

Hamiltonian analysis of nonprojectable Hořava-Lifshitz gravity with $U(1)$ symmetry

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(Received 28 April 2015; published 2 July 2015)

We study the nature of constraints and count the number of degrees of freedom in the nonprojectable version of the $U(1)$ extension of Hořava-Lifshitz gravity, using the standard method of Hamiltonian analysis in the classical field theory. This makes it possible for us to investigate the condition under which the scalar graviton is absent at a fully nonlinear level. We show that the scalar graviton does not exist at the classical level if and only if two specific coupling constants are exactly zero. The operators corresponding to these two coupling constants are marginal for any values of the dynamical critical exponent of the Lifshitz scaling and thus should be generated by quantum corrections even if they are eliminated from the bare action. We thus conclude that the theory in general contains the scalar graviton.

DOI: 10.1103/PhysRevD.92.024005

PACS numbers: 04.50.Kd

I. INTRODUCTION

Hořava [1] proposed a power-counting renormalizable theory of gravity in 2009. What renders the theory renormalizable in the power-counting sense is the anisotropic scaling, or Lifshitz scaling,

$$t \rightarrow b^z t, \quad \vec{x} \rightarrow b \vec{x}, \quad (1)$$

with the dynamical critical exponent $z \geq d$ at high energy, where d is the number of spatial dimensions. Because of this scaling, the theory is often called Hořava-Lifshitz gravity. The basic quantities are the lapse N , the shift N^i and the three-dimensional spatial metric g_{ij} . In the minimal theory called the projectable theory, the lapse is set to be a function of time only. Hence, the basic quantities in the projectable theory are

$$\begin{aligned} N &= N(t), & N^i &= N^i(t, \vec{x}), \\ g_{ij} &= g_{ij}(t, \vec{x}), & (\text{projectable}). \end{aligned} \quad (2)$$

On the other hand, in the nonprojectable theory the lapse may depend on the spatial coordinates as well as the time, and thus the basic quantities yield

$$\begin{aligned} N &= N(t, \vec{x}), & N^i &= N^i(t, \vec{x}), \\ g_{ij} &= g_{ij}(t, \vec{x}), & (\text{nonprojectable}). \end{aligned} \quad (3)$$

The fundamental symmetry of the theory is the invariance under the so-called foliation-preserving diffeomorphism,

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$$t \rightarrow t'(t), \quad \vec{x} \rightarrow \vec{x}'(t, \vec{x}). \quad (4)$$

Since the theory enjoys less symmetries than general relativity (GR), the number of propagating degrees of freedom is larger. It contains not only a tensor graviton but also a scalar graviton. The properties of the scalar graviton at the linearized level crucially depend on a parameter commonly denoted as λ . The positivity of the (time) kinetic term of the scalar graviton on a flat background requires that

$$\lambda < \frac{1}{3} \quad \text{or} \quad 1 < \lambda. \quad (5)$$

On the other hand, the dispersion relation of the scalar graviton in the projectable theory is

$$\begin{aligned} \omega^2 &= c_s^2 \vec{k}^2 + \alpha_2 \frac{(\vec{k}^2)^2}{M^2} + \alpha_3 \frac{(\vec{k}^2)^3}{M^4}, \\ c_s^2 &= -\frac{\lambda - 1}{3\lambda - 1} c_g^2, \quad (\text{projectable}), \end{aligned} \quad (6)$$

where M is a mass scale, $\alpha_{2,3}$ are dimensionless constants and c_g is the speed of gravitational waves. Short-wavelength perturbations are stable, provided that $\alpha_{2,3}$ are positive. However, for the range of λ shown in Eq. (5), the “sound speed squared” c_s^2 is negative and thus the flat background is unstable against long-wavelength perturbations.

One way to tame the instability in the infrared (IR) is to relax the projectability condition and thus to consider the nonprojectable theory (3). In this case the lapse can depend on spatial coordinates, and thus terms depending on spatial derivatives of the lapse may enter the action. This significantly increases the number of independent coupling constants in the theory but allows for a stable regime of

parameters in the IR. The “sound speed squared” is now given by

$$c_s^2 = \frac{\lambda - 1}{3\lambda - 1} \frac{2c_g^2 - \eta}{\eta} c_g^2, \quad (\text{nonprojectable}), \quad (7)$$

where η is the coefficient of the new term $g^{ij}\partial_i \ln N \partial_j \ln N$. (The formula for c_s^2 in the projectable theory is recovered in the limit $\eta \rightarrow \infty$.) Thus the scalar graviton is stable in the IR if the condition (5) is satisfied and if

$$0 < \eta < 2c_g^2. \quad (8)$$

Another way out is to keep the projectability condition and to impose the condition [2]

$$|c_s| \leq \max[\sqrt{|\Phi|}, HL] \quad \text{for} \\ L > \max[0.01 \text{ mm}, 1/M], \quad (\text{projectable}), \quad (9)$$

where H is the Hubble expansion rate of the background universe, L is the length scale of interest and $\Phi \sim -G_N \rho L^2$ is the Newtonian potential. This and the condition (5) imply that λ must approach 1 from above under the renormalization group (RG) flow as the Hubble expansion rate decreases. This would pose a nontrivial phenomenological constraint on properties of the RG flow.

In the projectable theory, in the limit $\lambda \rightarrow 1$ deep inside the condition (9), the solution to the momentum constraint equation for scalar-type perturbations around a flat background becomes singular. Due to this behavior, the perturbative expansion of the action for the scalar graviton breaks down as λ approaches 1 from above: there is an infinite series of terms in which the coefficients of terms of higher order in perturbative expansion have more negative powers of $(\lambda - 1)$ and thus are more singular in the limit. However, all those singular terms are kinetic terms, i.e. terms with two time derivatives, and terms without time derivatives are always independent of λ . For this reason, if the sum of kinetic terms can in principle be canonically normalized, then the potential terms written in terms of the canonically normalized field have only positive powers of $(\lambda - 1)$ and should be regular in the $\lambda \rightarrow 1$ limit. In this sense, the canonically normalized scalar graviton is weakly coupled and expected to be decoupled from the rest of the world in the $\lambda \rightarrow 1$ limit. Indeed, it has been shown for simple cases such as static, spherically symmetric configurations [2] and superhorizon perturbations in the expanding universe [3,4] that GR (plus dark matter [5]) is smoothly recovered in the limit if (and only if) nonlinearity is fully taken into account. For this reason the behavior of the scalar graviton in the $\lambda \rightarrow 1$ limit does not necessarily pose a conceptual problem at least in principle. Nonetheless, it is fair to say that the breakdown of perturbative expansion leads to technical difficulties as fully nonlinear behavior is not easy to analyze in general, beyond the above-mentioned

explicit examples. As pointed out in Ref. [2], the situation is similar to the Vainshtein screening mechanism in massive gravity and other modified gravity theories [6,7].

Even in the nonprojectable theory, if we impose observational constraints on λ , η and c_g^2 then one of the following two should happen: the perturbative expansion breaks down at a scale considerably lower than the Planck scale [8], or higher derivative corrections to the dispersion relation must kick in at an even lower scale [9]. The latter case may be in conflict with observation [10]. Therefore, one might have to consider the former case in which the perturbative expansion breaks down at a certain scale. As in the case of the projectable theory, since the perturbative expansion breaks down only in the kinetic terms of the scalar graviton, this is not necessarily a problem at least in principle. This situation is again similar to the Vainshtein screening mechanism.

As the third possibility to get around the issue of the scalar graviton, Hořava and Melby-Thompson [11] proposed a $U(1)$ extension of the projectable version of Hořava-Lifshitz gravity in which the scalar graviton is claimed to be absent. In the context of the $U(1)$ extension of Hořava-Lifshitz gravity it was originally claimed that the symmetry of the theory automatically fixes the parameter λ to 1 and that the scalar graviton is absent because of this value of λ . Later it was however shown that the theory with any value of λ can be constructed without spoiling the symmetry [12]. Nonetheless, at the level of linear perturbations it was shown that the scalar graviton is still absent for any λ . The proof of the absence of the scalar graviton was later extended to a fully nonlinear level [13].

While the original theory with the $U(1)$ symmetry is projectable, it is possible to extend it to a nonprojectable theory [14,15]. At linear perturbations around the Minkowski background it is known that the nonprojectable $U(1)$ extension has a scalar graviton for a generic choice of coupling constants [16]. On the other hand, if one of the parameters in the IR action (which is denoted as η_2 in the present paper) is set to zero, then linear perturbations in the theory do not contain the scalar graviton [14].

The purpose of the present paper is to perform the Hamiltonian analysis and to count the number of degrees of freedom in the nonprojectable version of the $U(1)$ extension. This makes it possible for us to investigate the condition under which the scalar graviton disappears at a fully nonlinear level. We shall show that the theory generically contains the scalar graviton: the scalar graviton does not exist if and only if two coupling constants are exactly zero. The terms corresponding to these two specific coupling constants are marginal for any values of the critical exponent z and the spatial dimension d , and thus should be generated by quantum corrections even if they are eliminated from the bare action by hand.

The rest of the present paper is organized as follows. In Sec. II we review the construction of the $U(1)$ extension of

Hořava-Lifshitz gravity. In Sec. III we adopt the method of Hamiltonian analysis in the classical field theory to study the nature of constraints in the theory. We then count the number of degrees of freedom in Sec. IV. Section V is devoted to a summary of this paper and to some discussions. As supporting materials, we show a useful technique to ease calculations by using the symmetry of spatial diffeomorphism in Appendix A, and briefly discuss the case of the projectable version of the theory in Appendix B.

II. NONPROJECTABLE HOŘAVA-LIFSHITZ GRAVITY WITH $U(1)$ SYMMETRY

In this section we provide a brief review of the construction of the $U(1)$ extension of Hořava-Lifshitz gravity, following the derivation presented in Appendix A of Ref. [16]. For details of the theory we refer readers to, e.g. Refs. [14,15].

