

Evaluation of CKM matrix elements from exclusive $P_{\ell 4}$ decaysC. S. Kim^{*}*Department of Physics and IPAP, Yonsei University, Seoul 120-749, Korea*G. López Castro[†] and S. L. Tostado[‡]*Departamento de Física, Centro de Investigación y de Estudios Avanzados,
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(Received 13 February 2017; published 10 April 2017)

We consider the exclusive $P_{\ell 4}$ decays, $P \rightarrow (P_1 P_2)_V \ell \nu_\ell$, where the subindex V means that the invariant mass of the pseudoscalar pair is taken within a small window around the mass of the vector meson V . Pole contributions beyond the dominant $P \rightarrow V(\rightarrow P_1 P_2) \ell \nu_\ell$ amplitude of $P_{\ell 4}$ decays are identified, which, in turn, affect the determination of the CKM matrix elements $|V_{qq'}|$. We evaluate the effects of those contributions in the extraction of bottom and charm quark mixings. An application to $B \rightarrow (\pi\pi)_\rho \ell \nu_\ell$ data from the Belle Collaboration shows an increase in the extracted value of $|V_{ub}|$ in better agreement with determinations based on $B \rightarrow \pi \ell \nu_\ell$ decays. The effect of the ρ and D^* pole contributions in the determination of $|V_{cd}|$ from the decay $D \rightarrow \pi \pi \ell^- \bar{\nu}_\ell$ has also been investigated.

DOI: [10.1103/PhysRevD.95.073003](https://doi.org/10.1103/PhysRevD.95.073003)**I. INTRODUCTION**

Precise measurements of the Cabibbo-Kobayashi-Maskawa (CKM) [1,2] matrix elements can shed light on new physics and are among the main targets of flavor factories. Deviations from the unitarity property of the CKM matrix would indicate the existence of additional degrees of freedom. While the determination of $|V_{ud}|$ and $|V_{us}|$ has been made with the impressive accuracy of 0.02% and 0.3%, respectively, the values of $|V_{ub}|$ and $|V_{cb}|$ are known at 5% and 2% accuracy only [3]. Better determinations of $|V_{qb}|$, among other standard-model parameters, are the most important for searches of new sources of CP violation beyond the one encoded in the CKM paradigm.

Currently, the most precise determinations [3] of the matrix elements, $|V_{cb}|$ and $|V_{ub}|$, indicate a tension between values extracted from exclusive and inclusive decay channels of b -flavored hadrons. Clearly, more theoretical work and refined measurements are required to solve this discrepancy and achieve better accuracy. One can gain some precision by combining values of $|V_{qb}|$ extracted from different decay channels of bottom hadrons, provided their measurements furnish a consistent set of data. While measurements of exclusive channels are better suited from an experimental point of view, the calculation of their form factors in the whole kinematical regime is still challenging. Among the preferred exclusive channels, the $B \rightarrow (P, V) \ell \nu_\ell$ decays, with P (V) a pseudoscalar (vector) meson, are the simplest ones to describe theoretically and the dominant final states of charmful and charmless

semileptonic decays of B mesons. While lattice QCD provides reliable results at low recoil of final-state mesons [4–7], other methods (like light-cone sum rules, see for example Refs. [8–10]) are better suited at larger recoil values. Finally, experimental data can be used as a guide to extrapolate between these two domains.

In this paper, we are concerned with the extraction of $|V_{qq'}|$ matrix elements from $B, D \rightarrow V \ell \nu_\ell$ and its related observable $B, D \rightarrow P_1 P_2 \ell \nu_\ell$ decay channel. As mentioned above, these exclusive decays provide complementary information on CKM matrix elements and a consistency test of values extracted from other exclusive and inclusive channels. Furthermore, a good understanding of the dominant exclusive channels is essential to describe how inclusive decays are built out from exclusive components.

While pseudoscalar mesons are quasistable states, with some of them directly detectable by experiments, vector mesons are highly unstable resonances, which are reconstructed from their detectable decay products. From the theoretical point of view, using vector mesons as asymptotic states of the S -matrix is an approximation which, in principle, is not justified owing to their very short lifetimes. A theoretical definition that is consistent with the experimental one can be used instead. In this paper, we will consider $P \rightarrow P_1 P_2 \ell \nu_\ell$ ($P_{\ell 4}$) transitions, where the $P_1 P_2$ pseudoscalar pair is produced *dominantly* from the decay of a single vector meson $V \rightarrow P_1 P_2$. The extraction of the decay observables associated with $P \rightarrow V \ell \nu_\ell$ decays is affected by the contributions of subdominant s - and d -wave configurations of the $P_1 P_2$ system [11–17] even if one chooses a narrow window in their invariant mass distribution around the V resonance mass. For an example, the extraction of the $|V_{ub}|$ matrix element from $B \rightarrow \pi \pi \ell \nu_\ell$, considering the resonances, backgrounds and rescattering

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effects in the $\pi\pi$ system, was studied in Refs. [12–14,18]. Those authors found that these effects can bring the determination of $|V_{ub}|$ from four-body semileptonic decays into better agreement with the value extracted from $B \rightarrow \pi\ell\nu_\ell$ decay. Also, additional kinematical distributions accessible in four-body semileptonic decays, as compared to three-body decays, allow us to explore further observables sensitive to new physics [13]. A study of $D^+ \rightarrow K^-\pi^+e^+\nu_e$ decays that incorporate the strong-interaction dynamics of the $K\pi$ system was recently reported in Ref. [19].

Here, we consider the effects of an additional pole contribution P^* in the observables associated with $P \rightarrow P_2P^* \rightarrow P_1P_2\ell\nu_\ell$ decays. Although four-body decays of heavy mesons have been considered before, including refinements in the treatment of the s wave and excited resonances in the p wave of final-state mesons [11–18], the effects of the P^* pole have not been considered in the literature. This pollution can affect the different invariant mass distribution of the P_1P_2 system and can modify the values of CKM matrix elements extracted from $P \rightarrow V\ell\nu_\ell$ transitions. Examples of these decays are $B \rightarrow (D\pi, \pi\pi)\ell\nu$ and $D \rightarrow (K\pi, \pi\pi)\ell\nu$, which are dominated by the (D^*, ρ) and (K^*, ρ) resonances, respectively. The presence of additional B^* and D^* poles can affect the determination of the CKM matrix elements to a few-percent level, which is important for present and future studies. We present the effects of these additional pole contributions in the invariant mass distribution of the meson pair and in the branching fractions to estimate their effects in the values of the relevant CKM matrix elements.

