,0)-field up to $\mathcal{O}(\frac{m}{E})$.

If we had instead made use of a longitudinal projector $\mathcal{P}_L$ applied onto $\mathcal{D}$, this would have yielded an equation of motion that is solved directly by A^L. The longitudinal degree of freedom by itself is a scalar and thus obeys the Klein-Gordon equation. In other words, Eq. (8) applied to $A_a^L \coloneqq (\mathcal{P}_L A)_a$ would automatically reduce such that $(\mathcal{P}_L\mathcal{D})_{ab} = (\Box - m^2)\eta_{ab}$, from which the GBET follows immediately: after applying the derivative, we use the Gupta-Bleuler gauge condition and replace $\partial_a A^a$ by ϕ, which agrees with the result before.

B. The GBET for massive spin-2 fields

To extend the GBET to spin-2 bosons, we will essentially follow the steps outlined in the previous section. Our starting point is the theory of a free (0,2)-tensor field h_{ab}, which acquires a mass from the Fierz-Pauli term [13]

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\partial_a\tilde{h}_{bc}\partial^a\tilde{h}^{bc} + \partial_b\tilde{h}_{ca}\partial^c\tilde{h}^{ba} - \partial_b\tilde{h}^{bc}\partial_c\tilde{h} \\
&+ \frac{1}{2}\partial_a\tilde{h}\partial^a\tilde{h} - \frac{1}{2}m^2(\tilde{h}_{ab}\tilde{h}^{ab} - \tilde{h}^2),
\end{aligned} \tag{16}
$$

with $h = tr_\eta(h_{ab})$ and m the mass of h_{ab}. A massive spin-2 field propagates 5 degrees of freedom and 1 ghost degree of freedom, the Boulware-Deser ghost [25], which disappears at the linearized level but reappears in the nonlinear theory. This instability comes from the loss of the Hamiltonian constraint; however, it has been shown that nonlinear theories without Boulware-Deser instability can be constructed by imposing a second constraint [26].

The kinetic operator $h_{ab}\mathscr{D}^{abcd}h_{cd}$ can be retrieved after an integration by parts of Eq. (16); thus we find the explicit form of the Lichnerowicz operator $\mathscr{D}^{ab}{}_{cd}$ to be

$$
\begin{aligned}
\mathscr{D}^{ab}{}_{cd} = &\frac{1}{2}(\Box - m^2)\left(\frac{1}{2}\eta^a{}_c\eta^b{}_d + \frac{1}{2}\eta^a{}_d\eta^b{}_c - \eta^{ab}\eta_{cd}\right) \\
&+ \eta^{ab}\partial_c\partial_d + \eta_{cd}\partial^a\partial^b \\
&- \frac{1}{2}(\eta^b{}_d\partial^a\partial_c + \eta^b{}_c\partial_d\partial^a + \eta^a{}_d\partial^b\partial_c + \eta^a{}_c\partial^b\partial_d).
\end{aligned} \tag{17}
$$

Similar to Proca theory, we enhance our symmetry using the Stückelberg formalism, which will then culminate into the Goldstone vectors and scalars for the massive (0,2)-tensor field. For the sake of clarity, we will perform the Stückelberg decomposition step by step, starting with the introduction of Goldstone vectors $\tilde{B}_a$, as follows:

$$
\tilde{h}_{ab} \to h_{ab} + \frac{1}{2m}\partial_a\tilde{B}_b + \frac{1}{2m}\partial_b\tilde{B}_a, \tag{18}
$$

The gauge symmetry is hence given by the transformations $\delta h_{ab} = \frac{1}{2}(\partial_a\zeta_b + \partial_b\zeta_a)$ and the shift in the Goldstone degree of freedom $\delta\tilde{B}_a = -m\zeta_a$. The redundancy (18) will produce mixed terms in $\mathcal{L}$, which we wish to eliminate by adding a suitable R_ξ-gauge,

$$
\mathcal{L}_{\mathrm{gf}} = \frac{1}{2\xi}(\partial^a h_{ab} - \partial_b h - \xi m\tilde{B}_b)^2, \tag{19}
$$

where the square shall be understood as a contraction in the free indices. As such, $\mathcal{L}$ is diagonal in the fields h_{ab} and $\tilde{B}_a$.

The above gauge condition could be viewed as an inhomogeneous de Donder gauge with the Goldstone boson as sourcing term; this is similar to the Proca case where the scalar field was sourcing the inhomogeneous Lorenz gauge. Since the massless part of Eq. (16) admits gauge invariance, only the Fierz-Pauli term changes, and we get an additional term for the Stückelberg fields. Note, $\mathcal{D}_{abcd}$ contains a longitudinal part, which is directly proportional to the Klein-Gordon equation; the other terms describe transversal degrees of freedom. On this level, it becomes evident that if the longitudinal degree of freedom dominates, then we will be left with a scalar field theory. By reducing the theory further, we will eventually arrive at a theory that incorporates a scalar degree of freedom, also known as the longitudinal Goldstone mode.

The first step is to show how the GBET employs the LSZ formalism to reduce a (0,2)-tensor field in the outgoing state. The rest-matrix element (cf., Sec. II A) reads

$$\mathcal{M}_{ab}(\psi_{\rm in}\to\psi_{\rm out}+h_{ab}) = C_h\int \mathrm{d}^4x e^{ikx} \times \mathscr{D}_{ab}{}^{cd}\langle\psi_{\rm out}|h_{cd}|\psi_{\rm in}\rangle, \qquad (20)$$

where C_h collects all constants because they are irrelevant for the GBET's mechanism. Hence, we will absorb all constants that appear in the course of the proof into C_h without further notice.

Following the steps in the previous section, we use the Ward-Takahashi identity and split the matrix element into polarization vectors and rest-matrix element $\mathcal{M}\to\mathcal{M}_{ab}\varepsilon^{ab}$. After splitting up the polarization tensor into two polarization vectors, we perform the high-energy replacement of one ε^a, such that $\mathcal{M}_{ab}$ is multiplied with a momentum k^a/m. Then, we recall the general Ward identity and find the equation for the longitudinal mode of the Goldstone vector,

$$\begin{aligned}-i\frac{k^a}{m}\mathcal{M}_{ab}(\psi_{\rm in}\to\psi_{\rm out}+h_{ab}) &= \frac{C_h}{m}\int \mathrm{d}^4x e^{ikx}\partial^a\mathscr{D}_{ab}{}^{cd}\langle\psi_{\rm out}|h_{cd}|\psi_{\rm in}\rangle\\ &= -\frac{C_h}{m}\int \mathrm{d}^4x e^{ikx}\left(\frac{\Box}{\xi}-m^2\right)\langle\psi_{\rm out}|m\xi\tilde{B}_b|\psi_{\rm in}\rangle, \end{aligned}\qquad (21)$$

for the first step (cf., Appendix B for details). Here, we used the gauge condition to replace $\partial^a h_{ab}-\partial_b h = m\tilde{B}_b$ derived from Eq. (19). This result holds, of course, in leading order of the high-energy limit, where the polarization tensor can be approximated by a product of polarization vectors in which high-energy behavior is dominated by the momentum ($\varepsilon^a_L\approx\frac{k^a}{m}+\mathcal{O}(\frac{E}{m})$). Therefore, we get for the first reduction

$$i\frac{k^a}{m}\mathcal{M}_{ab}(\psi_{\rm in}\to\psi_{\rm out}+h_{ab}) = \mathcal{M}_b(\psi_{\rm in}\to\psi_{\rm out}+\tilde{B}_b). \qquad (22)$$

We proceed as just before, however, with the new starting point given by the Lagrange density for the vector field

$$\mathcal{L} = -\frac{1}{8}F^{ab}F_{ab}+\frac{\xi}{2}m^2\tilde{B}^2. \qquad (23)$$

Our next step is to include a new degree of freedom ϕ via the Stückelberg transformation, as follows:

$$\tilde{B}_a\to B_a+\frac{\partial_a\phi}{m}. \qquad (24)$$

The scalar ϕ will play the role of the Goldstone scalar and again induces mixing terms in the Lagrange density, which are removed by a second gauge fixing similar to the inhomogenous Gupta-Bleuler condition in the R_ξ-gauge, as follows:

$$\mathcal{L}_{\rm gf2} = \frac{1}{2\lambda}(\partial_a B^a-\lambda m\phi)^2. \qquad (25)$$