The basic quantities of the theory are the lapse $N(t, \vec{x})$, the shift $N^i(t, \vec{x})$ and the spatial metric $g_{ij}(t, \vec{x})$, supplemented by the gauge field $A(t, \vec{x})$ and an auxiliary scalar $\nu(t, \vec{x})$, often called the Newtonian prepotential. The transformations of these quantities under the infinitesimal foliation-preserving diffeomorphism,

$$\delta t = f(t), \quad \delta x^i = \xi^i(t, \vec{x}), \quad (10)$$

are defined as

$$\begin{aligned} \delta g_{ij} &= f \partial_t g_{ij} + \mathcal{L}_\xi g_{ij}, & \delta N^i &= \partial_t (N^i f) + \partial_t \xi^i + \mathcal{L}_\xi N^i, \\ \delta N &= \partial_t (N f) + \xi^i \partial_i N, & \delta A &= \partial_t (A f) + \xi^i \partial_i A, \\ \delta \nu &= f \partial_t \nu + \xi^i \partial_i \nu, \end{aligned} \quad (11)$$

where $\mathcal{L}_\xi$ is the Lie derivative along ξ^i . On top of the symmetry under the transformation (11), we further introduce an Abelian symmetry, which is thus called the $U(1)$ symmetry, such that the action of the theory is invariant under the infinitesimal local transformation parametrized by the gauge parameter $\alpha(t, \vec{x})$. The basic quantities transform as,

$$\begin{aligned} \delta N_i &= N \partial_i \alpha, & \delta A &= N \partial_\perp \alpha, & \delta \nu &= \alpha, \\ \delta g_{ij} &= 0, & \delta N &= 0, \end{aligned} \quad (12)$$

where $\partial_\perp \equiv (1/N)(\partial_t - N^i \partial_i)$. Spatial indices are raised and lowered by g^{ij} and g_{ij} , respectively. The scaling dimensions are then assigned to coordinates and variables as follows:

$$\begin{aligned} [\partial_t] &= 1, & [\partial_i] &= z, & [dtd^d \vec{x}] &= -z - d, & [\partial_\perp] &= z, \\ [g_{ij}] &= 0, & [N_i] &= [N^i] = z - 1, & [N] &= 0, \\ [\alpha] &= z - 2, & [A] &= 2z - 2, & [\nu] &= z - 2, \end{aligned} \quad (13)$$

where d is the number of spatial dimensions, and $z \geq d$ is the dynamical critical exponent, which determines the scaling (1) in the UV regime.

In order to construct a $U(1)$ -invariant action, it is convenient to define the following $U(1)$ -invariant ingredients:

$$\begin{aligned} \tilde{K}_{ij} &= \frac{1}{2N} (\partial_t g_{ij} - D_i \tilde{N}_j - D_j \tilde{N}_i), \\ \tilde{N}^i &= N^i - N D^i \nu, \\ \sigma &= \frac{A}{N} - \partial_\perp \nu - \frac{1}{2} D^i \nu D_i \nu, \end{aligned} \quad (14)$$

where D_i is the spatial covariant derivative compatible with g_{ij} . The scaling dimensions of $\tilde{K}_{ij}$ and σ are

$$[\tilde{K}_{ij}] = z, \quad [\sigma] = 2z - 2. \quad (15)$$

The kinetic action for g_{ij} is then constructed from $\tilde{K}_{ij}$ as

$$\frac{M_{\text{Pl}}^2}{2} \int dtd^d x \sqrt{g} N [\tilde{K}^{ij} \tilde{K}_{ij} - \lambda \tilde{K}^2], \quad (16)$$

where M_{Pl} is a constant corresponding to the Planck scale at low energy, λ is a dimensionless constant, $g \equiv \det(g_{ij})$ and $\tilde{K} \equiv g^{ij} \tilde{K}_{ij}$. The scaling dimension of the kinetic terms is

$$[\tilde{K}^{ij} \tilde{K}_{ij}] = [\tilde{K}^2] = 2z, \quad (17)$$

and thus the power-counting renormalizability requires that the other terms in the action should have scaling dimensions equal to or less than $2z$. In particular, it is required that the dependence of the action on σ be at most linear. This is because

$$[\sigma^n] = 2n(z - 1) > 2z, \quad \text{for } n \geq 2, \quad z \geq d \geq 3, \quad (18)$$

and because derivatives and all other fundamental variables have non-negative scaling dimensions in $z \geq d \geq 3$. On the other hand, the linear term, σ , is always power-counting renormalizable (relevant) since $[\sigma] = 2z - 2 < 2z$. Other terms linear in σ , that is $R\sigma$, $a^i a_i \sigma$, and $D^i a_i \sigma$, are marginal, i.e.,

$$[R\sigma] = [a^i a_i \sigma] = [D^i a_i \sigma] = 2z, \quad (19)$$

where R is the Ricci scalar of g_{ij} and $a_i \equiv D_i \ln N$. We do not include in the action the time derivatives of N , N^i or A , since no relevant/marginal terms that are invariant under both the foliation-preserving diffeomorphism and the $U(1)$ transformation can be constructed.

The gravity action which respects foliation-preserving diffeomorphism, the $U(1)$ symmetry and the power-counting renormalizability in UV thus reads

$$\begin{aligned} I &= \frac{M_{\text{Pl}}^2}{2} \int dtd^d x \sqrt{g} N [\tilde{K}^{ij} \tilde{K}_{ij} - \lambda \tilde{K}^2 - \mathcal{L}_V \\ &\quad + (2\Omega - \eta_0 R + \eta_1 a^i a_i + \eta_2 D^i a_i) \sigma], \end{aligned} \quad (20)$$

where Ω and $\eta_{0,1,2}$ are constants, and the potential term $\mathcal{L}_V[g_{ij}, N]$ is constructed from a_i , g_{ij} , D_i and the Riemann tensor of g_{ij} , including all the terms that contain up to $2z$ spatial derivatives and are invariant under the diffeomorphism. One of the coupling constants $\eta_{0,1,2}$ is in fact redundant and one can, for example, set

$$\eta_0 = 1, \quad (21)$$

by a redefinition of A and Ω .

III. HAMILTONIAN ANALYSIS

It has been shown that the projectable version, i.e. the lapse N being a function only of time t , of the $U(1)$ extension of Hořava-Lifshitz gravity has the same number of degrees of freedom as GR, or equivalently one less degree of freedom than the original version of Hořava-Lifshitz theory [11–13]. However, once the nonprojectable version, i.e. $N = N(t, \vec{x})$, is considered with all the terms consistent with the symmetry included, an additional scalar degree of freedom, dubbed a scalar graviton, reappears already at the level of linearized perturbations [16]. In this section, in order to understand the complete structure of the theory, we perform the Hamiltonian analysis and count the number of degrees of freedom at the fully nonlinear order. Since we are interested in the physical degrees of freedom contained in the gravity sector, we shall omit matter fields.

We start with the $(d^2 + 3d + 6)$ -dimensional phase space $(g_{ij}, \pi^{ij}, N^i, \pi_i, N, \pi_N, A, \pi_A, \nu, \pi_\nu)$ at each point in spacetime, where $\pi^{ij}, \pi_i, \pi_N, \pi_A$ and π_ν are the canonical momenta conjugate to g_{ij}, N^i, N, A and ν , respectively. The Poisson bracket is defined in the standard way as

$$\begin{aligned} \{F, G\}_P \equiv & \int d^d x \left[\frac{\delta F}{\delta g_{ij}(\vec{x})} \frac{\delta G}{\delta \pi^{ij}(\vec{x})} + \frac{\delta F}{\delta N^i(\vec{x})} \frac{\delta G}{\delta \pi_i(\vec{x})} \right. \\ & + \frac{\delta F}{\delta N(\vec{x})} \frac{\delta G}{\delta \pi_N(\vec{x})} + \frac{\delta F}{\delta A(\vec{x})} \frac{\delta G}{\delta \pi_A(\vec{x})} \\ & + \frac{\delta F}{\delta \nu(\vec{x})} \frac{\delta G}{\delta \pi_\nu(\vec{x})} - \frac{\delta F}{\delta \pi^{ij}(\vec{x})} \frac{\delta G}{\delta g_{ij}(\vec{x})} \\ & - \frac{\delta F}{\delta \pi_i(\vec{x})} \frac{\delta G}{\delta N^i(\vec{x})} - \frac{\delta F}{\delta \pi_N(\vec{x})} \frac{\delta G}{\delta N(\vec{x})} \\ & \left. - \frac{\delta F}{\delta \pi_A(\vec{x})} \frac{\delta G}{\delta A(\vec{x})} - \frac{\delta F}{\delta \pi_\nu(\vec{x})} \frac{\delta G}{\delta \nu(\vec{x})} \right]. \quad (22) \end{aligned}$$

The canonical momenta are found as

$$\pi_i = 0, \quad \pi_N = 0, \quad \pi_A = 0, \quad \pi_\nu = -J_A, \quad (23)$$

$$\pi^{ij} = \frac{M_{\text{Pl}}^2}{2} \sqrt{g} \left[\frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl} \right] \tilde{K}_{kl}, \quad (24)$$

where

$$J_A \equiv \frac{M_{\text{Pl}}^2}{2} \sqrt{g} (2\Omega - \eta_0 R + \eta_1 a^i a_i + \eta_2 D^i a_i). \quad (25)$$

Notice that Eq. (23) constitutes the primary constraints of the system. We can invert the relation (24) to express $\tilde{K}_{ij}$ in terms of π^{ij} as

$$\tilde{K}_{ij} = \frac{2}{M_{\text{Pl}}^2 \sqrt{g}} \mathcal{G}_{ijkl} \pi^{kl}, \quad (26)$$

where for notational brevity we have defined

$$\mathcal{G}_{ijkl} \equiv \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{\lambda}{d\lambda - 1} g_{ij} g_{kl}. \quad (27)$$

As is clear from Eq. (23), such an inversion cannot be done for $\partial_i N^i, \partial_i N, \partial_i A$ and $\partial_i \nu$. While the action (20) does not contain $\partial_i N^i, \partial_i N$ or $\partial_i A$ and so there is no need for the inversion, $\partial_i \nu$ does not enter the Hamiltonian either once the constraint $\pi_\nu = -J_A$ in Eq. (23) is imposed. One can thus see that the $(d+3)$ primary constraints $\pi_i \approx \pi_N \approx \pi_A \approx \pi_\nu + J_A \approx 0$ are all independent.