II. FOUR-BODY SEMILEPTONIC DECAYS OF PSEUDOSCALAR MESONS

Let us consider the generic $P(p) \rightarrow P_1(p_1)P_2(p_2) \times \ell(p_3)\nu_\ell(p_4)$ decay, denoted as $P_{\ell 4}(P_1P_2)$, induced by the quark-level transition $q \rightarrow q'\ell\nu_\ell$, with (p, p_i) the particle four-momenta subject to the on-shell conditions ($p^2 = M^2$, $p_i^2 = m_i^2$). At the lowest level, using the local approximation (an infinitely heavy W boson), the decay amplitude can be written as

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{qq'} H_\mu \ell^\mu, \quad (1)$$

where $V_{qq'}$ is the quark mixing CKM matrix element, ℓ_μ is the leptonic $V - A$ charged current, and

$$\begin{aligned} H_\mu &= \langle P_1(p_1)P_2(p_2) | j_\mu | P(p) \rangle \\ &= V_\mu - A_\mu \end{aligned} \quad (2)$$

is the hadronic matrix element of the $V - A$ quark current.

Following Ref. [20], we can write the most general vector and axial-vector pieces of the hadronic matrix element as follows:

$$V_\mu = -\frac{H}{M^3} \epsilon_{\mu\nu\rho\sigma} L^\nu P^\rho Q^\sigma, \quad (3)$$

$$A_\mu = -\frac{i}{M} [FP_\mu + GQ_\mu + RL_\mu]. \quad (4)$$

The form factors H, F, G, R depend on the square of the momentum transfer to leptons and on two additional independent Lorentz scalars [20,21]. The hadronic vertices [Eqs. (3) and (4)] depend upon three independent Lorentz vectors which we chose as $P = p_1 + p_2$, $Q = p_1 - p_2$, and $L = p_3 + p_4 = p - p_1 - p_2$. Conservation of energy-momentum implies $p = P + L$. This choice is useful to fix the set of five independent kinematical variables to describe the four-body decay: $(s_{12} = P^2, s_{34} = L^2, \theta_p, \theta_\ell, \phi)$ (see definitions in Refs. [20,21]). The corresponding limits of integration are given (for massless neutrinos $m_4 = 0$) by $(m_1 + m_2)^2 \leq s_{12} \leq (M - m_3)^2$; $m_3^2 \leq s_{34} \leq (M - \sqrt{s_{12}})^2$; $0 \leq \theta_p, \theta_\ell \leq \pi$; and $0 \leq \phi \leq 2\pi$ [20,21].

One can get the decay rates by integrating over s_{12} , the invariant mass distribution of the pair of final-state pseudoscalar mesons:

$$\Gamma(P \rightarrow P_1P_2\ell\nu_\ell) = \int_{s_{12}^-}^{s_{12}^+} ds_{12} \frac{d\Gamma(P \rightarrow P_1P_2\ell\nu_\ell)}{ds_{12}}. \quad (5)$$

In the case that the invariant mass distribution is fully dominated by a single intermediate resonance R , namely $P \rightarrow R(\rightarrow P_1P_2)\ell\nu_\ell$, we can restrict the integration to the region defined by $s_{12}^\pm = (m_R \pm \Delta)^2$, where m_R is the mass of the resonance, and typically $\Delta = \Gamma_R/2$ or Γ_R , with Γ_R its decay width. In the case of a very narrow resonance ($\Gamma_R \rightarrow 0$), one recovers the usual result,

$$\Gamma(P \rightarrow P_1P_2\ell\nu_\ell) = \Gamma(P \rightarrow R\ell\nu_\ell) \times B(R \rightarrow P_1P_2). \quad (6)$$

This result is also a good approximation for wider resonances, provided no other contributions to the decay amplitude are present. It will be modified, however, by the contribution of additional pole contributions to the decay amplitude.

As shown in Fig. 1, there are three different contributions to the hadronic vertex of the $P \rightarrow P_1P_2\ell\nu$ decay (here we have chosen $P_2 = \pi$, for definiteness). We will assume that the two dominant contributions are given by single-pole contributions, namely $P \rightarrow P^*\pi \rightarrow P_1\pi\ell\nu$ [Fig. 1(a)] and $P \rightarrow P_{12}^*\ell\nu \rightarrow P_1\pi\ell\nu$ [Fig. 1(b)]. Additional resonances in these channels can contribute as well, and their contributions can be trivially added to our results; notice that usually the effects of heavier resonances similar to the one of interest in the P_{12}^* channel are taken into account and estimated as background in simulations. For simplicity, we will make the reasonable assumption that pole contributions are dominated by the exchange of vector meson resonances in Figs. 1(a) and 1(b).

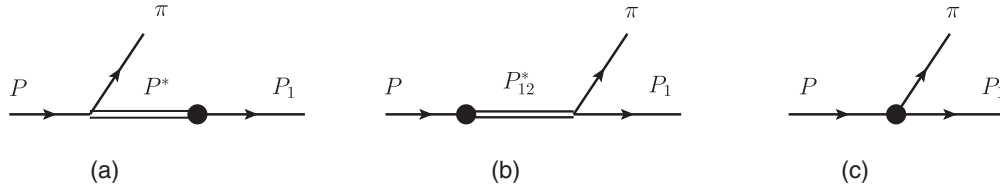


FIG. 1. Contributions to the hadronic vertex in $P \rightarrow P_1 \pi \ell^- \bar{\nu}_\ell$ decays: (a), (b) are pole terms and (c) is a contact term. Double lines are used for the intermediate vector resonances. The solid dot indicates the hadronic weak vertex.