The addition of Eq. (25) under consideration of Eq. (24) yields the desired diagonal form

$$\mathcal{L}+\mathcal{L}_{\rm gf2} = B_a\mathcal{D}^{ab}B_b+\phi D\phi, \qquad (26)$$

where $\mathcal{D}_{ab}$ and K in Eq. (26) are the kinetic operators with respect to the fields B^a and ϕ (with $\lambda\equiv 1$), as follows:

$$\mathcal{D}_{ab} = \frac{1}{4}\Box\eta_{ab}-\frac{1}{4}\partial_a\partial_b+\frac{1}{2}m^2\eta_{ab} \qquad (27)$$

$$D = -\frac{1}{2}(\Box+m^2). \qquad (28)$$

To reduce the scattering matrix element, we exploit again the Ward-Takahashi identity and multiply the scattering matrix element with a momentum k^a. Recalling the formula, we have already evaluated for spin-1 (cf., Sec. II A), we get

$$\begin{aligned}-i\frac{k^a}{m}\mathcal{M}_a(\psi_{\rm in}\to\psi_{\rm out}+B_a) &= C_B\int \mathrm{d}^4x e^{ikx}\frac{-i}{m}\partial_a\mathcal{D}^{ab}\langle\psi_{\rm out}\rangle B_b|\psi_{\rm in}\rangle\\ &= C_B\int \mathrm{d}^4x e^{ikx}\frac{i}{m}\langle\psi_{\rm out}|m\Box\phi-\lambda m^3\phi|\psi_{\rm in}\rangle\\ &= \mathcal{M}(\psi_{\rm in}\to\psi_{\rm out}+\phi). \end{aligned}\qquad (29)$$

In the last step, we inserted the high-energy approximation $\frac{k^a}{m}\approx\varepsilon^a_L$. Putting everything together, we have proven the GBET for massive spin-2 fields. The relation between the longitudinal degree of freedom in h_{ab} and the Goldstone scalar can be written in terms of the rest-matrix $\mathcal{M}_{ab}$ for h_{ab} and the amplitude $\mathcal{M}$ for ϕ as

$$\begin{aligned}&\varepsilon_L^a\varepsilon_L^b\mathcal{M}_{ab}(\psi_{\rm in}\to\psi_{\rm out}+h_{ab})\\&\quad\simeq\varepsilon_L^a\mathcal{M}_a(\psi_{\rm in}\to\psi_{\rm out}+B_a)\\&\quad\simeq\mathcal{M}(\psi_{\rm in}\to\psi_{\rm out}+\phi)\left[1+\mathcal{O}\left(\frac{m}{E}\right)\right],\end{aligned}\tag{30}$$

which successively confirms the GBET for spin-2 tensor fields in the high-energy limit on a flat background.

Although direct experimental measurements of graviton scattering amplitudes are still inaccessible, the S-matrix facilitates the probing of the high-energy structure in massive spin-2 theories. In particular, the GBET shows that the helicity-0 sector is the dominant degree of freedom at high energies, which is also responsible for many distinctive features of massive gravity, such as strong-coupling phenomena, decoupling limits, and potential instabilities.

III. CURVED BACKGROUNDS

Compared to Minkowski spacetime, curved spacetimes unveil a vast variety of subtleties pertinent to quantum field theory. To extend our results, we need a brief investigation of these peculiarities, which mostly root in the inapplicability of the Stone-von Neumann theorem. We recall that fields on curved spacetimes are not representations of the Poincaré group [27]; that is to say, there are infinitely many unitarily nonequivalent vacuum states, none of which needs to be preferred. In de Sitter spacetime, the Bunch-Davies vacuum represents such a preferred vacuum, because it fulfills the Hadamard condition [28,29], i.e., the light cone approaches the Minkowski light cone, while all isometries of the background are respected [30].

There exists another difficulty that arises from the lack of Poincaré symmetry: the Hilbert spaces at infinite past and infinite future are distinct, i.e., $\mathscr{H}_{\rm in}\neq\mathscr{H}_{\rm out}$. Therefore, the states in the initial Hilbert space $\mathscr{H}_{\rm in}$ have to be related with the final Hilbert space $\mathscr{H}_{\rm out}$ via a Bogoliubov transformation. In other words, even for free fields, the evolution may involve a scattering operator due to the nonequivalence of vacua [31]. Intuitively, curved spacetime represents an external quantity which is coupled to the specific quantum field theory. This motivates the presence of a scattering operator $\mathcal{S}:\mathscr{H}\to\mathscr{H}'$ even in persistency amplitudes in order to connect both Hilbert spaces, i.e., $\langle\psi_{\rm out}|\psi_{\rm in}\rangle=\langle\psi_{\rm in}\mathcal{S}|\psi_{\rm in}\rangle$. The geometric information is encoded in the Bogoliubov transformation and in principle also in the vacuum [32]. Another peculiarity comes with the definition of the Fourier transform. Flat spacetime descriptions provide plane waves as harmonics, which in turn serve as a complete basis for a Fourier expansion. In curved spacetime, this procedure cannot be applied in general. For spacetimes with certain symmetries, such as Bianchi I spacetimes, we could use plane waves, but e.g., for spherically symmetric spacetimes, we would need spherical harmonics as the basis.

Our starting point is the Lagrangian density for a massive spin-2 particle on an arbitrarily curved background, as follows:

$$\mathcal{L}=\mathcal{L}_{\rm EH}-\frac{1}{2}m^2(\tilde{h}_{ab}\tilde{h}^{ab}-\tilde{h}^2)-\frac{1}{2}\tilde{h}_{ab}T^{ab},\tag{31}$$

where $\mathcal{L}_{\rm EH}$ denotes the Einstein-Hilbert part of the Lagrangian, which is the kinetic part for h_{ab}. Massive gravity theories have been consistently formulated on curved spacetimes by de Rham *et al.* [18], who showed that the theory can be resummed and is ghost-free at all orders of the decoupling limit.

At this point, we take the position of studying a spin-2 field regardless if it describes gravity. Following the steps in Sec. II B, we introduce new degrees of freedom through the Stückelberg technique,

$$\tilde{h}_{ab}\to h_{ab}+\frac{\nabla_aA_b}{2m}+\frac{\nabla_bA_a}{2m},\tag{32}$$

with A_a being a Stückelberg vector and m the mass. Note that the derivative is replaced by a covariant derivative $\nabla_av_b=\partial_av_b+\Gamma^c_{ab}v_c$, with $\Gamma^c_{ab}=\frac{1}{2}g^{cd}\partial_bg_{ad}$ the Christoffel symbol. The Lagrange density $\mathcal{L}_{\rm EH}$ is not invariant under the Stückelberg transformation, and thus yields additional terms proportional to the vector field A_a and the Ricci curvature tensor R_{ab}. However, since the whole theory is gauge invariant, these terms cancel with terms coming from the coupling to the source $\tilde{h}_{ab}T^{ab}$.

Introducing Stückelberg fields provides additional symmetries [13] which provide a gauge freedom we use to bring the Lagrangian into diagonal form. We choose as gauge-fixing condition the following:

$$\mathcal{L}_{\rm gf}=\frac{1}{2\lambda}(\nabla^ah_{ab}-\nabla_bh-\lambda mA_b)^2.\tag{33}$$

By adding this gauge fixing to our original Lagrangian, we will be able to extract the kinetic operator $(\mathscr{D}^{ab}{}_{cd})_x$. The explicit form of $(\mathscr{D}^{ab}{}_{cd})_x$ can be found to be (cf., Appendix A for the explicit form of the Lagrange density)

$$\begin{aligned}\mathscr{D}_{abcd}=&-\frac{1}{2}m^2\left(\frac{1}{2}g_{ac}g_{bd}+\frac{1}{2}g_{bc}g_{ad}-g_{ab}g_{cd}\right)\\&-\frac{1}{2\lambda}g_{ab}g_{cd}\Box+\frac{1}{16\lambda}R_{(ac}g_{bd)}-\frac{1}{16\lambda}g_{(ad}\nabla_c\nabla_{b)}\\&+\frac{1}{8\lambda}(R_{dabc}+R_{dbac})+\frac{1}{4\lambda}g_{(ab}\nabla_c\nabla_{d)}.\end{aligned}\tag{34}$$

Here, we used the usual convention for the d'Alembert operator $\Box=\nabla_a\nabla^a$, where the contraction is performed by the background metric g_{ab}. Note that we use the metricity condition, i.e., the metric in unaffected by the covariant derivatives. The sum over all possible symmetric permutations of the indices is denoted by the round brackets in the index, e.g., $t_{(ab)} = \frac{1}{2}(t_{ab} + t_{ba})$. Note that due to the metric affinity of the Levi-Civita connection, $\nabla g = 0$, there are no terms coming from the metric in Eq. (A1). An important subtlety arises due to the fact that the covariant derivatives do not commute, which follows from the definition of the Riemann tensor $R(X, Y)Z$, as follows:

$$R^a{}_{bcd} X^b Y^c Z^d = (\nabla_d \nabla_c Z^a - \nabla_c \nabla_d Z^a) X^c Y^d, \quad (35)$$

for arbitrary vector fields X, Y, and Z.