To proceed, we first inspect the algebra of the primary constraints. The Poisson brackets of π_i and π_A with all the constraints trivially vanish:

$$\{\pi_i(\vec{x}), \Phi(\vec{y})\}_P = 0, \quad \{\pi_A(\vec{x}), \Phi(\vec{y})\}_P = 0, \quad (28)$$

where Φ stands for $\{\pi_i, \pi_A, \pi_N, \pi_\nu + J_A\}$. The Poisson brackets between π_N and $\pi_\nu + J_A$ are, on the other hand,

$$\{\pi_N(\vec{x}), \pi_N(\vec{y})\}_P = \{\pi_\nu(\vec{x}) + J_A(\vec{x}), \pi_\nu(\vec{y}) + J_A(\vec{y})\}_P = 0, \quad (29)$$

$$\begin{aligned} & \{\overline{\pi_N}[\chi], \overline{\pi_\nu + J_A}[\varphi]\}_P \\ &= \{\overline{\pi_N}[\chi], \overline{J_A}[\varphi]\}_P \\ &= -\frac{M_{\text{Pl}}^2}{2} \int d^d x \sqrt{g} \varphi (\eta_2 D^2 + 2\eta_1 a_i D^i) \frac{\chi}{N}, \quad (30) \end{aligned}$$

where the over-bar denotes

$$\overline{\Phi}[\chi] \equiv \int d^d x \Phi(\vec{x}) \chi(\vec{x}), \quad (31)$$

where Φ is any constraint of scalar type and χ is an arbitrary function. Hereafter we assume that the arbitrary functions corresponding to $\chi(\vec{x})$ are independent of the canonical variables; otherwise, some of the strong equalities ($=$) should be replaced by the weak ones ($\approx$).

The Hamiltonian of the system is now constructed as

$$\begin{aligned} H \equiv & \int d^d x [\pi^{ij} \partial_i g_{ij} + \pi_i \partial_i N^i + \pi_N \partial_i N + \pi_A \partial_i A + \pi_\nu \partial_i \nu - \mathcal{L}] \\ &= \int d^d x \left[N \sqrt{g} \left(\frac{2}{M_{\text{Pl}}^2 \sqrt{g}} \mathcal{G}_{ijkl} \frac{\pi^{kl}}{\sqrt{g}} + \frac{M_{\text{Pl}}^2}{2} \mathcal{L}_V \right) - N \mathcal{H}_i D^i \nu \right. \\ & \quad \left. + \frac{N}{2} J_A D_i \nu D^i \nu + N^i (\mathcal{H}_i - J_A D_i \nu) - A J_A \right], \quad (32) \end{aligned}$$

where $\mathcal{L}$ is the Lagrangian density and we have defined

$$\mathcal{H}_i \equiv -2\sqrt{g}D_j \left(\frac{\pi_i^j}{\sqrt{g}} \right). \quad (33)$$

Because of the presence of the constraints, the time evolution in the phase space is not completely determined by this Hamiltonian. Including the terms corresponding to the primary constraints, we thus redefine the Hamiltonian as

$$\hat{H} = H + \int d^d x [\lambda^i \pi_i + \lambda_A \pi_A + \lambda_N \pi_N + \lambda_J (\pi_\nu + J_A)], \quad (34)$$

where $\lambda^i, \lambda_A, \lambda_N, \lambda_J$ are Lagrange multipliers.

Now we need to impose the consistency conditions for the primary constraints against time evolution with respect to $\hat{H}$ to determine the secondary constraints. However, we observe from Eq. (30) that the constraint structure differs depending on whether the right-hand side of the equation vanishes identically or not. We therefore separate the analysis into the following two subsections, one corresponding to the generic case where the right-hand side of Eq. (30) is nonvanishing, and the other to the special case where it is identically zero.

A. Generic case

In this subsection we consider the case where the right-hand side of Eq. (30) does not vanish identically, namely the case with $(\eta_1, \eta_2) \neq (0, 0)$. We call it generic since even if η_1 and η_2 are zero at the classical level, they generically become nonzero when quantum corrections to the action are taken into account.

The consistency conditions of the primary constraints with the time evolution are

$$\begin{aligned} 0 &= \frac{d}{dt}(\pi_\nu + J_A) \approx \{\pi_\nu + J_A, \hat{H}\}_P \\ &= \{\pi_\nu + J_A, H\}_P + \{\pi_\nu + J_A, \overline{\pi_N}[\lambda_N]\}_P, \\ 0 &= \frac{d}{dt}\pi_N \approx \{\pi_N, \hat{H}\}_P \\ &= \{\pi_N, H\}_P + \{\pi_N, \overline{\pi_\nu + J_A}[\lambda_J]\}_P, \end{aligned} \quad (35)$$

and

$$\begin{aligned} 0 &= \frac{d}{dt}\pi_A \approx \{\pi_A, \hat{H}\}_P = \{\pi_A, H\}_P = -\frac{\delta H}{\delta A} = J_A, \\ 0 &= \frac{d}{dt}\pi_i \approx \{\pi_i, \hat{H}\}_P = \{\pi_i, H\}_P = -\frac{\delta H}{\delta N^i} = -\mathcal{H}_i + J_A D_i \nu, \end{aligned} \quad (36)$$

where $\approx$ denotes an equality in the weak sense, i.e. with all the constraints imposed. The first two equations (35) do not lead to additional constraints but are simply considered as equations to determine the multipliers λ_N and λ_J ,

$$\eta_2 D^2 \frac{\lambda_N}{N} + 2\eta_1 a_i D^i \frac{\lambda_N}{N} \approx -\frac{2}{M_{\text{Pl}}^2 \sqrt{g}} \{\pi_\nu + J_A, H\}_P,$$

$$\eta_2 D^2 \lambda_J - 2\eta_1 D^i (a_i \lambda_J) \approx \frac{2N}{M_{\text{Pl}}^2 \sqrt{g}} \{\pi_N, H\}_P. \quad (37)$$

On the other hand, the last two equations (36) give secondary constraints, $J_A \approx 0$ and $\mathcal{H}_i \approx 0$.

The set of constraints that we obtained so far is $(\pi_i, \pi_A, \pi_\nu, \mathcal{H}_i, \pi_N, J_A)$. Let us calculate the Poisson brackets among them. The Poisson brackets of π_i, π_A and π_ν with all the constraints trivially vanish:

$$\begin{aligned} \{\pi_i(\vec{x}), \Phi(\vec{y})\}_P &= 0, \\ \{\pi_A(\vec{x}), \Phi(\vec{y})\}_P &= 0, \\ \{\pi_\nu(\vec{x}), \Phi(\vec{y})\}_P &= 0, \end{aligned} \quad (38)$$

where $\Phi = \{\pi_j, \pi_A, \pi_\nu, \mathcal{H}_j, \pi_N, J_A\}$. On the other hand, the Poisson bracket between $\mathcal{H}_i$ and J_A can be calculated by using the formula

$$\{\overline{\mathcal{H}}[f], F\}_P = \int d^d x \left(\frac{\delta F}{\delta s} f^i \partial_i s + \frac{\delta F}{\delta V^i} [f, V]^i \right), \quad (39)$$

where $F = F[g_{ij}, \pi^{ij}, s(\text{scalar}), V^i(\text{vector})]$ is a functional invariant under time-independent spatial diffeomorphism and $[f, V]^i = f^j D_j V^i - V^j D_j f^i$ (see Appendix A for the proof). The result is

$$\begin{aligned} \{\overline{\mathcal{H}}[f], \overline{J_A}[\varphi]\}_P &= \overline{J_A}[f \partial \varphi] + \int d^d x \frac{\delta \overline{J_A}[\varphi]}{\delta N} f^i \partial_i N \\ &\approx -\{\overline{\pi_N}[f \partial N], \overline{J_A}[\varphi]\}_P, \end{aligned} \quad (40)$$

and this does not vanish weakly, where

$$\overline{\mathcal{H}}[f] \equiv \int d^d x \mathcal{H}_i f^i, \quad f \partial \varphi \equiv f^i \partial_i \varphi. \quad (41)$$

Similarly to the comment made after Eq. (31), we assume that the functions $f^i(\vec{x})$ are independent of the canonical variables; otherwise, the strong equality in the first line of Eq. (40) should be replaced by the weak one, and the same applies to the following calculations. We thus define the linear combination

$$\mathcal{H}_i^N \equiv \mathcal{H}_i + \pi_N D_i N, \quad (42)$$

so that the Poisson bracket of $\mathcal{H}_i^N$ with J_A vanishes weakly. Actually, the Poisson brackets of $\mathcal{H}_i^N$ with all the constraints vanish weakly:

$$\begin{aligned} \{\overline{\mathcal{H}}^N[f], \overline{\mathcal{H}}^N[g]\}_P &= \overline{\mathcal{H}}^N[[f, g]] \approx 0 \quad \text{for } \forall f^i, \forall g^i, \\ \{\overline{\mathcal{H}}^N[f], \overline{\Phi}[\varphi]\}_P &= \overline{\Phi}[f \partial \varphi] \approx 0 \quad \text{for } \forall f^i, \forall \varphi, \end{aligned} \quad (43)$$

where $\Phi = \{\pi_N, J_A\}$. The remaining Poisson brackets are those among π_N and J_A and are

$$\{\pi_N(\vec{x}), \pi_N(\vec{y})\}_P = 0, \quad \{J_A(\vec{x}), J_A(\vec{y})\}_P = 0, \quad (44)$$

and Eq. (30).

Hereafter we consider $(\pi_i, \pi_A, \pi_\nu, \mathcal{H}_i^N, \pi_N, J_A)$ as the basis for the set of constraints. Adding the terms corresponding to the secondary constraints, we thus redefine the Hamiltonian as

$$\tilde{H} = H + \int d^d x [\lambda^i \pi_i + \lambda_A \pi_A + \lambda_\nu \pi_\nu + \lambda_{\mathcal{H}} \mathcal{H}_i^N + \lambda_N \pi_N + \lambda_J J_A], \quad (45)$$

where $\lambda^i, \lambda_A, \lambda_\nu, \lambda_{\mathcal{H}}, \lambda_N, \lambda_J$ are Lagrange multipliers.

