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Misner-Sharp mass in n-dimensional $f(R)$ gravity

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We study the Misner-Sharp mass for the $f(R)$ gravity in an n-dimensional ($n \geq 3$) spacetime which permits three-type $(n-2)$-dimensional maximally symmetric subspace. We obtain the Misner-Sharp mass via two approaches. One is the inverse unified first law method and the other is the conserved charge method, which uses a generalized Kodama vector. In the first approach, we assume the unified first law still holds in n-dimensional $f(R)$ gravity, which requires a quasilocal mass form (we define it as the generalized Misner-Sharp mass). In the second approach, the conserved charge corresponding to the generalized local Kodama vector is the generalized Misner-Sharp mass. The two approaches, which are bridged by a constraint, are equivalent. This constraint determines the existence of a well-defined Misner-Sharp mass. In an important special case, we present the explicit form for the static space, and we calculate the Misner-Sharp mass for the Clifton-Barrow solution as an example.

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I. INTRODUCTION

In contrast to intuition, the mass (energy) of a gravity field is an extraordinarily involved problem. The existence of a tensor form of gravitational stress energy is forbidden by the equivalence principle. At the least, we can define the total mass at spacelike infinity (Arnowitt-Deser-Misner mass) and null infinity (Bondi-Sachs mass) for asymptotic flat manifolds. This situation cannot satisfy us, as we can consider neither the gravity energy flux between different regions nor the energy flux between a gravity field and other fields. It is thus hard to investigate the relativistical astrophysics processes. In view of this situation, physicists introduced the notion of quasilocal mass. Quasilocal mass is defined on a finite, compact two surface. Now we have several different quasilocal mass forms: for instance, Penrose mass, Hawking-Hayward mass, Chen-Nester mass, and Brown-York mass. For the motivations and the general rules that they should obey, see Ref. [1].

In a spherically symmetric spacetime, one of the leading mass forms is the Misner-Sharp mass. Misner and Sharp defined it in their studies of gravitational collapse [2]. Subsequent works have shown that it has several nice properties [3,4]. In the Newtonian limit, it reduces to Newtonian mass to the leading order and Newtonian mass plus Newtonian potential energy to the next-to-leading order. It is the conserved charge corresponding to the timelike Killing vector in static spherically symmetric spacetime and to the Kodama vector in a general spherically symmetric spacetime. Also, it is useful when the quantum effect is included [5]. The Misner-Sharp mass is just the Schwarzschild mass in the Schwarzschild space, which is independent of radius. We find that Misner-Sharp mass is the energy in an adiabatic system. Under this assumption, we can derive the Schwarzschild solution by the first law without invoking an Einstein field equation [6].

Recently, the Misner-Sharp mass was extended to four-dimensional $f(R)$ gravity [7]. The $f(R)$ gravity is a kind of higher derivative gravity theory. The Hilbert-Einstein action plus a cosmological constant term is the unique Lagrangian which can generate field equations without higher than two-order derivatives with respect to the metric in four-dimensional spacetime. However, the studies of renormalizability of gravity imply that we indeed need higher order derivatives in the field equation to control divergence [8]. Moreover, the higher order derivatives always appear in the low energy effective field equation when we consider quantum corrections [9] or string theory [10]. Such motivations urge people to consider the higher derivative gravity seriously. One of the most thoroughly studied models in higher derivative theories is $f(R)$ theory. In this model, the Lagrangian is an arbitrary (smooth) function of the Ricci scalar R. Compared to more generic and more complicated models including terms like $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, it has at least two distinct advantages: First of all, it is simple. The Lagrangian $f(R)$ is general enough to encode the fundamental structure of higher derivative theories and simple enough to handle. Second, it is the unique one which can evade the critical Ostrogradski

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instability among all higher derivative gravity theories [11]. For a review of the $f(R)$ model, see, for example, Ref. [12].

Higher dimensional theory has had a fairly long history since Kluza-Klein's proposal. Higher dimension is even treated as a universal feature of unified theory, including gravity. Similar to the four-dimensional case, we need higher order derivatives to control the divergence. Note that the combinations of Riemann tensor, Ricci tensor, and Ricci scalar which do not generate higher order derivatives in field equations (called Lovelock terms) are also widely investigated in higher dimensional theories. On the other hand, the lower dimensional gravity theory is also of interest. Three-dimensional gravity is very useful when studying two-dimensional conformal field theory and provides relatively easy analytical results; for a review, see [13]. Recently, we found black hole and gravity wave solutions in the frame of three-dimensional $f(R)$ gravity [14]. In this article, we shall explore the Misner-Sharp mass for n-dimensional $f(R)$ gravity ($n \geq 3$).