Next, we define the complete set of solutions by f^n_{ab} and d^n_{ab} with respect to the symplectic product $(\cdot,\cdot)$ as

$$(f^n_{cd}, f^m_{ab}) = \frac{1}{2}(g_{ac}g_{bd} + g_{bc}g_{ad})\delta_{nm}, \quad (36)$$

$$((f^n_{cd})^*, (f^m_{ab})^*) = -\frac{1}{2}(g_{ac}g_{bd} + g_{bc}g_{ac})\delta_{nm}, \quad (37)$$

$$(f^n_{cd}, (f^m_{ab})^*) = 0, \quad (38)$$

where g_{ab} represents the background metric; similar expressions hold for d^n_{cd}. Those mode functions can be used to expand the field into creation and annihilation operators. To do this, we define in- and out-vacuum states. In order to express the fields asymptotically as free states, we impose restrictions on the background spacetime, i.e., the spacetime must support asymptotically free states which are effectively not affected by the curvature. These setups are realized in asymptotically flat or asymptotically simple spacetimes.

From now on we specialize to spacetimes which support asymptotic theory such that the LSZ reduction can be performed [33]. Using this, we represent the interacting fields—using asymptotical freedom in the asymptotic regions—h^{in}_{ab} and h^{out}_{ab} in a complete basis, as follows:

$$\begin{aligned} h^{\text{in}}_{ab}(x) &= \sum_n [a^{\text{in}}_n f^n_{ab}(x) + (a^{\text{in}}_n)^\dagger (f^n_{ab}(x))^*] \\ &= \sum_n [b^{\text{in}}_n d^n_{ab}(x) + (b^{\text{in}}_n)^\dagger (d^n_{ab}(x))^*] \\ h^{\text{out}}_{ab}(x) &= \sum_n [a^{\text{out}}_n f^n_{ab}(x) + (a^{\text{out}}_n)^\dagger (f^n_{ab}(x))^*] \\ &= \sum_n [b^{\text{out}}_n d^n_{ab}(x) + (b^{\text{out}}_n)^\dagger (d^n_{ab}(x))^*]. \end{aligned} \quad (39)$$

Given the Bogoliubov transformation, the fields can be either expressed by the ladder operators a_n or b_n, which are the usual annihilation operators with respect to the individual vacuum and its corresponding eigenbases f^n_{ab} or d^n_{ab}. We wanted to mention that other dependencies, e.g., momentum, are stored in the metalabel n.

Quantization of the system requires constructing "in" and "out" Fock spaces. In static spacetimes, there will be a natural definition of positive frequency modes due to the existence of a global timelike Killing vector field. The positive frequency solutions f^n_{ab} fulfill the equation of motion and can be used to give a natural definition of the vacuum [32].

In curved spacetime, this basis is not unique, because Poincaré symmetry is not necessarily present. In fact, we have infinitely many nonequivalent vacua, thus, the decomposition into positive and negative frequency modes is not unique [31]. A second peculiarity is given by asymptotic theory, that is to say, fields become effectively free $h_{ab}(t,x) \to h^{\text{in}}_{ab}(t,x)$ in the limit $t \to -\infty$ and $h_{ab}(t,x) \to h^{\text{out}}_{ab}(t,x)$ in the limit $t \to +\infty$. Additionally, we assume they can be described as momentum eigenstates [34] such that we can substitute the creation or annihilation operators by expressions in terms of the field operator. This presumes the existence of asymptotic particle states or, in cosmological settings, an adiabatic particle interpretation in the WKB regime. Suppose the modes f^n_{ab} are positive frequency as $t \to -\infty$ and d^n_{ab} as $t \to +\infty$. Thus, the modes f^n_{ab} and $(f^n_{ab})^*$ can be related with d^n_{ab} and $(d^n_{ab})^*$ by a Bogoliubov transformation, as follows:

$$f^n_{ab} = \sum_m (\alpha^*_{nm} d^m_{ab} + \beta_{nm}(d^m_{ab})^*) \quad (40)$$

$$d^n_{ab} = \sum_m (\alpha_{nm} f^m_{ab} + \beta_{nm}(f^m_{ab})^*), \quad (41)$$

where the Bogoliubov coefficients α_{nm} and β_{nm} are time-independent complex matrices which obey

$$\sum_r (\alpha_{mr}\alpha^*_{nr} - \beta_{mr}\beta^*_{nr}) = \delta_{mn}. \quad (42)$$

Our aim is evaluating the amplitude $\langle \text{out}, n | m, \text{in} \rangle$. Note that due to the nontrivial vacuum structure in curved spacetimes, even the vacuum persistency amplitude $\langle \text{out}, n | m, \text{in} \rangle = \langle \text{out}, n | \mathcal{S} | m, \text{in} \rangle$ incorporates a scattering operator in order to translate between the states. The $\mathcal{S}$-operator contains vacuum expectation values of time-ordered field operators [32,34]

$$\begin{aligned} &\langle \text{out}, n^{(s)} | \mathcal{S} | m^{(r)}, \text{in} \rangle \\ &= \sum_\tau i^{\frac{r-l}{2}} \prod_{p=1}^{k} \mathcal{O}_{z_{\tau(p)}} \prod_{q=k+1}^{l} \alpha^{-1}_{m_{\tau(q)} n_{\tau(q)}} \prod_{j=l+1}^{r-1} \Lambda_{\tau(j)\tau(j+1)} \\ &\quad \times \sum_\sigma i^{\frac{s-l-w+k}{2}} \prod_{\iota=1}^{w} \mathcal{Q}_{z_{\sigma(\iota)}} \prod_{v=w+1}^{s-l+k-1} V_{\sigma(v)\sigma(v+1)} \\ &\quad \times \langle \text{out} | \mathcal{T}\{ h_{ab}(x_{\sigma(\iota)}) h_{ab}(x_{\tau(p)}) \} | \text{in} \rangle, \end{aligned} \quad (43)$$

where we have made several definitions: the integral operators $\mathcal{O}_{x_i}$ and $\mathcal{Q}_{x_i}$ are with respect to the two different mode functions d^{ab}_n and f^{ab}_l, as follows:

$$\mathscr{O}_{x_i} \equiv i\sum_l \alpha^{-1}_{m_i l}\int [d^{cd}_l(x_i)(\mathscr{D}^{ab}{}_{cd})_{x_i}]\sqrt{-g_{x_i}}\mathrm{d}^4x_i \tag{44}$$

$$\mathscr{Q}_{x_i} \equiv i\sum_l \alpha^{-1}_{ln_i}\int [f^{cd}_l(x_i)(\mathscr{D}^{ab}{}_{cd})_{x_i}]\sqrt{-g_{x_i}}\mathrm{d}^4x_i, \tag{45}$$

where the permutations of the indices are given by τ and σ, while the coefficients of the matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{V}$ are determined by the Bogoliubov coefficients

$$\Lambda_{ij} \equiv -i\sum_n \alpha^{-1}_{m_i n}\beta_{nm_j} \tag{46}$$

$$V_{ij} \equiv i\sum_n \beta^*_{m_j n}\alpha^{-1}_{nm_i}. \tag{47}$$

When we factor out the polarization in d^{ab}_n to get $d^{ab}_n = d_n\varepsilon^{ab}$, the index n no longer encodes the polarization of the fields. Our next step is to substitute the polarization tensor with the momentum, therefore we split ε^{ab} into its vector parts and perform the high-energy limit. Thus, we can replace ε^{ab} by the longitudinal polarized part ε^{ab}_L, which in turn yields[5]

$$\varepsilon^{ab}_L = \frac{k^a k^b}{m^2} + \mathcal{O}\left(\frac{m}{E}\right). \tag{48}$$