The consistency conditions of J_A, π_N with the time evolution with respect to the Hamiltonian $\tilde{H}$ are

$$\begin{aligned} 0 &= \frac{d}{dt} J_A \approx \{J_A, H\}_P + \{J_A, \overline{\pi_N}[\lambda_N]\}_P, \\ 0 &= \frac{d}{dt} \pi_N \approx \{\pi_N, H\}_P + \{\pi_N, \overline{J_A}[\lambda_J]\}_P. \end{aligned} \quad (46)$$

They yield no additional constraint but equations to determine the multipliers λ_N and λ_J as Eq. (37). The consistency conditions of π_A, π_i, π_ν and $\mathcal{H}_i^N$ with the time evolution are

$$\begin{aligned} 0 &= \frac{d}{dt} \pi_A \approx \{\pi_A, H\}_P = -\frac{\delta H}{\delta A} = J_A, \\ 0 &= \frac{d}{dt} \pi_i \approx \{\pi_i, H\}_P = -\frac{\delta H}{\delta N^i} = -\mathcal{H}_i + J_A D_i \nu, \\ 0 &= \frac{d}{dt} \pi_\nu \approx \{\pi_\nu, H\}_P = -\frac{\delta H}{\delta \nu} = -\partial_i (N \mathcal{H}^i + \tilde{N}^i J_A), \\ 0 &= \frac{d}{dt} \overline{\mathcal{H}^N}[f] \approx \{\overline{\mathcal{H}^N}[f], H\}_P \\ &= \int d^d x [-(N \mathcal{H}^i + \tilde{N}^i J_A) \partial_i (f \partial \nu) - J_A f \partial A \\ &\quad + (\mathcal{H}_i - J_A \partial_i \nu) [f, N]^i], \end{aligned} \quad (47)$$

where we have used the formula (39) to derive the last equality. They vanish weakly and thus do not give any additional constraints.

In summary, $(\pi_i, \pi_A, \pi_\nu, \mathcal{H}_i^N, \pi_N, J_A)$ forms the set of all constraints for the generic case studied in this subsection. Among them, $(2d+2)$ constraints $(\pi_i, \mathcal{H}_i^N, \pi_A, \pi_\nu)$ are first class and two constraints (π_N, J_A) are second class.

B. Special case with $\eta_1 = \eta_2 = 0$

In this subsection we consider the special case where the right-hand side of Eq. (30) vanishes identically, namely the case with $\eta_1 = \eta_2 = 0$. We consider that this is not generic but rather exceptional. Indeed, the choice of parameters $\eta_1 = \eta_2 = 0$ is not stable under radiative corrections. We

nonetheless investigate this case for completeness. Also, the study of this special case sheds some light on important roles of Eq. (37) for the structure of the theory: with (and only with) $\eta_1 = \eta_2 = 0$, Eq. (37) leads to secondary constraints instead of yielding equations to determine the Lagrange multipliers.

The consistency conditions of the primary constraints with the time evolution are

$$\begin{aligned} 0 &= \frac{d}{dt} \pi_A \approx \{\pi_A, H\}_P = -\frac{\delta H}{\delta A} = J_A, \\ 0 &= \frac{d}{dt} \pi_i \approx \{\pi_i, H\}_P = -\frac{\delta H}{\delta N^i} = -\mathcal{H}_i + J_A D_i \nu, \\ 0 &= \frac{d}{dt} \pi_N \approx \{\pi_N, H\}_P = -\frac{\delta H}{\delta N} = -\mathcal{H}_\perp + \mathcal{H}^i D_i \nu \\ &\quad - \frac{1}{2} J_A D^i \nu D_i \nu, \\ 0 &= \frac{d}{dt} (\pi_\nu + J_A) \approx -\frac{\delta H}{\delta \nu} + \{J_A, H\}_P \\ &= -\partial_i (N \mathcal{H}^i + \tilde{N}^i J_A) + \phi_A + \frac{2N}{M_{\text{Pl}}^2} g^{ij} \mathcal{G}_{ijkl} \pi^{kl} \frac{J_A}{\sqrt{g}} \\ &\quad + \sqrt{g} D_i \left(\tilde{N}^i \frac{J_A}{\sqrt{g}} \right), \end{aligned} \quad (48)$$

where

$$\mathcal{H}_\perp \equiv \frac{2}{M_{\text{Pl}}^2 \sqrt{g}} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} + \frac{M_{\text{Pl}}^2}{2} \frac{\delta}{\delta N} \int d^d x \sqrt{g} N \mathcal{L}_V, \quad (49)$$

$$\begin{aligned} \phi_A &\equiv 2\eta_0 \sqrt{g} \left[\frac{\lambda - 1}{d\lambda - 1} D^2 \left(\frac{\pi}{\sqrt{g}} N \right) + R^{ij} \mathcal{G}_{ijkl} \frac{\pi^{kl}}{\sqrt{g}} N \right. \\ &\quad \left. - D_i D_j \left(\frac{\pi^{ij}}{\sqrt{g}} N \right) \right], \end{aligned} \quad (50)$$

with $\pi \equiv g_{ij} \pi^{ij}$. They give secondary constraints $J_A \approx 0$, $\mathcal{H}_i \approx 0$, $\mathcal{H}_\perp \approx 0$ and $\phi_A \approx 0$.

The set of constraints that we obtained so far is $(\pi_i, \pi_A, \pi_\nu, \mathcal{H}_i, \pi_N, J_A, \mathcal{H}_\perp, \phi_A)$. Following the same logic as that in the generic case studied in the previous subsection, we define the linear combination $\mathcal{H}_i^N$ shown in Eq. (42). While the original constraint $\mathcal{H}_i$ has (weakly) nonvanishing Poisson brackets with $\mathcal{H}_\perp$ and ϕ_A as

$$\begin{aligned} \{\tilde{\mathcal{H}}[f], \overline{\mathcal{H}_\perp}[\varphi]\}_P &= \overline{\mathcal{H}_\perp}[f \partial \varphi] + \int d^d x \frac{\delta \overline{\mathcal{H}_\perp}[\varphi]}{\delta N} f^i \partial_i N \\ &\approx \int d^d x \frac{\delta \overline{\mathcal{H}_\perp}[\varphi]}{\delta N} f^i \partial_i N, \\ \{\tilde{\mathcal{H}}[f], \overline{\phi_A}[\varphi]\}_P &= \overline{\phi_A}[f \partial \varphi] + \int d^d x \frac{\delta \overline{\phi_A}[\varphi]}{\delta N} f^i \partial_i N \\ &\approx \int d^d x \frac{\delta \overline{\phi_A}[\varphi]}{\delta N} f^i \partial_i N, \end{aligned} \quad (51)$$

the Poisson brackets of $\mathcal{H}_i^N$ with all the constraints vanish weakly:

$$\begin{aligned} \{\overline{\mathcal{H}^N}[f], \overline{\mathcal{H}^N}[g]\}_P &= \overline{\mathcal{H}^N}[[f, g]] \approx 0 \quad \text{for } \forall f^i, \forall g^i, \\ \{\overline{\mathcal{H}^N}[f], \overline{\Phi}[\varphi]\}_P &= \overline{\Phi}[f\partial\varphi] \approx 0 \quad \text{for } \forall f^i, \forall \varphi, \end{aligned} \quad (52)$$

where $\Phi = \{\pi_N, J_A, \mathcal{H}_\perp, \phi_A\}$. Here, the formula (39) has been used to simplify the calculation of the Poisson brackets shown in Eqs. (51) and (52). The Poisson brackets of π_i , π_A and π_ν with all the constraints vanish:

$$\begin{aligned} \{\pi_i(\vec{x}), \Phi(\vec{y})\}_P &= 0, \\ \{\pi_A(\vec{x}), \Phi(\vec{y})\}_P &= 0, \\ \{\pi_\nu(\vec{x}), \Phi(\vec{y})\}_P &= 0, \end{aligned} \quad (53)$$

where $\Phi = \{\pi_j, \pi_A, \pi_\nu, \mathcal{H}_j^N, \pi_N, J_A, \mathcal{H}_\perp, \phi_A\}$. Finally, while the Poisson brackets between π_N and J_A vanish,

$$\begin{aligned} \{\pi_N(\vec{x}), \pi_N(\vec{y})\}_P &= \{\pi_N(\vec{x}), J_A(\vec{y})\}_P \\ &= \{J_A(\vec{x}), J_A(\vec{y})\}_P = 0, \end{aligned} \quad (54)$$

some of the Poisson brackets among π_N , J_A , $\mathcal{H}_\perp$ and ϕ_A are nonvanishing:

$$\begin{aligned} \{\overline{\pi_N}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P \\ = -\frac{M_{\text{Pl}}^2}{2} \int d^d x \chi \frac{\delta}{\delta N} \int d^d y \varphi \frac{\delta}{\delta N} \int d^d z \sqrt{g} N \mathcal{L}_V, \end{aligned} \quad (55)$$

$$M \equiv \begin{pmatrix} \{\overline{\pi_N}[\chi], \overline{\pi_N}[\varphi]\}_P & \{\overline{\pi_N}[\chi], \overline{J_A}[\varphi]\}_P & \{\overline{\pi_N}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P & \{\overline{\pi_N}[\chi], \overline{\phi_A}[\varphi]\}_P \\ \{\overline{J_A}[\chi], \overline{\pi_N}[\varphi]\}_P & \{\overline{J_A}[\chi], \overline{J_A}[\varphi]\}_P & \{\overline{J_A}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P & \{\overline{J_A}[\chi], \overline{\phi_A}[\varphi]\}_P \\ \{\overline{\mathcal{H}_\perp}[\chi], \overline{\pi_N}[\varphi]\}_P & \{\overline{\mathcal{H}_\perp}[\chi], \overline{J_A}[\varphi]\}_P & \{\overline{\mathcal{H}_\perp}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P & \{\overline{\mathcal{H}_\perp}[\chi], \overline{\phi_A}[\varphi]\}_P \\ \{\overline{\phi_A}[\chi], \overline{\pi_N}[\varphi]\}_P & \{\overline{\phi_A}[\chi], \overline{J_A}[\varphi]\}_P & \{\overline{\phi_A}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P & \{\overline{\phi_A}[\chi], \overline{\phi_A}[\varphi]\}_P \end{pmatrix}. \quad (60)$$