The different hadronic vertices that enter in decays of charged and neutral P mesons in Fig. 1 are related by isospin symmetry, which will be assumed to be a good approximation. We define the strong PVP' vertex as $ig_{PVP'}(p-p') \cdot \epsilon(p_V)$, using the convention $V(p_V, \epsilon_V) \rightarrow P(p)P'(p')$. The weak matrix elements required for our evaluations corresponding to the two vector pole contributions in Fig. 1 are (we use the convention of Ref. [22] for the $R' \rightarrow P_1$ transition)

$$\begin{aligned} \langle P_1(p_1) | j_\mu | R'(\epsilon, p'_R) \rangle = & -\frac{2iV'}{m_{R'} + m_1} \epsilon_{\mu\nu\rho\sigma} \epsilon^\nu p'_1 p_{R'}^\sigma - 2m_{R'} A'_0 \frac{q' \cdot \epsilon}{q'^2} q'_\mu \\ & - (m_{R'} + m_1) A'_1 \epsilon^\beta T_{\beta\mu}(q') - A'_2 \frac{\epsilon \cdot q'}{m_{R'} + m_1} (p_{R'} + p_1)^\beta T_{\beta\mu}(q') \end{aligned} \quad (7)$$

and

$$\begin{aligned} \langle R(\epsilon^*, p_R) | j_\mu | P(p) \rangle = & \frac{2iV}{m_R + M} \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\nu} p^\rho p_R^\sigma - 2m_R A_0 \frac{q \cdot \epsilon^*}{q^2} q_\mu \\ & - (m_R + M) A_1 \epsilon^{*\beta} T_{\beta\mu}(q) + A_2 \frac{\epsilon^* \cdot q}{m_R + M} (p + p_R)^\beta T_{\beta\mu}(q) \end{aligned} \quad (8)$$

for the $P \rightarrow R$ transition [22]. The weak current is $j_\mu = \bar{q}' \gamma_\mu (1 - \gamma_5) q$; we have assumed that the intermediate resonances $R(R')$ are vector mesons, with the polarization four-vector $\epsilon_\mu^*(\epsilon_\mu)$, such that $p_R \cdot \epsilon^* = p_{R'} \cdot \epsilon = 0$. The primed form factors for the $R' \rightarrow P_1$ weak transition depend upon $q'^2 = (p_{R'} - p_1)^2$, while those of $P \rightarrow R$ depend upon $q^2 = (p - p_R)^2$, owing to energy-momentum conservation $q = q' = L$. In the above expressions, $T_{\beta\mu}(q) \equiv g_{\beta\mu} - q_\beta q_\mu / q^2$ is the hadronic transverse tensor ($q^\beta T_{\beta\mu} = 0$).

As a concrete example, let us consider the $B(p) \rightarrow D(p_1) \pi(p_2) \ell(p_3) \nu_\ell(p_4)$ decay, which we assume to be dominated by the B^* and D^* pole contributions in the region where the $D\pi$ invariant mass is close to the D^* resonance. Using the definitions introduced previously, we can compute the form factors defined in Eqs. (3) and (4) and get the following results:

$$-\frac{H}{M_B^3} = 2i \left[-\frac{g_{BB^*\pi}}{D_{B^*}(p-p_2)} \frac{V'}{m_{B^*} + m_D} + \frac{g_{DD^*\pi}}{D_{D^*}(P)} \frac{V}{M_B + m_{D^*}} \right], \quad (9)$$

$$\begin{aligned} -\frac{iF}{M_B} = & \frac{g_{BB^*\pi}}{D_{B^*}(p-p_2)} \left[\frac{A'_2 X}{m_{B^*} + m_D} + (m_{B^*} + m_D) A'_1 X_1 \right] \\ & - \frac{g_{DD^*\pi}}{D_{D^*}(P)} \left[\frac{2A_2 Y}{M_B + m_{D^*}} + (M_B + m_{D^*}) A_1 \frac{P \cdot Q}{m_{D^*}^2} \right], \end{aligned} \quad (10)$$

$$\begin{aligned} -\frac{iG}{M_B} = & \frac{g_{BB^*\pi}}{D_{B^*}(p-p_2)} \left[\frac{A'_2 X}{m_{B^*} + m_D} + (m_{B^*} + m_D) A'_1 X_2 \right] \\ & + \frac{g_{DD^*\pi}}{D_{D^*}(P)} (M_B + m_{D^*}) A_1, \end{aligned} \quad (11)$$

$$\begin{aligned} -\frac{iR}{M_B} = & \frac{g_{BB^*\pi}}{D_{B^*}(p-p_2)} \left[2m_{B^*} \frac{A'_0 X}{L^2} - \frac{(P+Q) \cdot L}{L^2} \frac{A'_2 X}{m_{B^*} + m_D} + (m_{B^*} + m_D) A'_1 \left(A - \frac{X \cdot L}{L^2} \right) \right] \\ & + \frac{g_{DD^*\pi}}{D_{D^*}(P)} \left[2m_{D^*} \frac{A_0 Y}{L^2} + \frac{2A_2 Y}{M_B + m_{D^*}} \frac{P \cdot L}{L^2} - (M_B + m_{D^*}) A_1 \frac{Y \cdot L}{L^2} \right]. \end{aligned} \quad (12)$$

In the above expressions, we have defined the four-vectors $X^\beta = X_+ P^\beta + X_- Q^\beta + A L^\beta$, $Y^\beta = Q^\beta - (P \cdot Q / m_{D^*}^2) P^\beta$, and the Lorentz scalars $X = L \cdot X$, $Y = L \cdot Y$, $X_\pm = A/2 \pm 1$, with $A = 1 - p_{B^*}^2 / m_{B^*}^2 [1 + (P - Q) \cdot p_{B^*} / p_{B^*}^2]$, where $p_{B^*} = P + L - (P - Q)/2$. We have used the notation $D_{B^*}(q) = q^2 - m_{B^*}^2$ and $D_{D^*}(q) = q^2 - m_{D^*}^2 + i m_{D^*} \Gamma_{D^*}$. It is easy to check that for $P \rightarrow P_1 \pi \ell \nu_\ell$ decays, we have to replace $B^{(*)} \rightarrow P^{(*)}$, $D^{(*)} \rightarrow P_1(P_{12}^*)$ and the corresponding weak form factors and strong coupling constants in the previous expressions.