Let us proceed by using only one momentum at a time. Contrary to the flat metric, we substitute k_a with covariant derivatives to be consistent with the equations of motions in curved spacetime. In order to finally prove the GBET, we get the Goldstone fields and relate them to the spin-2 tensor field $h_{ab}(x)$. The underlying symmetry provides the Gupta-Bleuler condition and the gauge fixing, as follows:

$$\nabla^a h_{ab} = \nabla_b h + \lambda m A_b. \tag{49}$$

After inserting the gauge condition (49) into Eq. (31), we get a part of the Lagrange density only depending on the Goldstone vector, that is, $-\mathcal{L}_A = A_a\mathcal{D}^{ab}A_b$, with $\mathcal{D}^{ab}$ being the differential operator for A_a, as follows:

$$\mathcal{D}_{ab} = \left[\frac{1}{4}g_{ab}\Box - \frac{1}{8}\nabla_b\nabla_a - \frac{1}{8}\nabla_a\nabla_b + \frac{3}{8}R_{ab} + \frac{\lambda}{2}m^2 g_{ab}\right]. \tag{50}$$

After using the gauge freedom, we find the following condition to hold:

$$\nabla^b\mathscr{D}^{cd}{}_{ba}h_{cd} = -m\mathcal{D}^b{}_a A_b, \tag{51}$$

where the explicit calculations are found in Appendix B. Thus, we could formulate the first step of the GBET on curved spacetimes, i.e., a reduction to Goldstone vectors

$$\begin{aligned}\varepsilon^b k^a\mathcal{M}_{ab}(\psi_{\mathrm{in}} + h_{ab} \to \psi_{\mathrm{out}}) \\ = C_i\langle \mathrm{out}, n^{(s)}|m\int [\varepsilon^b\mathcal{D}^a{}_b A_a]\sqrt{-g}\mathrm{d}^4x|m^{(r)} - m_i, \mathrm{in}\rangle\end{aligned} \tag{52}$$

for the rest-matrix element. In the high-energy limit, we use the condition that the longitudinal polarization dominates and we find the desired formulation of the GBET, as follows:

$$\varepsilon^b\varepsilon^a_L\mathcal{M}_{ab}(\psi_{\mathrm{in}} + h_{ab} \to \psi_{\mathrm{out}}) = C\varepsilon^b\mathcal{M}_b(\psi_{\mathrm{in}} + A_b \to \psi_{\mathrm{out}}). \tag{53}$$

In the high-energy limit, the theory is—according to Ref. (53)—accurately described by a spin-1 particle; that is, the Goldstone vector field. In this regime, our theory reduces effectively to

$$\mathcal{L}_{\tilde{A}} = -\frac{1}{8}F_{ab}F^{ab} + \frac{1}{2}\tilde{A}^a R_{ab}\tilde{A}^b + \frac{1}{2}\lambda m^2\tilde{A}_a\tilde{A}^a. \tag{54}$$

Again, we make use of the Stückelberg formalism and introduce an additional scalar degree of freedom, as follows:

$$\tilde{A}_a \to A_a + \frac{\nabla_a\phi}{m}. \tag{55}$$

In order to extract the kinetic operators, we need to diagonalize (54). Hence, we choose for the gauge fixing

$$\mathcal{L}_{\mathrm{gf}} = \frac{1}{2\zeta}\left[\nabla^b\left(\frac{A^a R_{ab}}{m} + \lambda m A_b\right) + \zeta m\phi\right]^2, \tag{56}$$

where the gauge parameter ζ denotes the gauge coupling. Our effective theory $\mathcal{L}_A + \mathcal{L}_{\mathrm{gf}}$ including the Goldstone scalar is then described by

$$\begin{aligned}\mathcal{L}_A + \mathcal{L}_\phi = A^b\bigg[&\frac{1}{4}\Box g^a_b - \frac{1}{8}\nabla_b\nabla^a - \frac{1}{8}\nabla^a\nabla_b + \frac{3}{8}R^a_b \\ &+ \frac{\lambda m^2 g^a_b}{2} - \frac{(R^a_c + \lambda m^2 g^a_c)\nabla^c\nabla^d(R_{bd} + \lambda m^2 g_{bd})}{2\zeta m^2}\bigg]A_a \\ &+ \phi\left[-\frac{1}{2m^2}\nabla^a(R_{ac}\nabla^c) - \frac{\lambda}{2}\Box + \frac{\zeta m^2}{4}\right]\phi.\end{aligned} \tag{57}$$

Consider the matrix element for the massive spin-1 particle—but this time in the high-energy limit. For

[5]The decoupling limit in curved spacetimes is rather local than global; additionally one may need to consider low curvature, i.e., $R \ll E^2$ such that some operators become curvature suppressed as R/m^2.

convenience, we call the expression in brackets $\bar{\mathcal{D}}_{ab}$. With Eq. (56), we get

$$\nabla^b \bar{\mathcal{D}}^a_b A_a = -\frac{1}{4}\zeta m^3 \phi + \frac{1}{2m}\nabla^b [R_{bc}\nabla^c \phi] + \frac{1}{2}\lambda m \Box \phi. \quad (58)$$

Comparison to the kinetic operator for the scalar field (57) shows that we have picked up an extra term that is proportional to the Ricci curvature tensor. This term inflicts a direct modification of the kinetic operator, such that $\overline{\Box} = \mathfrak{g}^{ab}\nabla_a\nabla_b$ where the effective metric $\mathfrak{g}$ contains a Ricci tensor modification

$$\mathfrak{g}_{ab} = g_{ab} + \frac{1}{\lambda m^2} R_{ab}, \quad (59)$$

which comes with a mass suppression. This feature is important, because it will introduce a limit to the validity of the Stückelberg procedure within curved spacetimes. Additionally, this extra term highlights the difference between the high-energy limit and high-curvature region. In curved spacetimes, these two limits do not necessarily coincide, which is due to the fact that no Fourier conjugation exists between configuration and momentum space. In this sense, we need to be careful with both limits, and the additional Ricci term keeps track of this, as we will see for the FLRW spacetime in the next section.

We would expect such a limit to be prevalent, since we work in a linearized theory with weakly coupled degrees of freedom. However, as effective field theory studies of quantum fields on curved backgrounds show, even the free field description can, in certain setups, reach the boundaries of validity of the description no matter the initial conditions. This may happen when, due to curvature effects, the field experiences spacetime friction that renders the effective description insignificant [35].

On the other hand, the effective metric does not necessarily need to support a well-defined kinetic term since, in principle, the sign in $\mathfrak{g}$ is not *a priori* fixed, leading to a possibly wrong sign in the kinetic term, which is associated with an instability in the system. From this perspective, the methodology we used serves as a sensitive indicator to study these bounds explicitly.

As such, this term is potentially harmful, as it may trigger several instabilities. Since the term itself remains second order in derivatives, Ostrogradsky instabilities are not expected to occur. Although the LSZ reduction formalism has reduced the linearized spin-2 tensor field to only the healthy Goldstone scalar degree of freedom (the sixth ghost degree of freedom was excluded), the modified kinetic operator could still cause instabilities.

Hence, constructing the kinetic operator D with $\mathfrak{g}$ gives the effective differential operator $\bar{D}$. Nevertheless, due to the locality of the decoupling limit in curved spacetimes, such operators could also be considered to decouple in the high-energy limit as long as $R \ll E^2$ and, thus, these terms become curvature suppressed by R/m^2. In this sense, we find

$$\nabla_b \bar{\mathcal{D}}^{ab} A_a = -m\bar{D}\phi. \quad (60)$$

Putting everything together, it turns out that through successively reducing the spin-2 tensor field, we can formulate the Goldstone boson equivalence theorem for deformed spin-2 tensors on curved spacetimes, as follows:

$$\begin{aligned} &\varepsilon_L^{cd}\mathcal{M}_{cd}(\psi_{\text{in}} + h_{ab} \to \psi_{\text{out}}) \\ &\quad = -C\mathcal{M}(\psi_{\text{in}} + \phi \to \psi_{\text{out}})\left[1 + \mathcal{O}\left(\frac{E}{m}\right) + \mathcal{O}\left(\frac{R}{m^2}\right)\right], \end{aligned} \quad (61)$$

which concludes that the Goldstone boson equivalence theorem is valid for high enough energies and sufficiently large curvature radii.