With Eq. (54), M can be partitioned with 2×2 matrices B, C, D as

$$M \equiv \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}, \quad (61)$$

where

$$\begin{aligned} B &= \begin{pmatrix} \{\overline{\pi_N}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P & \{\overline{\pi_N}[\chi], \overline{\phi_A}[\varphi]\}_P \\ \{\overline{J_A}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P & \{\overline{J_A}[\chi], \overline{\phi_A}[\varphi]\}_P \end{pmatrix}, \\ C &= -(B^T \text{ with } \chi \leftrightarrow \varphi), \end{aligned} \quad (62)$$

and 0 is the 2×2 matrix with all vanishing components. We thus have

$$\begin{aligned} \{\overline{\pi_N}[\chi], \overline{\phi_A}[\varphi]\}_P \\ = -2\eta_0 \int d^d x \chi \left(\frac{\lambda-1}{d\lambda-1} \pi D^2 + R^{ij} \mathcal{G}_{ijkl} \pi^{kl} - \pi^{ij} D_i D_j \right) \varphi, \end{aligned} \quad (56)$$

$$\begin{aligned} \{\overline{J_A}[\chi], \overline{\mathcal{H}_\perp}[\varphi]\}_P \\ \approx +2\eta_0 \int d^d x \varphi \left(\frac{\lambda-1}{d\lambda-1} \pi D^2 + R^{ij} \mathcal{G}_{ijkl} \pi^{kl} - \pi^{ij} D_i D_j \right) \chi, \end{aligned} \quad (57)$$

$$\begin{aligned} \{\overline{J_A}[\chi], \overline{\phi_A}[\varphi]\}_P \\ \approx \eta_0^2 M_{\text{Pl}}^2 \int d^d x \sqrt{g} N [R^{ij} \chi + (g^{ij} D^2 - D^i D^j) \chi] \\ \times \mathcal{G}_{ijkl} [R^{kl} \varphi + (g^{kl} D^2 - D^k D^l) \varphi]. \end{aligned} \quad (58)$$

Hereafter we consider $(\pi_i, \pi_A, \pi_\nu, \mathcal{H}_i^N, \pi_N, J_A, \mathcal{H}_\perp, \phi_A)$ as the basis for the set of constraints. Adding the terms corresponding to the secondary constraints, we thus redefine the Hamiltonian as

$$\begin{aligned} \tilde{H} &= H + \int d^d x (\lambda^i \pi_i + \lambda_A \pi_A + \lambda_\nu \pi_\nu \\ &\quad + \lambda_{\mathcal{H}}^i \mathcal{H}_i^N + \lambda_N \pi_N + \lambda_J J_A + \lambda_\perp \mathcal{H}_\perp + \lambda_\phi \phi_A), \end{aligned} \quad (59)$$

where $\lambda^i, \lambda_A, \lambda_\nu, \lambda_{\mathcal{H}}^i, \lambda_N, \lambda_J, \lambda_\perp, \lambda_\phi$ are Lagrange multipliers.

The consistency conditions of $\pi_N, J_A, \mathcal{H}_\perp$ and ϕ_A with the time evolution yield no additional constraint but equations to determine the Lagrange multipliers $\lambda_\phi, \lambda_\perp, \lambda_J$ and λ_N if the determinant of the following matrix does not vanish weakly [17]:

$$\det M = (\det B)(\det C) = (\det B)(\det B \text{ with } \chi \leftrightarrow \varphi). \quad (63)$$

Equations (55)–(58) show that each component of the 2×2 matrix B is weakly nonvanishing and that only Eq. (55) among them depends on the potential term $\mathcal{L}_V[g_{ij}, a_i]$, which is a linear combination of $a^i a_i, R^{ij} a_i a_j, D^2 a^i D^2 a_i$ and so forth. This means that the component (55) linearly depends on the coefficients of those potential terms but that other components are independent of them. Moreover, $\det B \not\approx 0$ when all coefficients in $\mathcal{L}_V[g_{ij}, a_i]$ are set to zero. Inspecting Eqs. (55)–(58), we thus conclude that

det $B \neq 0$, for any choices of the coupling constants in $\mathcal{L}_V$. Hence Eq. (63) leads to $\det M \neq 0$ and the consistency conditions of π_N , J_A , $\mathcal{H}_\perp$ and ϕ_A do not yield any additional constraints. The consistency conditions of π_A , π_i , π_ν and $\mathcal{H}_i^N$ with the time evolution are

$$\begin{aligned} 0 &= \frac{d}{dt} \pi_A \approx \{\pi_A, H\}_P = -\frac{\delta H}{\delta A} = J_A, \\ 0 &= \frac{d}{dt} \pi_i \approx \{\pi_i, H\}_P = -\frac{\delta H}{\delta N^i} = -\mathcal{H}_i + J_A D_i \nu, \\ 0 &= \frac{d}{dt} \pi_\nu \approx \{\pi_\nu, H\}_P = -\frac{\delta H}{\delta \nu} = -\partial_i (N \mathcal{H}^i + \tilde{N}^i J_A), \\ 0 &= \frac{d}{dt} \overline{\mathcal{H}^N}[f] \approx \{\overline{\mathcal{H}^N}[f], H\}_P \\ &= \int d^d x [- (N \mathcal{H}^i + \tilde{N}^i J_A) \partial_i (f \partial \nu) - J_A f \partial A \\ &\quad + (\mathcal{H}_i - J_A \partial_i \nu) [f, N]^i]. \end{aligned} \quad (64)$$

They do not give any additional constraints since the right-hand sides are weakly vanishing.

Therefore, $(\pi_i, \pi_A, \pi_\nu, \mathcal{H}_i^N, \pi_N, J_A, \mathcal{H}_\perp, \phi_A)$ forms the set of all constraints for the special case with $\eta_1 = \eta_2 = 0$. Among them, $(2d+2)$ constraints $(\pi_i, \mathcal{H}_i^N, \pi_A, \pi_\nu)$ are first class and 4 constraints $(\pi_N, J_A, \mathcal{H}_\perp, \phi_A)$ are second class.

IV. NUMBER OF DEGREES OF FREEDOM

A. Generic case

Our analysis in Sec. III on the nonprojectable Hořava-Lifshitz theory with the $U(1)$ symmetry shows that we have $C_1 = 2d+2$ first-class constraints and $C_2 = 2$ second-class constraints in the generic case (Sec. III A) and $C_2 = 4$ in the special case (Sec. III B) in the phase-space dimension $\dim P = d^2 + 3d + 6$. Hence the number of degrees of freedom, $\mathcal{N}$, is

$$\mathcal{N} = \frac{1}{2} (\dim P - 2C_1 - C_2) = \frac{1}{2} (d^2 - d + 2 - C_2). \quad (65)$$

In the special case with $C_2 = 4$, $\mathcal{N} = (d-2)(d+1)/2$ counts the same as the number of transverse traceless polarizations of a massless graviton in $(d+1)$ -dimensional spacetime, which shows the absence of the scalar graviton, the mode present in the original version of Hořava-Lifshitz gravity. On the other hand, in the generic nonprojectable case with $C_2 = 2$, there exists an additional degree of freedom in the theory, corresponding to a scalar graviton. One thus observes that despite an additional $U(1)$ symmetry, the scalar graviton generically remains present due to the lack of two second-class constraints, and the number of the propagating degrees of freedom is the same as that in the original Hořava-Lifshitz theory.

Since the first-class constraints $\pi_i \approx \mathcal{H}_i^N \approx \pi_A \approx \pi_\nu \approx 0$ are related to the generators of the gauge transformations under which the theory is invariant, the motion of the canonical variables is not completely determined by the Hamiltonian $\tilde{H}$ due to the unfixed multipliers $\lambda^i, \lambda_{\mathcal{H}}^i, \lambda_A$ and λ_ν . In order to remove this subtlety, one can in general fix the gauge by imposing additional constraints

$$\mathcal{G}^i \approx 0, \mathcal{F}^i \approx 0, \mathcal{G}_A \approx 0, \mathcal{G}_\nu \approx 0, \quad (i = 1, 2, \dots, d), \quad (66)$$

such that the determinant

$$\det \begin{pmatrix} \frac{\delta \mathcal{G}^i(\vec{x})}{\delta N^j(\vec{y})} & \{\mathcal{G}^i(\vec{x}), \mathcal{H}_j^N(\vec{y})\}_P & \frac{\delta \mathcal{G}^i(\vec{x})}{\delta A(\vec{y})} & \frac{\delta \mathcal{G}^i(\vec{x})}{\delta \nu(\vec{y})} \\ \frac{\delta \mathcal{F}^i(\vec{x})}{\delta N^j(\vec{y})} & \{\mathcal{F}^i(\vec{x}), \mathcal{H}_j^N(\vec{y})\}_P & \frac{\delta \mathcal{F}^i(\vec{x})}{\delta A(\vec{y})} & \frac{\delta \mathcal{F}^i(\vec{x})}{\delta \nu(\vec{y})} \\ \frac{\delta \mathcal{G}_A(\vec{x})}{\delta N^j(\vec{y})} & \{\mathcal{G}_A(\vec{x}), \mathcal{H}_j^N(\vec{y})\}_P & \frac{\delta \mathcal{G}_A(\vec{x})}{\delta A(\vec{y})} & \frac{\delta \mathcal{G}_A(\vec{x})}{\delta \nu(\vec{y})} \\ \frac{\delta \mathcal{G}_\nu(\vec{x})}{\delta N^j(\vec{y})} & \{\mathcal{G}_\nu(\vec{x}), \mathcal{H}_j^N(\vec{y})\}_P & \frac{\delta \mathcal{G}_\nu(\vec{x})}{\delta A(\vec{y})} & \frac{\delta \mathcal{G}_\nu(\vec{x})}{\delta \nu(\vec{y})} \end{pmatrix} \quad (67)$$

does not vanish weakly. For the generic case $(\eta_1, \eta_2) \neq (0, 0)$, we define the total Hamiltonian by including these gauge-fixing conditions as

$$\begin{aligned} H_{\text{tot}} &= \int d^d x [C + N^i (\mathcal{H}_i - J_A D_i \nu) - A J_A + \lambda^i \pi_i + \lambda_A \pi_A \\ &\quad + \lambda_\nu \pi_\nu + \lambda_{\mathcal{H}}^i \mathcal{H}_i^N + \lambda_N \pi_N + \lambda_J J_A + n_i \mathcal{G}^i \\ &\quad + \lambda_i^{\mathcal{F}} \mathcal{F}^i + n_A \mathcal{G}_A + n_\nu \mathcal{G}_\nu] \end{aligned} \quad (68)$$

where $(n_i, \lambda_i^{\mathcal{F}}, n_A, n_\nu)$ are Lagrange multipliers, and we have defined

$$\begin{aligned} C &\equiv N \left[\sqrt{g} \left(\frac{2}{M_{\text{Pl}}^2} \frac{\pi^{ij}}{\sqrt{g}} \mathcal{G}_{ijkl} \frac{\pi^{kl}}{\sqrt{g}} + \frac{M_{\text{Pl}}^2}{2} \mathcal{L}_V \right) \right. \\ &\quad \left. - \mathcal{H}_i D^i \nu + \frac{1}{2} J_A D_i \nu D^i \nu \right]. \end{aligned} \quad (69)$$

The set of all Lagrange multipliers $(\lambda^i, \lambda_A, \lambda_\nu, \lambda_{\mathcal{H}}^i, n_i, \lambda_{\mathcal{F}^i}, n_A, n_\nu, \lambda_N, \lambda_J)$ would be fully determined by imposing the consistency conditions on the constraints, which are now all second class, with H_{tot} instead of $\tilde{H}$.