III. DISCUSSIONS ON EXPERIMENTAL OBSERVABLES OF $P_{\ell 4}$

Several experiments have reported measurements of branching ratios and invariant mass distributions of $B_{\ell 4}$ [23–25] and $D_{\ell 4}$ [19,26,27] decays. The corresponding analysis to extract the CKM matrix elements from the $D, B \rightarrow V \ell \nu_\ell$ observables differs in several ways: (i) the form factors used to model the weak transition, (ii) the window of the hadronic mass distribution $|\sqrt{s_{12}} - m_V|$ chosen to isolate the V vector-meson signal, and (iii) the inclusion of several wave configurations and resonance contributions in the hadronic system. As an illustration of the second item, the following cuts in the hadronic invariant mass distribution have been used by different experiments for $B \rightarrow \pi \pi \ell \nu_\ell$ decays: $|\sqrt{s_{\pi\pi}} - m_\rho| \leq \Gamma_\rho$ [23] ($2\Gamma_\rho$ [25]), $0.650 \text{ GeV} \leq \sqrt{s_{\pi\pi}} \leq 0.850 \text{ GeV}$ [24], and $0.60 \text{ GeV} \leq \sqrt{s_{\pi\pi}} \leq 1.00 \text{ GeV}$ [28], which prevent a direct comparison of reported values for the branching fractions. In addition, some experiments report values of the combined results from neutral and charged B -meson branching fractions using isospin symmetry. Isospin-symmetry-breaking effects should be duly taken into account in analyses when measurements reach 1% accuracy. Furthermore, the contributions of the additional pole contribution considered in this paper become relevant at the few-percent-level accuracy determinations of CKM matrix elements.

In our previous paper [29], we have shown that the effects of the B^* pole contribution in $B \rightarrow D \pi \ell \nu_\ell$ are negligibly small compared to those of the D^* pole, owing to the very narrow width of the D^* resonance, which fully dominates the $D\pi$ invariant mass distribution close to the D^* mass. The effect of the B^* pole in the extraction of the ratio $R(D^*)$ from $B_{\ell 4}(D\pi)$ decays is also negligible [29]. This leads to the interesting question of how large this effect can be for wider resonances and how it affects the extraction of the CKM matrix elements when using $P \rightarrow V \ell \nu$ decays. Here we study the effects of the pole diagram of Fig. 1(a) in the hadronic invariant mass distribution, and the branching fraction of $P_{\ell 4}$ decays in the region close to the P_{12} resonance and its consequences for the extraction of $|V_{q q'}|$.

In our calculations, we use the following phase convention for pseudoscalar meson states [30]: $|\pi^+\rangle = -u\bar{d}$, $|\pi^0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, $|\pi^-\rangle = d\bar{u}$, $|K^+\rangle = u\bar{s}$, $|K^0\rangle = d\bar{s}$, $|\bar{K}^0\rangle = -s\bar{d}$, $|K^-\rangle = s\bar{u}$, $|D^+\rangle = -c\bar{d}$, $|D^0\rangle = c\bar{u}$, $|\bar{D}^0\rangle = u\bar{c}$, $|D^-\rangle = d\bar{c}$. The conventions for B mesons are similar to those for K mesons under the replacement $s \rightarrow b$. With these conventions, isospin symmetry provides the following relations among different couplings:

$$\begin{aligned} g_{\rho\pi\pi} &= g_{\rho^+\pi^+\pi^0} = -g_{\rho^0\pi^+\pi^-}, \\ g_{K^*K\pi} &= g_{K^{*+}K^0\pi^+} = \sqrt{2}g_{K^{*+}K^+\pi^0} = -g_{K^{*0}K^+\pi^-} \\ &= \sqrt{2}g_{K^{*0}K^0\pi^0}, \\ g_{D^*D\pi} &= g_{D^{*+}D^0\pi^+} = -\sqrt{2}g_{D^{*+}D^+\pi^0} = -g_{D^{*0}D^+\pi^-} \\ &= -\sqrt{2}g_{D^{*0}D^0\pi^0}, \\ g_{B^*B\pi} &= -g_{B^{*-}B^0\pi^-} = -\sqrt{2}g_{B^{*0}B^0\pi^0}. \end{aligned} \quad (13)$$

For our numerical evaluations, we will use the values $g_{\rho\pi\pi} = 5.98 \pm 0.03$, $g_{K^*K\pi} = 3.28 \pm 0.03$ and $g_{D^*D\pi} = 8.39 \pm 0.08$, extracted from the experimental widths of resonances [3], and $g_{B^*B\pi} = 20.0 \pm 1.2$ from the most recent lattice calculations [31]. The masses and widths of the vector resonances are taken from Ref. [3].

For the weak form factors, we use the following results: (i) for the $B \rightarrow \rho$ transitions, we rely on lattice calculations of Ref. [32]; (ii) the form factors for semileptonic decays of charmed mesons $D \rightarrow K^*$ and $D \rightarrow \rho$ are taken from the experimental data of Refs. [19] and [27], respectively; and (iii) for the evaluation of the $P^* \rightarrow P_1$ form factors, we use the relativistic harmonic oscillator potential model of Refs. [33,34] (WSB). In this model, the q^2 dependence of all the form factors is assumed to have a monopolar form:

$$F_i(q^2) = \frac{F_i(0)}{1 - q^2/m_i^2(J^P)}. \quad (14)$$

The form factors at $q^2 = 0$ are computed from the overlap of relativistic wave functions in this model; the values of pole masses are chosen to correspond to the lightest resonances with appropriate quantum numbers that allow coupling to the weak currents.

TABLE I. Form factors of the weak transition $P^* \rightarrow P_1$ at $q^2 = 0$ in the Wirbel-Stech-Bauer model [33,34]. Values of pole masses are given in GeV units.