IV. COSMOLOGICAL SPACETIME

The FLRW spacetime is, in cosmological time, given by

$$g = -\mathrm{d}t \otimes \mathrm{d}t + a^2(t)\left(\frac{\mathrm{d}r \otimes \mathrm{d}r}{1-\kappa r^2} + r^2 \mathrm{d}\mathbb{S}_2\right), \quad (62)$$

where $a(t)$ denotes the scale factor and $\mathrm{d}\mathbb{S}_2$ the line element of the 2-sphere. The sectional curvature is determined through κ. It captures three cases: a flat geometry (flat universe) with $\kappa = 0$, a spherical geometry (closed universe) for $\kappa = 1$, and a hyperbolic geometry for $\kappa = -1$ (open universe).

Regarding particular examples, we study the radiation-dominated universe, i.e., $a(t) = \sqrt{t}$ and the de Sitter scenario $a(t) = e^{\mathcal{H}_0 t}$ with $\mathcal{H}_0$ a constant. Radiation-dominated universes describe highly energetic regions such as right after the big bang, while the de Sitter phase captures inflationary scenarios.

We would like to understand the impact of the effective kinetic operator on an FLRW background. For the sake of simplicity, we choose the flat case and work in Cartesian coordinates. In this geometry, the Ricci tensor becomes diagonal with nonzero components, as follows:

$$R^{tt} = -3(\dot{\mathcal{H}} + \mathcal{H}^2), \qquad R^{ii} = (\dot{\mathcal{H}} + 3\mathcal{H})g^{ii}, \quad \forall\, i \in \{x, y, z\}, \quad (63)$$

where we defined the Hubble parameter $\mathcal{H}(t) = \frac{\dot{a}(t)}{a(t)}$. Using Eq. (59) and the ϕ-part of the Lagrange density (57),

$$\mathcal{L} \supset \mathcal{L}_\phi = -\frac{1}{2}\phi\left[\mathfrak{g}^{ab}\nabla_a\nabla_b - \frac{\zeta m^2}{2}\right]\phi, \quad (64)$$

we discover several potential cases for instabilities. They are related to the components of $\mathfrak{g}$. The first instability would be a ghost instability originating from the tt-component. This pathology is prevented as long as

$$0 < 1 + \frac{3(\dot{\mathcal{H}} + \mathcal{H}^2)}{\lambda m^2}. \tag{65}$$

The spatial coefficients could generate a similar instability due to a sign change. Hence, whenever

$$0 < 1 + \frac{\dot{\mathcal{H}} + 3\mathcal{H}^2}{\lambda m^2}, \tag{66}$$

the setup remains healthy. A violation of Eq. (66) would result in an exponential growth for large-k modes. The third instability is connected to the propagation speed of fluctuations. A modified propagator will trickle down to the dispersion relation

$$\omega^2 = c_s^2 \frac{k^2}{a^2} + \frac{\zeta m^2}{2}, \tag{67}$$

where we defined the effective signal speed c_s as

$$c_s^2 = \frac{\lambda m^2 + \dot{\mathcal{H}} + 3\mathcal{H}^2}{\lambda m^2 + 3\dot{\mathcal{H}} + 3\mathcal{H}^2}. \tag{68}$$

It is clear that whenever $c_s^2 \leq 0$, we develop a gradient instability that leads to an exponentially growing mode, while the condition $c_s > 1$ would violate the rules of information processing due to superluminal propagation speed. Regarding the latter, we would retrieve a condition

$$2\dot{\mathcal{H}} < 3\mathcal{H}^2 \quad \Leftrightarrow \quad 2\frac{\ddot{a}}{a} < 3\left(\frac{\dot{a}}{a}\right)^2. \tag{69}$$

To appreciate the information coming from these inequalities, we consider two examples of specific matter content: a cosmological constant (de Sitter universe) and the radiation-filled universe.

A. De Sitter universe

Starting with the first example, $a(t) = e^{\mathcal{H}_0 t}$, where $\mathcal{H} = \mathcal{H}_0$ and $\dot{\mathcal{H}} = 0$. In this case, most conditions are trivially obeyed (at least under meaningful assumptions, e.g., $\mathcal{H}^2 > 0$ and $m^2 > 0$). The bounds from both the temporal as well as the spatial gradient fulfill $\lambda m^2 + 3\mathcal{H}^2 > 0$; therefore, there exists neither a ghost nor a gradient instability. Since, $\dot{\mathcal{H}} \equiv 0$ everywhere, $c_s = 1$ for de Sitter patches. This resonates with the simulation-based analysis in Ref. [35].

B. Radiation-filled universe

The radiation-filled universe, that is, $a(t) = \sqrt{t}$, offers a much richer phenomenology with the big bang singularity at $t = 0$. As a consequence, the condition for a lack of ghosts is not necessarily fulfilled because $R_{00} = -3\frac{\ddot{a}}{a} = -\frac{3}{4t^2}$ is negative definite for $t > 0$. Hence, Eq. (65) is violated whenever

$$t < \sqrt{\frac{3}{4m^2\lambda}}, \tag{70}$$

because the sign of the kinetic term changes. To turn it around, the Goldstone description becomes predictive, after a certain distance from the big bang. In regions where the curvature radius becomes small with respect to the mass of the scalar, we see that the GBET breaks down because it becomes a ghost degree of freedom. The sign change only occurs in the tt-component while the spatial components remain positive definite, and therefore Eq. (66) is always fulfilled such that no gradient instability is triggered.

The late time consistency is maintained as well for the signal propagation speed, which reads

$$c_s = \frac{4\lambda m^2 t^2 + 1}{4\lambda m^2 t^2 - 3}, \tag{71}$$

which for large times becomes approximately 1, while small times create huge digressions from the speed of light, such that for a similar range $t \sim \frac{1}{m}$, the description shows pathological features. However, this deformation of the signal speed is parametrically small, i.e., $c_s^2 - 1 \sim \frac{\mathcal{H}^2}{m^2}$, which is within the errors induced by the GBET itself, that is, $\mathcal{O}(\frac{R}{m^2})$.

C. Open and closed universes

Finally, we classify our findings with respect to the strong coupling regime with cutoff $\Lambda_3 \sim (m^2 m_{\mathrm{pl}})^{1/3}$, where m_{pl} denotes the Planck mass [18]. Effectively, the curvature contributions admit a mass suppression of the kind $\frac{R}{m^2}$. The spin-0 mode enters the strong coupling regime once $R \sim \mathcal{H}^2 \gtrsim m^2$. For radiation-dominated universes, a majority of the modes will not obey this condition for small times [35]; hence, the effective field theory description will break down long before the ghost regime will be reached. When analyzing the effective field theory bound for de Sitter, we consider the Higuchi bound [36], $m^2 \geq 2\mathcal{H}_0^2$, which limits our description beyond the limits set by the effective metric.

Let us comment on the possibilities to incorporate the other sectional curvatures $\kappa = \pm 1$ as well. The issue is that we cannot rely on the Fourier transformation, since the harmonic functions u_p in spaces with nonzero sectional curvature are different; hence, we must use the harmonic equation $\Delta u_p = \lambda_p u_p$ with $p \in \{k, n, q\}$, which accounts for all cases. The precise spectrum depends on the spatial curvature:

$$\kappa = 0 \quad \Rightarrow \quad \lambda_k = k^2, \qquad k \in \mathbb{R}^+, \tag{72}$$

$$\kappa = +1 \quad \Rightarrow \quad \lambda_n = n(n+2), \qquad n \in \mathbb{N}_0, \tag{73}$$

$$\kappa = -1 \quad \Rightarrow \quad \lambda_q = q^2 + 1, \qquad q \in \mathbb{R}_>. \tag{74}$$

The quantity $\sqrt{\lambda_p}$ therefore plays the role of the comoving momentum magnitude, while the associated physical momentum would be $p = \sqrt{\lambda_p}/a(t)$. In the short-wavelength regime $p \ll H$, the harmonics admit a local WKB interpretation and derivative operators may be replaced parametrically by powers of $\sqrt{\lambda_p}$, reproducing the standard flat-space momentum correspondence.