As an explicit example of the gauge fixing, let us consider the following gauge:

$$\begin{aligned} \mathcal{G}^i &= N^i, & \mathcal{F}^i &= \mathcal{F}^i[g_{ij}, N, \nu, \pi^{kl}, \pi_N, \pi_\nu], \\ \mathcal{G}_A &= A, & \mathcal{G}_\nu &= \mathcal{G}_\nu[g_{ij}, N, \nu, \pi^{kl}, \pi_N, \pi_\nu], \end{aligned} \quad (70)$$

for which the constraints $\mathcal{F}^i$ and $\mathcal{G}_\nu$ satisfy

$$\{\mathcal{F}^i(\vec{x}), \mathcal{H}_j^N(\vec{y})\}_P \neq 0, \quad \frac{\delta \mathcal{G}_\nu(\vec{x})}{\delta \nu(\vec{y})} \neq 0. \quad (71)$$

In this case, the consistency conditions

$$\frac{d}{dt}\mathcal{G}^i \approx 0, \quad \frac{d}{dt}\pi_i \approx 0, \quad \frac{d}{dt}\mathcal{G}_A \approx 0, \quad \frac{d}{dt}\pi_A \approx 0 \quad (72)$$

tell us that the $2d+2$ multipliers $(\lambda^i, \lambda_A, n_i, n_A)$ are determined as

$$\lambda^i = 0, \quad n_i = -\mathcal{H}_i + J_A D_i \nu, \quad \lambda_A = 0, \quad n_A = J_A. \quad (73)$$

Then the total Hamiltonian in this gauge is

$$H_{\text{tot}} = \int d^d x [C + \lambda_\nu \pi_\nu + \lambda_{\mathcal{H}}^i \mathcal{H}_i^N + \lambda_i^{\mathcal{F}} \mathcal{F}^i + n_\nu \mathcal{G}_\nu + \lambda_N \pi_N + \lambda_J J_A]. \quad (74)$$

From the absence of (N^i, A, π_i, π_A) in this gauge-fixed total Hamiltonian, we can see that we have excluded the canonical pairs (N^i, π_i) and (A, π_A) from the phase space in this gauge. The dimension of the reduced phase space $(g_{ij}, N, \nu, \pi^{ij}, \pi_N, \pi_\nu)$ is $d^2 + d + 4$. As usual with second-class constraints, all the remaining Lagrange multipliers $(\lambda_\nu, \lambda_{\mathcal{H}}^i, \lambda_i^{\mathcal{F}}, n_\nu, \lambda_N, \lambda_J)$ are fully determined by imposing

$$\begin{aligned} \{\pi_\nu(\vec{x}), H_{\text{tot}}\}_P &\approx 0, & \{\mathcal{G}_\nu(\vec{x}), H_{\text{tot}}\}_P &\approx 0, \\ \{\mathcal{H}_i^N(\vec{x}), H_{\text{tot}}\}_P &\approx 0, & \{\mathcal{F}^i(\vec{x}), H_{\text{tot}}\}_P &\approx 0, \\ \{\pi_N(\vec{x}), H_{\text{tot}}\}_P &\approx 0, & \{J_A(\vec{x}), H_{\text{tot}}\}_P &\approx 0. \end{aligned} \quad (75)$$

There remains the following set of $2d+4$ second-class constraints acting on the $(d^2 + d + 4)$ -dimensional reduced phase space:

$$\begin{aligned} \pi_\nu \approx 0, \quad \mathcal{G}_\nu \approx 0, \quad \mathcal{H}_i^N \approx 0, \quad \mathcal{F}^i \approx 0, \\ \pi_N \approx 0, \quad J_A \approx 0, \quad (i = 1, 2, \dots, d). \end{aligned} \quad (76)$$

Hence, in the generic case $(\eta_1, \eta_2) \neq (0, 0)$, we end up with a $(d^2 - d)$ -dimensional physical phase space, and the number of propagating degrees of freedom is $d(d-1)/2$, which shows that there exists a scalar graviton in the system.

B. Special case with $\eta_1 = \eta_2 = 0$

The constraint structure and the number of degrees of freedom differ in a nontrivial way in the special case described in Sec. III B, i.e. the case with $(\eta_1, \eta_2) = (0, 0)$. Since J_A is now independent of N , the time consistency conditions of π_N and J_A introduce two additional secondary constraints $\mathcal{H}_\perp$ and ϕ_A , shown in Eq. (48), which are absent in the generic case. The total Hamiltonian contains these new constraint terms, and formally, we simply need to add two terms

$$\int d^d x (\lambda_\perp \mathcal{H}_\perp + \lambda_\phi \phi_A) \quad (77)$$

to Eq. (68). We can fix the gauge and eliminate the canonical pairs (N^i, π_i) and (A, π_A) from the phase space

by the same procedure as in the generic case. In this special case, however, we have two additional second-class constraints, $\mathcal{H}_\perp \approx 0$ and $\phi_A \approx 0$. They reduce the dimension of the phase space by two, and the total number of propagating degrees of freedom becomes $(d^2 - d - 2)/2$, the same as the number of transverse traceless polarizations of a graviton. Hence this shows that the scalar graviton disappears in this special case, at least classically.

An apparent absence of the scalar graviton may arise in the perturbation theory even in the generic case. For example, if one sets $\eta_2 = 0$ in Eq. (25) and considers the vanishing background of σ and the background of N independent of the spatial coordinates, then the constraint structure appears to be the same as the special case up to the level of linear perturbations. This apparent behavior at the linear order is, however, only an artifact of the perturbative expansion around a specific background and does not mean that the system is free from a scalar graviton. In the situations of this sort, care must be taken, and one needs to consider the nonlinear orders of perturbations, at which the scalar graviton becomes dynamical.

C. Projectable theory

For completeness, we briefly discuss the projectable case, in which the lapse N is a function only of time t . This restriction implies that the lapse is no longer a local degree of freedom, and thus we can consistently take it out of the phase space of the theory. As summarized in Appendix B, the phase space consists of $(g_{ij}, N^i, A, \nu, \pi^{ij}, \pi_i, \pi_A, \pi_\nu)$ with $2d+2$ first-class and 2 second-class constraints, leading to $(d-2)(d+1)/2$ dynamical degrees of freedom. This equals the number of traceless transverse polarizations of a graviton, exhibiting the absence of the scalar graviton in the projectable case, in sharp contrast to the generic nonprojectable version of the theory (see Sec. IV A).

The total action, including the gauge-fixing conditions, of the projectable version of the $U(1)$ Hořava-Lifshitz theory is, from Eqs. (32) and (B3),

$$\begin{aligned} H_{\text{tot}}^{\text{proj}} = \int d^d x [&\mathcal{C}^{\text{proj}} + N^i (\mathcal{H}_i - J_A D_i \nu) - A J_A + \lambda^i \pi_i \\ &+ \lambda_A \pi_A + \lambda_\nu \pi_\nu + \lambda_{\mathcal{H}}^i \mathcal{H}_i + \lambda_J J_A + \lambda_\phi \phi_A \\ &+ n_i \mathcal{G}^i + \lambda_i^{\mathcal{F}} \mathcal{F}^i + n_A \mathcal{G}_A + n_\nu \mathcal{G}_\nu], \end{aligned} \quad (78)$$

where $\mathcal{G}^i, \mathcal{F}^i, \mathcal{G}_A$ and $\mathcal{G}_\nu$ are the gauge-fixing terms as in Eq. (66), and $\mathcal{C}^{\text{proj}}$ is the projectable counterpart of $\mathcal{C}$, defined in Eq. (69). Note that the potential term $\mathcal{L}_V$ and the constraint J_A now depend only on g_{ij} , since $\partial_i \ln N = 0$ and all of their dependence on N vanishes in the projectable case.