Transition	$A'_0(0)$	$A'_1(0)$	$A'_2(0)$	$V'(0)$	$m(0^-)$	$m(1^-)$	$m(1^+)$
$D^* \rightarrow K$	0.78	1.02	0.40	0.90	1.97	2.11	2.53
$D^* \rightarrow \pi$	0.75	1.08	0.37	0.76	1.87	2.01	2.42
$B^* \rightarrow \pi$ [22]	0.34	0.38	0.29	0.34	5.27	5.32	5.71
$B^* \rightarrow D$ [22]	0.63	0.66	0.56	0.70	6.30	6.34	6.73

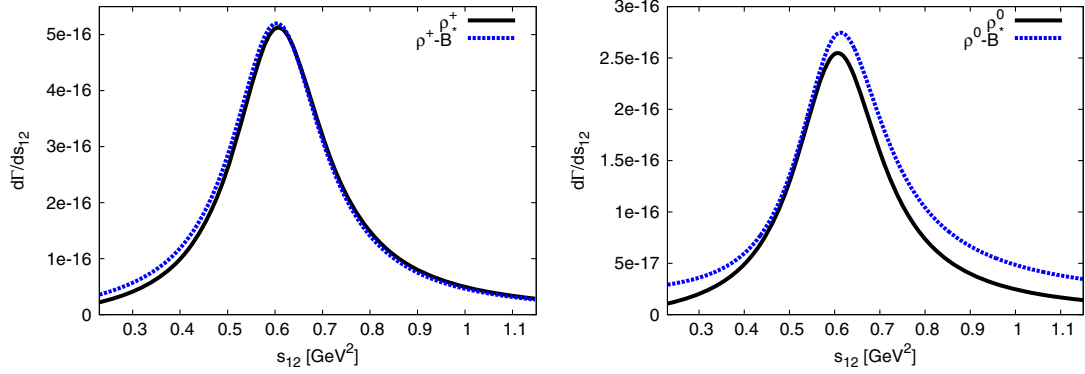


FIG. 2. Invariant mass distribution of $\pi\pi$ in $B \rightarrow \pi\pi\ell\bar{\nu}$ ($\ell = e\mu$) decays. Left (right) panel is for decays of neutral (charged) B mesons. The solid (dotted) lines describe the dominant ρ ($\rho + B^*$) pole contributions to the hadronic vertex.

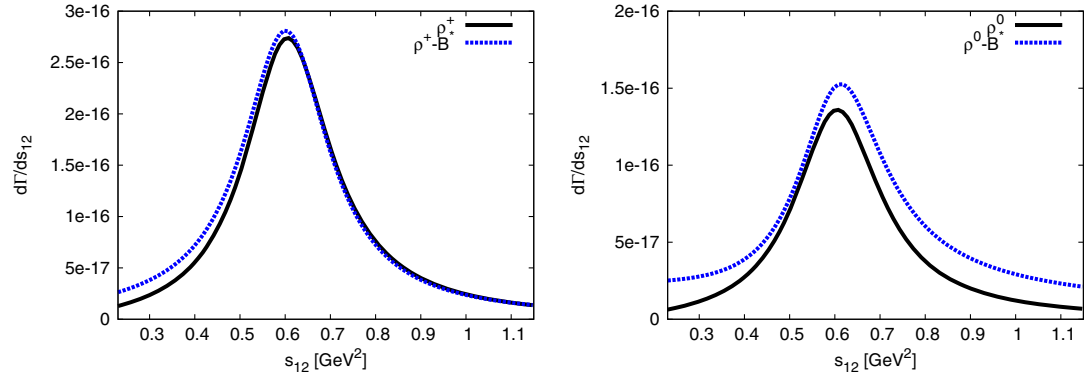


FIG. 3. Same as Fig. 2 for $B \rightarrow \pi\pi\tau\bar{\nu}$ decays.

In the case of $B^* \rightarrow D$ and $B^* \rightarrow \pi$ transitions, the form factors at $q^2 = 0$ have been evaluated in Ref. [22] using this model. We have checked these values of form factors at $q^2 = 0$ and have evaluated within the same model the form factors corresponding to the $D^* \rightarrow K$ and $D^* \rightarrow \pi$ weak transitions. The results for the different form factors and the values of pole masses used in our evaluations are shown in Table I. As long as the P^* pole contribution to the decay

amplitude is subleading, we should take the numerical contribution due to the $P^* \rightarrow P_1$ form factors as a good estimate of their true values.

A. Hadronic invariant mass distributions

In Figs. 2, 3, 4 and 5, we plot the hadronic invariant mass distributions of $B \rightarrow \pi\pi\ell\nu_\ell$ and $D \rightarrow (K\pi, \pi\pi)\ell\nu_\ell$ decays and compare the single dominant resonance contribution

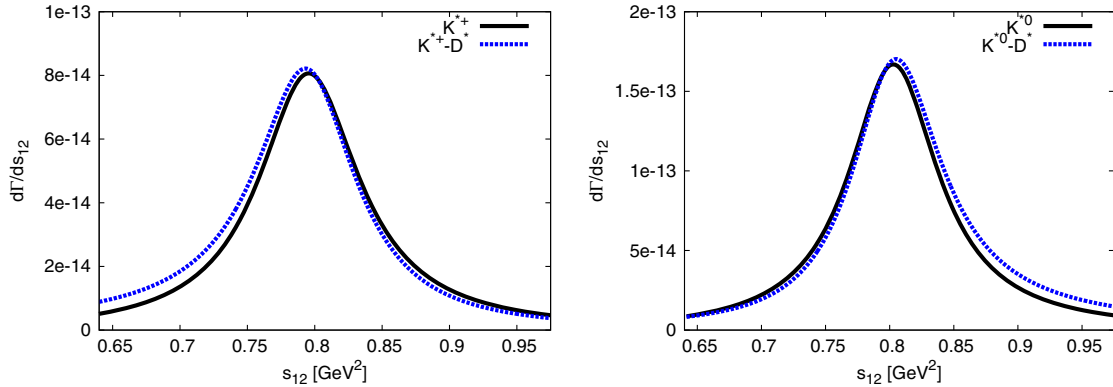
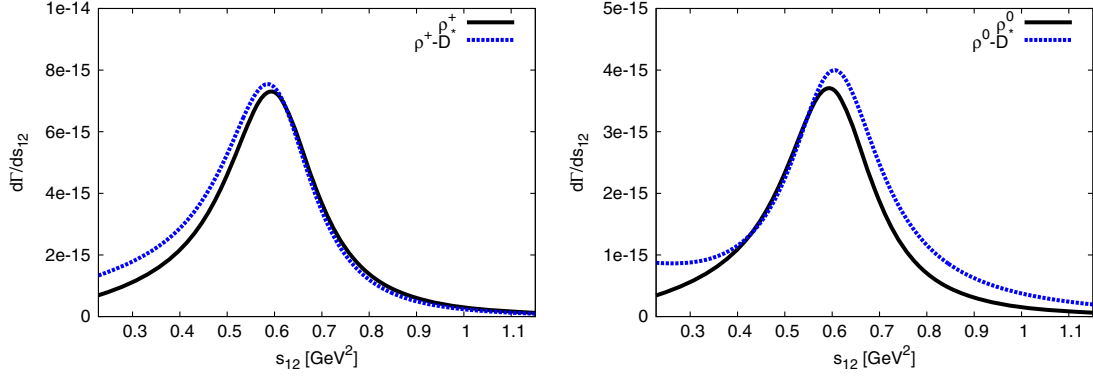