It is important to stress that the harmonics are not truly eigenfunctions of the single derivative operator in the same sense as are plane waves. Rather, they diagonalize the Laplace-Beltrami operator, which is the covariant second-order spatial operator compatible with homogeneity and isotropy. Consequently, the identification $\sqrt{\lambda_p} \sim |\boldsymbol{k}|$ should be understood in terms of spectral theory: it characterizes the inverse wavelength or gradient scale of the mode, not the eigenvalue of an individual covariant derivative.

In this sense, due to the essential self-adjointness of Δ, one may take the formal square root of the harmonic equation to associate a momentum with the single derivative; i.e., $\sqrt{\Delta} = \sqrt{\lambda_p}$ then implies $|\nabla| \to \sqrt{\lambda_p}$ mode by mode.

The physical justification originates from the short-wavelength or WKB regime. We assume that this approximation holds such that we are able to define instantaneous vacuum states [30]. Upon a formal harmonic decomposition of the field,

$$\phi(x) = \int \mathrm{d}^3 p \varphi_p(t) u_p(\boldsymbol{x}), \tag{75}$$

the Klein-Gordon equations transform into a free massive scalar mode obeying the wave equation

$$\ddot{\varphi}_p + 3H\dot{\varphi}_p + \left(m^2 + \frac{\lambda_p}{a^2(t)}\right)\varphi_p = 0. \tag{76}$$

Assuming that the mode frequency varies slowly compared with the oscillation time scale, one can use the ansatz

$$\varphi_p(t) = A(t) \exp\left(-i\int_0^t \omega_p(\tau)\mathrm{d}\tau\right), \tag{77}$$

where $A(t)$ denotes the amplitude here. At zeroth order in adiabaticity [37,38], this yields

$$\omega_p^{(0)}(t) \simeq \sqrt{m^2 + \frac{\lambda_p}{a(t)^2}}. \tag{78}$$

Thus, in the adiabatic expansion and within the WKB approximation, the harmonic mode behaves locally as a field on a flat background with physical momentum $p = \sqrt{\lambda_p}/a(t)$. Note that the validity of the WKB expansion is determined through $|\dot{\omega}_p| \ll |\omega_p^2|$, which for high-energetic modes reduces parametrically to

$$\mathcal{H} \ll \frac{\sqrt{\lambda_p}}{a(t)}. \tag{79}$$

In that limit, the mode probes only a local region of spacetime for which curvature effects are negligible.

This observation is particularly relevant for the GBET, because what is referred to as high energies should in fact be small distances in curved spacetimes. Hence, the theorem should be understood as holding in the joint high-energy and short-wavelength regime for which longitudinal polarizations reduce locally to their Goldstone counterparts up to corrections. While the harmonic decomposition provides the natural curved-space replacement of Fourier momentum methods, the WKB limit supplies the dynamical justification for their use.

In fact, the harmonics provide then a spectral replacement that is valid mode by mode, and becomes asymptotically exact in the UV (or WKB) limit. In this sense, the GBET for generic FLRW spacetime would read

$$\begin{aligned} &\varepsilon^{cd}\mathcal{M}_{cd}(\psi_{\mathrm{in}} + h_{ab} \to \psi_{\mathrm{out}}) \\ &= C\mathcal{M}(\psi_{\mathrm{in}} + \phi \to \psi_{\mathrm{out}}) \\ &\quad \times \left[1 + \mathcal{O}\left(\frac{ma(t)}{\sqrt{\lambda_p}}\right) + \mathcal{O}\left(\frac{\mathcal{H}(t)a(t)}{\sqrt{\lambda_p}}\right)\right], \end{aligned} \tag{80}$$

where we expressed the physical quantities through the spectral values of the harmonics.

V. CONCLUSION

We showed that the Goldstone boson equivalence theorem can be formulated on a certain class of flat spacetimes. Within the high-energy regime, even a spin-2 theory is well described by the longitudinal scalar degree of freedom. Our analysis unveiled additionally that the GBET can only be trusted as long as the curvature radius is large compared to the energy of the field. Once this threshold is exceeded, the Goldstone degree of freedom may cause instabilities, signaling a failure of the description. We would interpret this as the need for incorporating backreaction. Altogether, this shows that the GBET is a remarkably powerful tool and provides robust qualitative and quantitative estimates for an effective high-energy theory. Colloquially speaking, the GBET is a proxy for the full theory at energy scales that are much higher than the particles' mass, and at curvature radii that are larger than the particles' energy.

In this article, we have used a versatile method to derive the GBET based on the LSZ reduction formalism. In particular, we showed that the formalism extends the validity of the GBET for the subclass of cosmological spacetimes such as the flat FLRW background up to the case of spin-2 tensor fields. The generalization of the GBET to spin-2 particles justifies that there shall exist a GBET for general higher spin theories as well; i.e., at high energies massive (infrared deformed) (r, s)-tensor fields shall be well described by the longitudinal scalar degree of freedom which is responsible for strong-coupling effects, decoupling limits, and potential instabilities. In this sense, the significance of the result lies not primarily in the computation of graviton scattering observables, but rather in the insights into the dynamics of the longitudinal sector.

Moreover, the outlined treatment does not require an underlying Abelian gauge symmetry. For the Standard Model, an analog technique can be applied [23]. Instead of using the Gupta-Bleuler condition, non-Abelian theories require BRST charges and the addition of a Fadeev-Popov Lagrange density in order to cancel anomalies. Besides this, the GBET works similar to the procedure outlined in this article and can also be implemented in curved spacetimes.

However, we saw that the formalism propagates potential instabilities coming from a curvature contribution in the kinetic operator. For the FLRW spacetimes, we investigated how the additional term inflicts instabilities and compared them with the effective field theory bound. It turned out that for a de Sitter (inflationary) universe, there are no potential threats, while a radiation-dominated universe may admit potential instabilities. However, those pathologies were only triggered in the high curvature regime, i.e., close to the singularity, and hence, they lie potentially outside the effective field theoretic description. In this sense, the GBET offers a probing mechanism for the range of validity of the Goldstone description on curved spacetimes.

Here, we made use of the fact that we can replace the momentum by a derivative which is a consequence of the Ward identity. Since our analysis has been performed in spatially flat spacetime, we were able to use a replacement *à la* Weinberg, where we traded the partial derivative that comes from the Fourier inversion of the polarization by a covariant derivative. The idea works also for FLRW spacetimes with different sectional curvature, that is, in situations where the spatial slices are spheres or hyperbolic spaces. By using the spectrum of the Laplace-Beltrami operator, we can associate the eigenvalues with a comoving momentum in the high-energy limit (or better in the small-distance limit).

However, in general curved spacetimes, this method cannot be assumed to go through so easily. In fact, one would probably work to identify the vector field of the momentum and then use the harmonics and the related eigenvalue equation as we have done for the nonflat FLRW spacetimes. However, another complication imposes the Ward identities and therefore the LSZ reduction which in generally curved spacetimes may not be straightforward [33,39,40]. While our method works for highly symmetric spacetimes like FLRW, we do not claim that it is universally applicable. The present construction may be generalized to a broader class of backgrounds. Of particular interest would be Einstein backgrounds, which allow a consistent formulation of massive spin-2 theories [41]. In such an approach, the role of the momentum variable would be replaced by the spectral parameter of the relevant Laplace-Beltrami operator—provided it admits an essentially self-adjoint extension—with the associated momentum scale determined by the square root of its eigenvalues. However, a rigorous implementation requires a careful analysis of the spectral properties of the underlying differential operators. Since these issues are strongly geometry dependent, the question how our method can be adapted to more generic backgrounds will be a matter of further research.

ACKNOWLEDGMENTS

Thank you to Sophia Zielinski and Stefan Hofmann for helpful discussions during the early stages of this investigation. This work was supported by the Basque Government Grants No. IT1628-22 and by No. PID2021-123226NB-I00 (funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe").