Performing the gauge fixing in the same manner as in the previous subsection, we choose

$$\begin{aligned} \mathcal{G}^i &= N^i, & \mathcal{F}^i &= \mathcal{F}^i[g_{ij}, \nu, \pi^{kl}, \pi_\nu], \\ \mathcal{G}_A &= A, & \mathcal{G}_\nu &= \mathcal{G}_\nu[g_{ij}, \nu, \pi^{kl}, \pi_\nu], \end{aligned} \quad (79)$$

imposing the conditions

$$\{\mathcal{F}^i(\vec{x}), \mathcal{H}_j(\vec{y})\} \not\approx 0, \quad \frac{\delta \mathcal{G}_\nu(\vec{x})}{\delta \nu(\vec{y})} \not\approx 0, \quad (80)$$

similar to Eq. (71). Then the time consistency conditions for $\mathcal{G}^i$, π_i , $\mathcal{G}_A$ and π_A determine the multipliers $(\lambda^i, \lambda_A, n_i, n_A)$ in exactly the same way as in Eq. (73). After this gauge fixing, we can completely eliminate the gauge modes N^i , A , π_i and π_A from the phase space, resulting in the Hamiltonian

$$H_{\text{tot}}^{\text{proj}} = \int d^d x [\mathcal{C}^{\text{proj}} + \lambda_\nu \pi_\nu + \lambda_{\mathcal{H}}^i \mathcal{H}_i + \lambda_J J_A + \lambda_\phi \phi_A + \lambda_i^{\mathcal{F}} \mathcal{F}^i + n_\nu \mathcal{G}_\nu]. \quad (81)$$

The evolution of the system with the reduced phase space $(g_{ij}, \nu, \pi^{ij}, \pi^\nu)$ is governed by this Hamiltonian. Due to the conditions in Eq. (80), all of the constraints $(\pi_\nu, \mathcal{H}_i, J_A, \phi_A, \mathcal{F}^i, \mathcal{G}_\nu) \approx 0$ are now second class. Imposing them then reduces the phase-space dimension from $d(d+1)+2$ to $(d-2)(d+1)$, or equivalently $(d-2)(d+1)/2$ degrees of freedom, as desired. Similarly to Eq. (75), the Lagrange multipliers are determined by the time consistency conditions for the constraints.

The Hamiltonian structure of the projectable case is closely similar to the special version of the nonprojectable case, described in Sec. IV B. The only differences at the classical level are that the former excludes the lapse N from the local degrees of freedom (and thus the corresponding second-class constraints are absent) and that $\mathcal{L}_V$ and J_A in the former have no dependence on N . However, quantum effects are expected to detune such a special choice of parameters as in the special nonprojectable case, and fine-tunings would be required to keep those quantum corrections from arising. Thus even if one starts from the classical action without a scalar graviton in the nonprojectable theory, it would generically reappear at the quantum level. On the other hand, the projectable case is constructed in such a way that N is constant at any constant-time hypersurface, and the absence of a scalar graviton is consistently guaranteed at the full order. The number of propagating degrees of freedom in the gravity sector is thus the same as that of a massless spin-2 graviton in GR.

V. DISCUSSION

In this paper we have applied the standard method of Hamiltonian analysis in the classical field theory to the $U(1)$ extension of Hořava-Lifshitz gravity without the projectability condition. We have studied the nature of constraints and counted the number of physical degrees of freedom. In particular, we have shown that the theory contains the scalar graviton unless the coefficients of $a^i a_i \sigma$ and $D^i a_i \sigma$, which we denoted as η_1 and η_2 , are set to zero exactly.

It is known that the scalar graviton is absent in linear perturbations around a flat background even if $\eta_1 \neq 0$, as far as $\eta_2 = 0$. Actually, on a flat background the term $a^i a_i \sigma$ is of the cubic order in perturbation and thus does not contribute to the equations of motion for linear perturbations. This is the reason why the linear perturbation analysis is insensitive to the value of η_1 . On the other hand, the absence of the scalar graviton at the fully nonlinear level requires not only $\eta_2 = 0$ but also $\eta_1 = 0$.

The reason why η_1 and η_2 determine the presence/absence of the scalar graviton is understood as follows. The Poisson bracket between two constraints, that we denoted as π_N and J_A , vanishes identically if and only if $\eta_1 = \eta_2 = 0$. Thus, only in this special case, the consistency of the two constraints with the time evolution does not determine Lagrange multipliers but instead yields two secondary constraints. These secondary constraints are responsible for the elimination of the canonical pair corresponding to the scalar graviton.

We have identified the condition under which the scalar graviton is absent, i.e. $\eta_1 = \eta_2 = 0$. However, the operators corresponding to the two coupling constants are $a^i a_i \sigma$ and $D^i a_i \sigma$, and both of them are marginal for any values of the dynamical critical exponent z . Therefore, even if η_1 and η_2 are set to zero by hand, they should be generated by quantum corrections. In this sense the condition $\eta_1 = \eta_2 = 0$ is unstable under radiative corrections. We thus conclude that the scalar graviton is generically present in the theory.

Contrary to the generic nonprojectable theory, the projectable version of the $U(1)$ extension of Hořava-Lifshitz gravity does not contain the scalar graviton (see Sec. IV C). An important point is that, unlike the above-mentioned condition $\eta_1 = \eta_2 = 0$ in the nonprojectable theory, the projectability condition provides a consistent truncation and thus is expected to be stable under radiative corrections. Hence, the number of physical degrees of freedom in the projectable theory is the same as in GR.¹ Nonetheless, physical properties of the propagating degrees of freedom, e.g. the dispersion relation, are different from those in GR. It would certainly be of theoretical interest to determine whether there are any other such theories.

ACKNOWLEDGMENTS

The authors thank C. M. Melby-Thompson and T. Sotiriou for useful discussions. S. M. was supported in part by Grant-in-Aid for Scientific Research 24540256, and Y. W. was supported by the Program for Leading Graduate Schools, MEXT, Japan. The authors acknowledge support by the WPI Initiative, MEXT Japan. Part of this work has been done within the Labex ILP (reference ANR-10-LABX-63) part of the Idex SUPER, and received financial state aid managed by the Agence Nationale de la

¹Lovelock's theorem [18] is evaded because of the presence of auxiliary fields and nontrivial constraints.

Recherche, as part of the programme Investissements d'avenir under the reference ANR-11-IDEX-0004-02. The authors are thankful to Institut Astrophysique de Paris for warm hospitality during their stay.

APPENDIX A: CALCULATION OF $\{\tilde{\mathcal{H}}[f], F\}_{\text{P}}$

In this appendix we calculate the Poisson bracket $\{\tilde{\mathcal{H}}[f], F\}_{\text{P}}$, where $F = F[g_{ij}, \pi^{ij}, s, V^i]$ is a functional of the spatial metric g_{ij} , its canonical momenta π^{ij} , a scalar s and a vector V^i on a constant- t surface and we assume that F is invariant under time-independent spatial diffeomorphism. We do *not* require the invariance of F under a time-dependent spatial diffeomorphism *nor* assume any properties of s and V^i under the time-dependent spatial diffeomorphism.

We consider the time-independent spatial diffeomorphism,

$$\vec{x} \rightarrow \vec{x} + \vec{\xi}(\vec{x}), \quad (\text{A1})$$

where $\vec{\xi}(\vec{x})$ is a time-independent d -dimensional spatial vector. Under this infinitesimal transformation, the spatial metric g_{ij} , its canonical momenta π^{ij} , a scalar s and a vector V^i transform as

$$\begin{aligned} g_{ij} &\rightarrow g_{ij} + D_i \xi_j + D_j \xi_i, \\ \frac{\pi^{ij}}{\sqrt{g}} &\rightarrow \frac{\pi^{ij}}{\sqrt{g}} + \xi^k D_k \left(\frac{\pi^{ij}}{\sqrt{g}} \right) - \frac{\pi^{ik}}{\sqrt{g}} D_k \xi^j - \frac{\pi^{kj}}{\sqrt{g}} D_k \xi^i, \\ s &\rightarrow s + \xi^k \partial_k s, \\ V^i &\rightarrow V^i + \xi^k D_k V^i - V^k D_k \xi^i, \end{aligned} \quad (\text{A2})$$

where D_k is the spatial covariant derivative compatible with g_{ij} and we have used the fact that $\pi^{ij}/\sqrt{g}$ is a tensor. The variation of a functional F of these variables is thus calculated as

$$\begin{aligned} \delta F &= \int d^d x \left[\left(\frac{\delta F}{\delta g_{ij}} \right)_{\pi/\sqrt{g}} \delta g_{ij} + \frac{\delta F}{\delta(\pi^{ij}/\sqrt{g})} \delta \left(\frac{\pi^{ij}}{\sqrt{g}} \right) + \frac{\delta F}{\delta s} \delta s + \frac{\delta F}{\delta V^i} \delta V^i \right] \\ &= \int d^d x \xi^i \left\{ -2g_{ik} \sqrt{g} D_j \left[\frac{1}{\sqrt{g}} \left(\frac{\delta F}{\delta g_{kj}} \right)_{\pi/\sqrt{g}} \right] + \frac{\delta F}{\delta(\pi^{jk}/\sqrt{g})} D_i \left(\frac{\pi^{jk}}{\sqrt{g}} \right) \right. \\ &\quad \left. + 2\sqrt{g} D_j \left(\frac{1}{\sqrt{g}} \frac{\delta F}{\delta(\pi^{il}/\sqrt{g})} \frac{\pi^{il}}{\sqrt{g}} \right) + \frac{\delta F}{\delta s} D_i s + \frac{\delta F}{\delta V^j} D_i V^j + \sqrt{g} D_j \left(\frac{1}{\sqrt{g}} \frac{\delta F}{\delta V^i} V^j \right) \right\}, \end{aligned} \quad (\text{A3})$$

where the subscript $\pi/\sqrt{g}$ in $(\delta F/\delta g_{ij})_{\pi/\sqrt{g}}$ indicates that the functional derivative is taken with $\pi^{kl}/\sqrt{g}$ (instead of π^{kl}) fixed. Therefore, the diffeomorphism invariance of F , i.e. $\delta F = 0$ for $\forall \xi^i$, implies that

$$\begin{aligned} 2D_j \left[\frac{1}{\sqrt{g}} \left(\frac{\delta F}{\delta g_{ij}} \right)_{\pi/\sqrt{g}} \right] &= \frac{1}{\sqrt{g}} \frac{\delta F}{\delta(\pi^{jk}/\sqrt{g})} D^i \left(\frac{\pi^{jk}}{\sqrt{g}} \right) + 2g^{ik} D_j \left(\frac{1}{\sqrt{g}} \frac{\delta F}{\delta(\pi^{kl}/\sqrt{g})} \frac{\pi^{kl}}{\sqrt{g}} \right) \\ &\quad + \frac{1}{\sqrt{g}} \frac{\delta F}{\delta s} D^i s + \frac{1}{\sqrt{g}} \frac{\delta F}{\delta V^j} D^i V^j + g^{ik} D_j \left(\frac{1}{\sqrt{g}} \frac{\delta F}{\delta V^k} V^j \right). \end{aligned} \quad (\text{A4})$$