FIG. 4. Invariant mass distribution of $K\pi$ in $D \rightarrow K\pi\ell\bar{\nu}$ decays. Left (right) panel is for decays of the $\bar{D}^0$ (D^-) meson. The solid (dotted) lines represent the $K^*(892)$ ($K^* + D^*$) pole contribution.

FIG. 5. Same as Fig. 4 for $D \rightarrow \pi\pi\ell^-\bar{\nu}$ decays.

(solid line) with the full calculation including both poles (dotted line). In the case of B -meson decays, we have plotted separately these distributions for light and heavy τ leptons given the interest for a test of lepton universality. We do not show the corresponding plots for $B \rightarrow D\pi\ell\nu_\ell$ decays, because the effect of the additional pole in that case is indistinguishable.

A comparison of the left and right panels in Figs. 2–5 shows important isospin breaking effects: the full contributions shift the peak of the distributions to the left (right) of the single dominant pole contribution for decays of neutral (charged) mesons. The origin of this asymmetry lies in the relative signs and different isospin factors for couplings of charged and neutral resonances coupled to two pseudoscalar mesons. A fit to the P_1P_2 invariant mass distribution, aiming to extract the resonance parameters of the P_{12}^* intermediate state in semileptonic decays, should take into account the two pole contributions. The P^* pole contribution, in this case, will play the role of a nonresonant background. A visual inspection of the plots in Figs. 2–5 indicates that the P^* pole will increase (decrease) the mass of the P_{12}^* resonance when extracted from neutral (charged) heavy pseudoscalar meson decays with respect to the case where the contribution of Fig. 1(a) is neglected.

B. Branching fractions

We can compute the integrated rates of $P \rightarrow P_1P_2\ell\nu_\ell$ decays by integrating the hadronic invariant mass distributions as shown in Eq. (5). We restrict this integration to the region close to the mass of the dominant vector resonance $R \rightarrow P_1P_2$, namely $s_{12}^\pm = (m_R \pm \Gamma_R/2)^2$. Our results are shown in Table II. We can identify the resulting decay rate¹ with $\Gamma(P \rightarrow P_1P_2\ell\nu_\ell) = \Gamma(P \rightarrow V\ell\nu_\ell) \times B(V \rightarrow P_1P_2)$ only in the case that the contribution of Fig. 1(a) is neglected (second column in Table II). When

the contribution of the diagram in Fig. 1(a) is included, the correct formula necessary to extract the branching fraction of the semileptonic $P \rightarrow V$ transition is

$$B(P \rightarrow V\ell\nu_\ell) = \frac{\tau_P \cdot \Gamma(P \rightarrow P_1P_2\ell\nu_\ell)}{B(V \rightarrow P_1P_2) \cdot (1 + \delta_{P^*})}, \quad (15)$$

where τ_P is the lifetime of the decaying particle and δ_{P^*} is the small correction due to a subdominant pole contribution.

The numerical value of δ_{P^*} is obtained from the ratio of the fourth/third columns in Table II (see last column); this correction can be as large as 15% for $B^- \rightarrow \pi^+\pi^-\tau^-\bar{\nu}_\tau$.

TABLE II. Integrated rates in units of 10^{-15} (10^{-17}) GeV for D (B) meson decays. The dominant (full) pole contribution is shown in the second (third) column. The quoted uncertainties arise from uncertainties in form factor inputs. Within parentheses we have indicated the results obtained in the narrow width approximation (see text), except for the last two rows, which do not change.

Channel	$\Gamma(P_{12})$	$\Gamma(P_{12}^* + P^*)$	δ_{P^*}
$D^- \rightarrow K^+\pi^-\ell^-\bar{\nu}_\ell$	22.2 (22.3)	22.7 ± 1.2 (22.6 ± 1.2)	2.2% (1.3%)
$\bar{D}^0 \rightarrow K^+\pi^0\ell^-\bar{\nu}_\ell$	11.5 (11.5)	11.7 ± 0.6 (11.7 ± 0.6)	1.7% (1.7%)
$D^- \rightarrow \pi^+\pi^-\ell^-\bar{\nu}_\ell$	1.33 (1.34)	1.47 ± 0.16 (1.45 ± 0.16)	10.5% (8.2%)
$\bar{D}^0 \rightarrow \pi^+\pi^0\ell^-\bar{\nu}_\ell$	2.62 (2.64)	2.69 ± 0.29 (2.69 ± 0.29)	2.7% (1.9%)
$B^- \rightarrow \pi^+\pi^-\ell^-\bar{\nu}_\ell$	9.22 (9.23)	10.10 ± 0.34 (9.86 ± 0.33)	9.5% (6.4%)
$B^- \rightarrow \pi^+\pi^-\tau^-\bar{\nu}_\tau$	4.92 (4.93)	5.67 ± 0.44 (5.49 ± 0.43)	15.2% (10.2%)
$\bar{B}^0 \rightarrow \pi^+\pi^0\ell^-\bar{\nu}_\ell$	18.57 (18.59)	18.91 ± 0.70 (18.91 ± 0.70)	1.8% (1.7%)
$\bar{B}^0 \rightarrow \pi^+\pi^0\tau^-\bar{\nu}_\tau$	9.91 (9.93)	10.23 ± 0.90 (10.21 ± 0.90)	3.2% (2.7%)
$\bar{B}^0 \rightarrow D^+\pi^0\ell^-\bar{\nu}_\ell$	606.2	606.2 ± 31.6	0%
$\bar{B}^0 \rightarrow D^+\pi^0\tau^-\bar{\nu}_\tau$	152.8	152.8 ± 8.0	0%