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

APPENDIX A: EXPLICIT FORM OF THE LAGRANGE DENSITY

The Lagrange density for a spin-2 tensor field on a curved background including the gauge-fixing condition is found to be

$$\mathcal{L}=\mathcal{L}_{\mathrm{EH}}-\frac{1}{2}h_{ab}T^{ab}+h^{ab}\Big[-\frac{m^2}{4}(g^c_a g^d_b+g^c_a g^c_b-2g_{ab}g^{cd})-\frac{1}{16\lambda}(\nabla_a\nabla^c g^d_b+\nabla_b\nabla^c g^d_a+\nabla_a\nabla^d g^c_b+\nabla_b\nabla^d g^c_a)$$
$$-\frac{1}{16\lambda}(\nabla^c\nabla_a g^d_b+\nabla^c\nabla_b g^d_a+\nabla^d\nabla_a g^c_b+\nabla^d\nabla_b g^c_a-R^c{}_a g^d_b-R^c{}_b g^d_a-R^d{}_a g^c_b-R^d{}_b g^c_a)+\frac{1}{8\lambda}(R_b{}^{dc}{}_a+R_b{}^{cd}{}_a)$$
$$+\frac{1}{4\lambda}(\nabla_a\nabla_b g^{cd}+\nabla_b\nabla_a g^{cd}+g_{ab}\nabla^c\nabla^d+g_{ab}\nabla^d\nabla^c)-\frac{1}{2\lambda}(g_{ab}\Box g^{cd})\Big]h_{cd}-\frac{1}{8}F_{ab}F^{ab}+\frac{1}{2}R_{ab}A^aA^b+\frac{\lambda m^2}{2}A_aA^a, \quad \text{(A1)}$$

where $\mathcal{L}_{\mathrm{total}}=L+\mathcal{L}_{\mathrm{gf}}$ with the gauge-fixing term (49).

APPENDIX B: CALCULATIONS IN FLAT SPACETIME

Our starting point of the calculation that is involved in the proof of the GBET for flat spacetime will be the kinetic operator $\mathscr{D}^{ab}{}_{cd}$ in Eq. (B3), which looks, after adding the gauge-fixing term (19), as follows:

$$\mathscr{D}^{ab}{}_{cd}=\frac{1}{2}(\Box-m^2)\left(\frac{1}{2}\eta^a{}_c\eta^b{}_d+\frac{1}{2}\eta^a{}_d\eta^b{}_c-\eta_{cd}\eta^{ab}\right)+\frac{1}{2}\left(1+\frac{1}{\xi}\right)(\eta^{ab}\partial_c\partial_d+\eta_{cd}\partial^a\partial^b)-\frac{1}{2\xi}\eta^{ab}\Box\eta_{cd}$$
$$-\frac{1}{4}\left(1+\frac{1}{2\xi}\right)(\eta^b{}_d\partial^a\partial_c+\eta^b{}_c\partial_d\partial^a+\eta^a{}_d\partial^b\partial_c+\eta^a{}_c\partial^b\partial_d), \quad \text{(B1)}$$

where we omitted the B_a-dependent term. By applying a partial derivative, the kinetic operator can be rewritten as

$$\partial_a\mathscr{D}^{ab}{}_{cd}=\frac{1}{2}(\Box-m^2)\left(\frac{1}{2}\eta^b{}_d\partial_c+\frac{1}{2}\eta^b{}_c\partial_d-\eta_{cd}\partial^b\right)+\frac{1}{2}\left(1+\frac{1}{\xi}\right)(\partial^b\partial_c\partial_d+\eta_{cd}\Box\partial^b)-\frac{1}{2\xi}\eta_{cd}\Box\partial^b$$
$$-\frac{1}{4}\left(1+\frac{1}{2\xi}\right)(\eta^b{}_d\Box\partial_c+\eta^b{}_c\Box\partial_d+\partial_d\partial^b\partial_c+\partial_c\partial^b\partial_d). \quad \text{(B2)}$$

When applying this onto h^{cd}, we can extract the following result:

$$\partial_a\mathscr{D}^{ab}{}_{cd}h^{cd}=\frac{1}{2}(\Box-m^2)(\partial_c h^{bc}-\partial^b h)+\frac{1}{2}\left(1+\frac{1}{\xi}\right)(\partial^b\partial_c\partial_d h^{cd}+\Box\partial^b h)-\frac{1}{2\xi}\Box\partial^b h-\frac{1}{2}\left(1+\frac{1}{2\xi}\right)(\Box\partial_c h^{cb}+\partial_c\partial^b\partial_d h^{cd}).$$
$$=\left[(\Box-m^2)-\left(1-\frac{1}{\xi}\right)\Box\right](\partial_c h^{bc}-\partial^b h)$$
$$=\left(\frac{\Box}{\xi}-m^2\right)\xi m\tilde{B}^b. \quad \text{(B3)}$$

In the last step, we used the gauge condition to replace the spin-2 field by its longitudinal, spin-1 Goldstone boson.

APPENDIX C: CALCULATIONS IN CURVED SPACETIME

To determine the explicit form of $\nabla^a\mathscr{D}_{ab}{}^{cd}h_{cd}$, we use the relation between h_{ab} and the Goldstone (Stückelberg) vector field $\nabla^a h_{ab}=\nabla_b h+\lambda mA_b$ to get

$$\nabla^a\mathscr{D}_{ab}{}^{cd}h_{cd}=-\frac{1}{2}\lambda m^3A_b-\frac{1}{8\lambda}(\Box\nabla_b h+\lambda m\Box A_b+\underbrace{\nabla^a\nabla_b\nabla^c h_{ca}}_{1.}+\underbrace{\nabla^a\nabla^c\nabla_a h_{cb}}_{2.}-4\nabla_b\Box h$$
$$+\underbrace{\nabla^a\nabla^c\nabla_b h_{ca}}_{3.}-(\nabla^a R^c{}_a)h_{cb}-R^c{}_a\nabla^a h_{cb}-(\nabla^a R^c{}_b)h_{ca}-R^c{}_b\nabla^a h_{ca}-2\Box\nabla_b h$$
$$-2(\nabla^a R_b{}^{dc}{}_a)h_{cd}-2R_b{}^{dc}{}_a\nabla^a h_{cd}-2\underbrace{\nabla^a\nabla_b\nabla_a h}_{4.}-4\nabla_b\Box h-4\lambda m\nabla_b\nabla^a A_a).$$

We provide the results for the underlined terms below:

(1)

$$\nabla^a\nabla_b\nabla^c h_{ca} = \nabla_b\nabla_a\nabla_c h^{ca} + R_{ab}{}^{ad}\nabla^c h_{cd} = \nabla_b\Box h + \lambda m\nabla_b\nabla A + R_b{}^d\nabla_d h + \lambda m R_b{}^d A_d$$

(2)

$$\begin{aligned}\nabla^a\nabla^c\nabla_a h_{cb} &= \Box\nabla^c h_{cb} + (\nabla_a R^{da})h_{db} + R^{da}\nabla_a h_{db} + (\nabla_a R_b{}^{dca})h_{cd} + R_b{}^{dca}\nabla_a h_{cd}\\ &= \Box\nabla_b h + \lambda m\Box A_b + (\nabla_a R^{da})h_{db} + R^{da}\nabla_a h_{db} + (\nabla_a R_b{}^{dca})h_{cd} + R_b{}^{dca}\nabla_a h_{cd}\end{aligned}$$

(3)

$$\begin{aligned}\nabla^a\nabla^c\nabla_b h_{ca} &= \nabla_a\nabla_b\nabla_c h^{ca} + (\nabla_a R_{db})h^{da} + R_{db}\nabla_a h^{da} + (\nabla_a R^a{}_{dcb})h^{cd} + R^a{}_{dcb}\nabla_a h^{cd}\\ &= \nabla_b\nabla_a\nabla_c h^{ca} + R_{bd}\nabla_c h^{cd} + (\nabla_a R_{db})h^{da} + R_{db}\nabla_a h^{da} + (\nabla_a R^a{}_{dcb})h^{cd} + R^a{}_{dcb}\nabla_a h^{cd}\\ &= \nabla_b\Box h + \lambda m\nabla_b\nabla A + R_{bd}\nabla^d h + \lambda m R_{bd}A^d + (\nabla_a R_{db})h^{da} + R_{db}\nabla^d h + \lambda m R_{db}A^d\\ &\quad + (\nabla_a R^a{}_{dcb})h^{cd} + R^a{}_{dcb}\nabla_a h^{cd}\end{aligned}$$

(4)

$$\nabla^a\nabla_b\nabla_a h = \nabla_a\nabla_b\nabla^a h = \nabla_b\Box h + R_{ba}\nabla^a h.$$

In the above calculations, the symmetries of the Riemann tensor are used as well as the commutation relation between two covariant derivatives.