For practical purposes, it is convenient to express $(\delta F/\delta g_{ij})_{\pi/\sqrt{g}}$ in terms of $(\delta F/\delta g_{ij})_{\pi}$, where the subscript π indicates that the functional derivative is taken with π^{kl} (instead of $\pi^{kl}/\sqrt{g}$) fixed. By writing down the variation δF in two different ways as

$$\left(\frac{\delta F}{\delta g_{ij}} \right)_{\pi} \delta g_{ij} + \frac{\delta F}{\delta \pi^{ij}} \delta \pi^{ij} + \frac{\delta F}{\delta s} \delta s + \frac{\delta F}{\delta V^i} \delta V^i = \left(\frac{\delta F}{\delta g_{ij}} \right)_{\pi/\sqrt{g}} \delta g_{ij} + \frac{\delta F}{\delta(\pi^{ij}/\sqrt{g})} \delta \left(\frac{\pi^{ij}}{\sqrt{g}} \right) + \frac{\delta F}{\delta s} \delta s + \frac{\delta F}{\delta V^i} \delta V^i, \quad (\text{A5})$$

and equating the coefficients of δg_{ij} and $\delta \pi^{ij}$, we obtain

$$\left(\frac{\delta F}{\delta g_{ij}} \right)_{\pi/\sqrt{g}} = \left(\frac{\delta F}{\delta g_{ij}} \right)_{\pi} + \frac{1}{2} \frac{\delta F}{\delta \pi^{kl}} \pi^{kl} g^{ij}, \quad \frac{\delta F}{\delta(\pi^{ij}/\sqrt{g})} = \sqrt{g} \frac{\delta F}{\delta \pi^{ij}}. \quad (\text{A6})$$

By substituting Eq. (A6) into Eq. (A4), we thus obtain

$$\begin{aligned}
2D_j \left[\frac{1}{\sqrt{g}} \left(\frac{\delta F}{\delta g_{ij}} \right)_\pi \right] &= -D^i \left(\frac{\delta F}{\delta \pi^{kl}} \frac{\pi^{kl}}{\sqrt{g}} \right) + \frac{\delta F}{\delta \pi^{jk}} D^i \left(\frac{\pi^{jk}}{\sqrt{g}} \right) \\
&+ 2g^{ik} D_j \left(\frac{\delta F}{\delta \pi^{kl}} \frac{\pi^{kl}}{\sqrt{g}} \right) + \frac{1}{\sqrt{g}} \frac{\delta F}{\delta s} D^i s \\
&+ \frac{1}{\sqrt{g}} \frac{\delta F}{\delta V^j} D^i V^j + g^{ik} D_j \left(\frac{1}{\sqrt{g}} \frac{\delta F}{\delta V^k} V^j \right).
\end{aligned} \tag{A7}$$

This formula will be used below.

For the calculation of $\{\tilde{\mathcal{H}}[f], F\}_P$, we first need to calculate the functional derivatives of $\mathcal{H}[f]$. A straightforward calculation leads to

$$\begin{aligned}
\frac{1}{\sqrt{g}} \frac{\delta \tilde{\mathcal{H}}[f]}{\delta g_{ij}} &= \frac{\pi^{ik}}{\sqrt{g}} D_k f^j + \frac{\pi^{kj}}{\sqrt{g}} D_k f^i - D_k \left(\frac{\pi^{ij}}{\sqrt{g}} f^k \right), \\
\frac{\delta \tilde{\mathcal{H}}[f]}{\delta \pi^{ij}} &= D_i f_j + D_j f_i.
\end{aligned} \tag{A8}$$

Hence, we obtain

$$\begin{aligned}
\{\tilde{\mathcal{H}}[f], F\}_P &= \int d^d x \left[\frac{\delta \tilde{\mathcal{H}}[f]}{\delta g_{ij}} \frac{\delta F}{\delta \pi^{ij}} - \frac{\delta \tilde{\mathcal{H}}[f]}{\delta \pi^{ij}} \left(\frac{\delta F}{\delta g_{ij}} \right)_\pi \right] \\
&= \int d^d x \sqrt{g} \left\{ \left[\frac{\pi^{ik}}{\sqrt{g}} D_k f^j + \frac{\pi^{kj}}{\sqrt{g}} D_k f^i - D_k \left(\frac{\pi^{ij}}{\sqrt{g}} f^k \right) \right] \frac{\delta F}{\delta \pi^{ij}} \right. \\
&\quad \left. + 2f_i D_j \left[\frac{1}{\sqrt{g}} \left(\frac{\delta F}{\delta g_{ij}} \right)_\pi \right] \right\}.
\end{aligned} \tag{A9}$$

By using Eq. (A7), we thus obtain

$$\{\tilde{\mathcal{H}}[f], F\}_P = \int d^d x \left(\frac{\delta F}{\delta s} f^i \partial_i s + \frac{\delta F}{\delta V^i} [f, V]^i \right). \tag{A10}$$

Here, f^i is assumed to be independent of canonical variables. If f^i depends on canonical variables then we instead obtain

$$\begin{aligned}
\{\tilde{\mathcal{H}}[f], F\}_P &= \tilde{\mathcal{H}}[\{f, F\}_P] + \int d^d x \left(\frac{\delta F}{\delta s} f^i \partial_i s + \frac{\delta F}{\delta V^i} [f, V]^i \right) \\
&\approx \int d^d x \left(\frac{\delta F}{\delta s} f^i \partial_i s + \frac{\delta F}{\delta V^i} [f, V]^i \right).
\end{aligned} \tag{A11}$$

APPENDIX B: HAMILTONIAN ANALYSIS OF THE PROJECTABLE THEORY

In order to compare the analysis in the present paper with that under the projectable condition $N = N(t)$, i.e. $a_i = 0$, performed in Ref. [13], in this appendix we sketch the analysis for the projectable theory using our notation. In the projectable case, the lapse N is not a local degree of

freedom and the Hamiltonian constraint, i.e. the equation of motion for $N(t)$, is not a local equation but an equation integrated over the whole space. Since the whole space here includes not only our patch of the universe but also many other patches, the Hamiltonian constraint of this form does not restrict the behavior of local degrees of freedom [5]. We thus do not have to impose the Hamiltonian constraint to count the number of degrees of freedom. In the following, we thus simply consider $N(t)$ as a fixed positive function of t and we do not vary it. The phase space $(g_{ij}, \pi^{ij}, N^i, \pi_i, A, \pi_A, \nu, \pi_\nu)$ is $(d^2 + 3d + 4)$ dimensional.

The primary constraints are

$$\pi_i = 0, \quad \pi_A = 0, \quad \pi_\nu + J_A = 0, \tag{B1}$$

whose Poisson brackets all vanish. The consistency conditions of the primary constraints are

$$\begin{aligned}
0 &= \frac{d}{dt} \pi_A \approx \{\pi_A, H\}_P = -\frac{\delta H}{\delta A} = J_A, \\
0 &= \frac{d}{dt} \pi_i \approx \{\pi_i, H\}_P = -\frac{\delta H}{\delta N^i} = -\mathcal{H}_i + J_A D_i \nu, \\
0 &= \frac{d}{dt} (\pi_\nu + J_A) \approx -\frac{\delta H}{\delta \nu} + \{J_A, H\}_P \\
&= -\partial_i (N \mathcal{H}^i + \tilde{N}^i J_A) + \phi_A + \frac{2}{M_{\text{Pl}}^2 \sqrt{g}} g^{ij} g_{ijkl} \pi^{kl} N J_A \\
&\quad + \sqrt{g} D_i (\tilde{N}^i J_A / \sqrt{g}).
\end{aligned} \tag{B2}$$

So far the constraints are $(\pi_i, \pi_A, \pi_\nu, \mathcal{H}_i, J_A, \phi_A)$. Only the Poisson bracket between J_A and ϕ_A does not vanish weakly [Eq. (58)] among them. We redefine the Hamiltonian as

$$\begin{aligned}
\tilde{H} &= H + \int d^d x (\lambda^i \pi_i + \lambda_A \pi_A + \lambda_\nu \pi_\nu + \lambda_{\mathcal{H}}^i \mathcal{H}_i \\
&\quad + \lambda_J J_A + \lambda_\phi \phi_A),
\end{aligned} \tag{B3}$$

where $\lambda^i, \lambda_A, \lambda_\nu, \lambda_{\mathcal{H}}^i, \lambda_J, \lambda_\phi$ are Lagrange multipliers.

The consistency conditions of J_A and ϕ_A with the time evolution do not yield additional constraints but simply result in equations to determine the Lagrange multipliers λ_J and λ_ϕ . The consistency conditions of π_i, π_A, π_ν and $\mathcal{H}_i$ with the time evolution are

$$\begin{aligned}
0 &= \frac{d}{dt} \pi_A \approx \{\pi_A, H\}_P = -\frac{\delta H}{\delta A} = J_A, \\
0 &= \frac{d}{dt} \pi_i \approx \{\pi_i, H\}_P = -\frac{\delta H}{\delta N^i} = -\mathcal{H}_i + J_A D_i \nu, \\
0 &= \frac{d}{dt} \pi_\nu \approx \{\pi_\nu, H\}_P = -\frac{\delta H}{\delta \nu} = -\partial_i (N \mathcal{H}^i + \tilde{N}^i J_A), \\
0 &= \frac{d}{dt} \tilde{\mathcal{H}}[f] \approx \{\tilde{\mathcal{H}}[f], H\}_P \\
&= \int d^d x [-(N \mathcal{H}^i + \tilde{N}^i J_A) \partial_i (f \partial \nu) \\
&\quad - J_A f \partial A + (\mathcal{H}_i - J_A \partial_i \nu) [f, N]^i].
\end{aligned} \tag{B4}$$

They weakly vanish and thus do not give any additional constraints. Therefore $(\pi_i, \mathcal{H}_i, \pi_A, \pi_\nu)$ are $(2d+2)$ first-class constraints and J_A and ϕ_A are two second-class constraints. The number degrees of freedom $\mathcal{N}$ is

$$\mathcal{N} = \frac{1}{2} [(d^2 + 3d + 4) - 2(2d + 2) - 2] = \frac{1}{2} (d^2 - d - 2), \quad (\text{B5})$$

which implies the absence of the scalar graviton.

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