¹Of course, this is true in the case that experiments have removed the contributions of excited resonances in the P_1P_2 system or the case that they are well separated from the dominant resonance region.

decays. Since the effect of the additional P^* pole is to increase the decay rates compared to the cases where it is neglected, the values extracted for $|V_{qq'}|$ will be decreased by $\delta_{P^*}/2$ when comparing the experimental and theoretical values of $P \rightarrow V\ell\nu$.

For comparison, we also show in Table II the results obtained in the narrow width approximation for the dominant resonant contribution (figures within parenthesis). We have implemented this limit by replacing the propagator of the P_{12}^* resonance as follows:

$$\frac{1}{|s_{12} - m_R^2 + im_R\Gamma_R|^2} \rightarrow \frac{\pi}{m_R\Gamma_R} \delta(s_{12} - m_R^2). \quad (16)$$

Using this approximation in the integrand of Eq. (5), the integration over the five-dimensional phase space reduces to an integration over four dimensions. As can be observed, the corresponding results change only slightly compared to the ones obtained by integrating over the finite range $(m_R - \Gamma_R/2)^2 \leq s_{12} \leq (m_R + \Gamma_R/2)^2$, except for $B^0 \rightarrow \pi^+\pi^-\ell\nu_\ell$ decays, where the largest variations are obtained.

IV. EFFECTS ON THE EVALUATION OF CKM MATRIX ELEMENTS

As discussed in Refs. [12–14], the value of $|V_{ub}|$ is increased if one uses the four-body $B \rightarrow (\pi\pi)_\rho\ell\nu_\ell$ ($B_{\ell 4}$) decays,² instead of the corresponding three-body $B \rightarrow \rho\ell\nu_\ell$ decay in its determination. This happens owing to the dynamics of the strong interactions manifested as rescattering effects and orbital angular configurations of the $\pi\pi$ system different from $L = 1$. In this section, we consider the additional modification of the $|V_{qq'}|$ mixing owing to strong interactions in the initial state of $P_{\ell 4}$ decays described in this paper. As a general trend, those effects tend to decrease the value of the CKM matrix element extracted from $P_{\ell 4}$ decays.

As an illustrative example, let us estimate the effect on the extraction of $|V_{ub}|$ due to the additional pole contribution in the case of charmless $B \rightarrow (\pi\pi)_\rho\ell\nu$ decays as measured by the Belle Collaboration in Ref. [25]. A rigorous procedure should include a fit to the measured $q^2 = s_{34}$ distribution in order to determine the free constants of a given form-factor model and then extract the value of $|V_{ub}|$ from the measured branching fraction. Instead, we estimate the effect of the B^* pole contribution using Eq. (15), which is equivalent to the formula given in Ref. [25] in the absence of the δ_{P^*} term:

$$|V_{ub}| = \sqrt{\frac{B(B \rightarrow (\pi\pi)_\rho\ell\nu_\ell)}{\tau_B \cdot \Delta\zeta \cdot (1 + \delta_{P^*})B(\rho \rightarrow \pi\pi)}}. \quad (17)$$

²The notation $(P_1P_2)_V$ means that the invariant mass of the pair of pseudoscalar mesons is taken in a small window around the V -meson mass.

Here, $B(B \rightarrow (\pi\pi)_\rho\ell\nu_\ell)$ is the measured branching fraction, $\Delta\zeta = \int d\Gamma/|V_{ub}|^2$ is the normalized (to the squared $|V_{ub}|^2$ quark mixing matrix element) rate integrated over the $s_{12}^\pm = (m_\rho \pm 2\Gamma_\rho)^2$ window, and τ_B is the B -meson lifetime. The quantity $\Delta\zeta$ depends of the model used to describe the form factors of the $B \rightarrow \rho$ transition. Using the model of Ref. [32], as done by Belle in Ref. [25], which we have used also in our evaluations of the hadronic spectrum and branching fractions, we have obtained $\Delta\zeta = (13.8 \pm 2.9) \text{ ps}^{-1}$ for the range $|\sqrt{s_{12}} - m_\rho| < 2\Gamma_\rho$. As a check of our calculation, using the narrow width approximation and the value of $|V_{ub}|$ as in Ref. [25], we reproduce the value $\Delta\zeta = (16.5 \pm 3.5) \text{ ps}^{-1}$ as reported in that reference for the form-factor model of Ref. [32].

Using the branching fractions $B(B^- \rightarrow (\pi\pi)_{\rho^0}\ell^-\bar{\nu}_\ell) = (1.83 \pm 0.10 \pm 0.10) \times 10^{-4}$ and $B(\bar{B}^0 \rightarrow (\pi\pi)_{\rho^+}\ell^-\bar{\nu}_\ell) = (3.22 \pm 0.27 \pm 0.24) \times 10^{-4}$ as reported in Ref. [25] for $|\sqrt{s_{12}} - m_\rho| < 2\Gamma_\rho$, and using $\delta_{B^*} = 2.8(24.0)\%$ for the B^* pole correction in the same range of the invariant mass of the $\pi^+\pi^-(\pi^+\pi^0)$ system, we get

$$|V_{ub}| = \begin{cases} (3.87 \pm 0.46) \times 10^{-3} & \text{from } B^- \text{ decay} \\ (3.62 \pm 0.40) \times 10^{-3} & \text{from } \bar{B}^0 \text{ decay} \end{cases}, \quad (18)$$

where the B^* pole effects mainly affect the decays of the charged B meson. The weighted average of the above results is $|V_{ub}| = (3.73 \pm 0.30) \times 10^{-3}$, which is closer to the determination obtained from $B \rightarrow \pi\ell\nu$ decays $|V_{ub}| = (3.72 \pm 0.19) \times 10^{-3}$ as reported by the PDG [3]. Let us mention that using the narrow width approximation as in Ref. [25], the effect becomes smaller; using the same input data, and the corresponding values of the B^* pole correction, $\delta_{B^*} = 1.7(6.4)\%$ as shown in Table II, we would have obtained $|V_{ub}| = (3.57 \pm 0.29) \times 10^{-3}$ for the average from B^- and $\bar{B}^0$ decays.