For the spin-1 field, we can perform a very similar calculation, which involves again the application of the covariant derivative

$$\begin{aligned}\nabla^b\mathcal{D}^a{}_b A_a &= \frac{1}{2}\nabla^b(R^a_b A_a) + \frac{1}{2}\lambda m^2\nabla A - \frac{1}{2\zeta m^2}\nabla_a[R^a_c\nabla^c\nabla^c(R_{bc}A^b)] - \frac{\lambda}{2\zeta}\nabla^b[R_{bc}\nabla^c\nabla A]\\ &\quad - \frac{1}{2\zeta}\nabla^b[\lambda\nabla_b\nabla^c(R_{ac}A^a)] - \frac{\lambda^2 m^2}{2\zeta}\nabla^b[\nabla_b\nabla A].\end{aligned} \tag{C1}$$

By using the gauge condition from above given by $\nabla^c(A^a R_{ac}) = -\lambda m^2\nabla^a A_a - \zeta m^3\varphi$, we can transform the whole expression into the following form:

$$\begin{aligned}\nabla^b\mathcal{D}^a{}_b A_a &= -\frac{1}{2}(\lambda m^2\nabla A + \zeta m^3\varphi) + \frac{1}{2}\lambda m^2\nabla A + \frac{1}{2\zeta m^2}\nabla^b[R_{bc}\nabla^c(\lambda m^2\nabla A + \zeta m^3\varphi)] - \frac{\lambda}{2\zeta}\nabla^b[R_{bc}\nabla^c\nabla A]\\ &\quad + \frac{1}{2\zeta}\nabla^b[\lambda\nabla_b(\lambda m^2\nabla A + \zeta m^3\varphi)] - \frac{\lambda^2 m^2}{2\zeta}\nabla^b[\nabla_b\nabla A]\\ &= -\frac{1}{4}\zeta m^3\varphi + \frac{1}{2m}\nabla^b[R_{bc}\nabla^c\varphi] + \frac{1}{2}\lambda m\Box\varphi,\end{aligned} \tag{C2}$$

which yields the desired result for the Stückelberg scalar.

[1] H. Lehmann, K. Symanzik, and W. Zimmermann, On the formulation of quantised field theories—II, Nuovo Cimento (1955–1965) **6**, 319 (1957).

[2] A. Farzinnia, Prospects for discovering the Higgs-like Pseudo-Nambu-goldstone boson of the classical scale symmetry, Phys. Rev. D **92**, 095012 (2015).

[3] G. Cuomo, L. Vecchi, and A. Wulzer, Goldstone equivalence and high energy electroweak physics, SciPost Phys. **8**, 078 (2020).

[4] J. Cornwall, D. Levin, and G. Tiktopoulos, Derivation of gauge invariance from high-energy unitarity bounds on the s-matrix, Phys. Rev. **10**, 1145 (1974).

[5] M. Chanowitz and M. Gaillard, The TeV physics of strongly interacting W's and Z's, Nucl. Phys. **B261**, 951 (1985).

[6] B. Lee, C. Quigg, and H. Thacker, Weak interactions at very high energies: The role of the Higgs boson mass, Phys. Rev. D **16**, 1519 (1977).

[7] G. Gounaris, R. Kögerler, and H. Neufeld, Relationship between polarised vector bosons and their scalar partners, Phys. Rev. D **34**, 295 (1986).

[8] Y. Yao and C. Yuan, Modification of the equivalence theorem due to loop corrections, Phys. Rev. D **38**, 2237 (1988).

[9] J. Bagger and C. Schmidt, Equivalence theorem redux, Phys. Rev. D **41**, 264 (1990).

[10] H. He, Y. Kuang, and X. Li, On the precise formulation of the equivalence theorem, Phys. Rev. Lett. **69**, 2619 (1992).

[11] H. He, Y. Kuang, and X. Li, Further investigation on the precise formulation of the equivalence theorem, Phys. Rev. D **49**, 4842 (1994).

[12] H. He and W. Kilgore, Equivalence theorem and its radiative-correction-free formulation for all r_ξ gauges, Phys. Rev. D **55**, 1515 (1997).

[13] N. Arkani-Hamed, H. Georgi, and M. Schwartz, Effective field theorie for massive gravitons and gravity in theory space, Ann. Phys. (Amsterdam) **305**, 96 (2003).

[14] F. Berkhahn, D. D. Dietrich, and S. Hofmann, Self-protection of massive cosmological gravitons, J. Cosmol. Astropart. Phys. 11 (2010) 018.

[15] F. Berkhahn, D. D. Dietrich, S. Hofmann, F. Kühnel, and P. Moyassari, Island of stability for consistent deformations of Einstein's gravity, Phys. Rev. Lett. **108**, 131102 (2012).

[16] M. Fierz and W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, Proc. R. Soc. B **173**, 211 (1939).

[17] K. Hinterbichler, Theoretical aspects of massive gravity, Rev. Mod. Phys. **84** (2012).

[18] C. De Rham, G. Gabadadze, and A. J. Tolley, Resummation of massive gravity, Phys. Rev. Lett. **106**, 231101 (2011).

[19] D. Green, K. Gupta, and Y. Huang, A goldstone boson equivalence for inflation, J. High Energy Phys. 09 (2024) 117.

[20] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (Dover, New York, 1980).

[21] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory: Volume 1, Introduction* (Cambridge University Press, Cambridge, England, 2012).

[22] C. Becchi, A. Rouet, and R. Stora, Renormalisation of gauge theories, Ann. Phys. (Amsterdam) **98**, 287 (1976).

[23] J. Hořejší, Electroweak interactions and high-energy limit, Czech. J. Phys. **47**, 951 (1997).

[24] R. Delbourgo, S. Twisk, and G. Thompson, Massive Yang-Mills theory: Renormalizability versus unitarity, Int. J. Mod. Phys. A **03**, 435 (1988).

[25] D. G. Boulware and S. Deser, Can gravitation have a finite range?, Phys. Rev. D **6**, 3368 (1972).

[26] S. F. Hassan and R. A. Rosen, Resolving the ghost problem in nonlinear massive gravity, Phys. Rev. Lett. **108**, 041101 (2012).

[27] B. DeWitt, Quantum field theory in curved spacetime, Phys. Rep. **19**, 295 (1975).

[28] C. Dappiaggi, V. Moretti, and N. Pinamonti, Cosmological horizons and reconstruction of quantum field theories, Commun. Math. Phys. **285**, 1129 (2009).

[29] C. Dappiaggi, V. Moretti, and N. Pinamonti, Distinguished quantum states in a class of cosmological spacetimes and their hadamard property, J. Math. Phys. (N.Y.) **50**, 062304 (2009).

[30] I. Agullo, W. Nelson, and A. Ashtekar, Preferred instantaneous vacuum for linear scalar fields in cosmological spacetimes, Phys. Rev. D **91**, 064051 (2015).

[31] A. Ashtekar and A. Magnon, Quantum fields in curved space-times, Proc. R. Soc. A **346**, 375 (1975).

[32] N. Birrell and J. Taylor, Analysis of interacting quantum field theory in curved spacetimes, J. Math. Phys. (N.Y.) **21**, 1740 (1980).

[33] J. L. Friedman, N. J. Papastamatiou, and J. Z. Simon, Unitarity of interacting fields in curved spacetime, Phys. Rev. D **46**, 4442 (1992).

[34] N. Birrell and P. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[35] K. H. Choi, S. Hofmann, and M. Schneider, Local diagnostic program for unitary evolution in general spacetimes, Phys. Rev. D **111**, 025019 (2025).

[36] A. Higuchi, Forbidden mass range for spin-2 field theory in de Sitter spacetime, Nucl. Phys. **B282**, 397 (1987).

[37] L. Parker and S. A. Fulling, Adiabatic regularization of the energy-momentum tensor of a quantized field in homogeneous spaces, Phys. Rev. D **9**, 341 (1974).

[38] P. R. Anderson and L. Parker, Adiabatic regularization in closed robertson-walker universes, Phys. Rev. D **36**, 2963 (1987).

[39] S. Mandal, Ward identities under the frame transformations in curved space-time, Commun. Theor. Phys. **76**, 095406 (2024).

[40] V. De Luca, J. Khoury, and S. S. Wong, Gravitational memory and ward identities in the local detector frame, Phys. Rev. D **112**, 024032 (2025).

[41] I. L. Buchbinder, D. M. Gitman, V. Krykhtin, and V. Pershin, Equations of motion for massive spin 2 field coupled to gravity, Nucl. Phys. **B584**, 615 (2000).