The effect of the additional pole considered in this paper will also be non-negligible for improved measurements of Cabibbo-suppressed $D \rightarrow \pi\pi\ell^-\bar{\nu}_\ell$ decays. The invariant mass distribution of the $\pi\pi$ system measured by CLEO [27] for $|m_{\pi\pi} - m_\rho| \leq 150 \text{ MeV}$ is dominated by the $\rho(770)$ resonance. By assuming a monopolar form of the different form factors and assuming $|V_{cd}| = 0.2252 \pm 0.0007$ from the unitarity of the CKM matrix, the values $V(0) = 0.84 \pm 0.09^{+0.05}_{-0.06}$, $A_1(0) = 0.56 \pm 0.01^{+0.02}_{-0.003}$, and $A_2(0) = 0.47 \pm 0.06 \pm 0.04$ were derived from the measured branching fractions and invariant mass distributions [27].

In order to estimate the effect of the D^* pole contribution in the determination of $|V_{cd}|$, we can use the same form factors and branching fractions measured in Ref. [27] using Eq. (17). Since Ref. [27] uses a different resonant shape of the $\pi\pi$ invariant mass than ours, for the purposes of estimating the effect of the D^* pole contribution we will use our results in the narrow width approximation [in this

case, $\Delta\zeta = (4.0 \pm 0.4) \times 10^{10} \text{ s}^{-1}$], and the values of δ_{D^*} are given in Table II. By including the ρ and D^* poles, we obtain

$$|V_{cd}|_{\rho+D^*} = 0.224 \pm 0.011 \quad (19)$$

from the average of D^- and $\bar{D}^0$ semileptonic decays. For comparison, the value obtained by including only the ρ -meson resonance is $|V_{cd}| = 0.230 \pm 0.012$; namely, the effect of including the D^* pole shifts downwards the value of this CKM matrix element by 2.7%. Although this effect is small compared to the statistical uncertainty of current measurements, it will become relevant in analyses of improved measurements in the future.

Let us emphasize that the aim of our evaluations is to estimate the shift produced by the additional pole contribution in the determination of $|V_{ub}|$ and $|V_{cd}|$ mixing matrix elements. More refined analysis which includes the effects of the s -wave $\pi\pi$ system and more precise measurements of the $D(B) \rightarrow \pi\pi\ell^-\bar{\nu}_\ell$ branching fractions would allow us to assess correctly the size of the additional $D^*(B^*)$ pole contribution. Conversely, using precise measurements of $P_{\ell 4}$ observables combined with the most reliable determinations of the quark mixing elements would allow us to test the form-factor models describing the dominant $B \rightarrow V$ weak transition as well as to understand some underlying dynamics of the hadronic system.

V. SUMMARY AND CONCLUSIONS

Precise determinations of the b -quark mixing matrix elements are necessary to find possible sources of CP violation beyond the CKM mechanism. One way to reach this goal is to combine mixing values extracted from different decays of b -flavored hadrons. Solving current discrepancies between the most precise determinations of $|V_{qb}|$ ($q = u, c$) from exclusive and inclusive channels, combined with more precise calculations of form factors and refined measurements at b factories will provide a consistency test of the SM and look for possible effects of

new physics. In this paper, we have studied the $P \rightarrow P_1P_2\ell\nu_\ell$ semileptonic decays of B and D mesons by modeling the weak hadronic matrix element of the $P \rightarrow P_1P_2$ transition with two poles contributions. The well-known $P \rightarrow V(\rightarrow P_1P_2)\ell\nu_\ell$, with V a resonant vector meson pole, gives the dominant contribution for invariant masses of the P_1P_2 within a small window around the V -meson mass. A subleading tree-level additional pole contribution is identified, which becomes relevant for decay observables at the few-percent level.

We have considered the effects of the subleading pole contribution in the invariant mass distribution of the meson pair in the $D \rightarrow (K\pi, \pi\pi)\ell\nu_\ell$ and $B \rightarrow (D\pi, \pi\pi)\ell\nu_\ell$ semileptonic decays. This correction shifts the invariant mass distribution differently for decays of charged and neutral heavy mesons, owing mainly to different isospin factors of the strong vertex involved in each case. We have also evaluated the correction in the branching fractions of these four-body decays induced by this subleading pole. We have illustrated how these corrections affect the determination of the $|V_{ub}|$ matrix element extracted by Belle [25] from $B \rightarrow \pi\pi\ell\nu_\ell$ decays, using a window of $\pm 2\Gamma_\rho$ around the ρ peak in the $\pi\pi$ invariant mass. The shift in the value of $|V_{ub}|$ is not significant compared to current experimental uncertainties, although it becomes in better agreement with the determination based on $B \rightarrow \pi\ell\nu_\ell$ decays. Similar considerations can be applied to the extraction of CKM matrix elements from other four-body decays of B and D mesons. Analysis of improved data expected in future measurements of these semileptonic decays must consider the effect of the additional pole contributions discussed in this paper.

ACKNOWLEDGMENTS

G. L. C. and S. L. T. are grateful to Conacyt for financial support under Projects No. 236394, No. 250628 (Ciencia Básica), and No. 296 (Fronteras de la Ciencia). The work of C. S. K. was supported by the NRF grant funded by the Korean government of the MEST (Grant No. 2016R1D1A1A02936965).

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