This paper is organized as follows. In Sec. 2, we review some previous results of Misner-Sharp mass, stressing the inverse unified first law method [7]. In Sec. 3, we obtain the Misner-Sharp mass for $f(R)$ gravity in n-dimensional spacetime ($n \geq 3$) by the inverse unified first law method and the conserved charge method, respectively. In Sec. 4, we present the special result in a static spacetime. In Sec. 5, we conclude the article.

II. MISNER-SHARP MASS: SOME PREVIOUS RESULTS

In four-dimensional spherically symmetric spacetime, we adopt the following coordinates:

$$ds^2 = I_{ab}dx^a dx^b + r^2 d\Omega_2^2, \tag{1}$$

where Ω_2 denotes a unit two sphere, I_{ab} are general functions which are independent on the inner coordinates of the two sphere, and a and b run from 0 to 1. This metric can be dynamic or static. In this spacetime, the Misner-Sharp mass can be defined as

$$M_{ms} = \frac{r}{2G}(1 - I^{ab}\partial_a r \partial_b r). \tag{2}$$

With this definition of Misner-Sharp mass, we have the unified first law [4],

$$dM_{ms} = A\Psi_a dx^a + WdV, \tag{3}$$

where $V = 4\pi r^3/3$ is the volume of the space region in consideration and $A = 4\pi r^2$ is its surface area. W is the work term:

$$W = -\frac{1}{2}I^{ab}T_{ab}. \tag{4}$$

Here T_{ab} denotes the stress-energy tensor of the matter field. Ψ_a labels the energy supply term:

$$\Psi_a = T_a{}^b\partial_b r + W\partial_a r. \tag{5}$$

The unified first law has been used in several different situations of Einstein gravity and in modified gravity [15,16]. The logic in the unified first law is as follows: We first define the Misner-Sharp mass as shown in (2), then we find that it satisfies the unified first law (3). However, in the modified gravity theories, we have no prior definition of Misner-Sharp mass. The unified first law requires a mass in the lhs of (3). Thus we can define the mass in (3) as an extension of Misner-Sharp mass in modified gravity theories by using the rhs of (3) [7].

Now we demonstrate the inverse first law method for a concrete metric in modified gravity. We work in an n-dimensional spacetime. Without losing generality, we write an n-dimensional spacetime with an $(n-2)$-dimensional maximally symmetric submanifold in double-null coordinates:

$$ds^2 = -2e^{-\varphi(u,v)}dudv + r^2(u,v)\gamma_{ij}dz^i dz^j. \tag{6}$$

Here γ_{ij} denotes the metric of the $(n-2)$-dimensional maximally symmetric submanifold with sectional curvature $k = -1, 0, +1$. The whole manifold $\mathcal{M}$ is a direct product of this maximally $(n-2)$-dimensional manifold $(\mathcal{K}, \gamma)$ and a two-dimensional manifold $(\mathcal{T}, I)$, where $I_{uv} = I_{vu} = -e^{-\varphi(u,v)}$.

Similar to the case of Einstein gravity (3), we suppose that the unified first law takes the same form,

$$dM_{ms} = A\Psi_a dx^a + WdV, \tag{7}$$

but A and V correspond to the region of the $(n-2)$-dimensional maximally symmetric submanifold with radius r, i.e., $A = V_{n-2}^k r^{n-2}$ and $V = V_{n-2}^k r^{n-1}/(n-1)$. The energy supply term Ψ_a and work density W are defined on $(\mathcal{T}, I)$. The concrete Ψ_a and W take the same forms of Einstein gravity. In the coordinates of (6), the rhs in (7) reads

$$A\Psi_a dx^a + WdV = A(u,v)du + B(u,v)dv, \tag{8}$$

where

$$A(u,v) = V_{n-2}^k r^{n-2} e^{\varphi}(r_{,u} T_{uv} - r_{,v} T_{uu}), \tag{9}$$

$$B(u,v) = V_{n-2}^k r^{n-2} e^{\varphi}(r_{,v} T_{uv} - r_{,u} T_{vv}). \tag{10}$$

Here a comma denotes a partial derivative. Substituting (7) into (8), we reach

$$F \equiv dM_{ms} = A(u,v)du + B(u,v)dv. \tag{11}$$

The above expression is only a formal equality. M_{ms} is well defined only if F is a locally exact form. Thus F is a closed form, from which we can obtain the constraint for the existence of a well-defined M_{ms}. The essence of the above discussion is that we assume the unified first law takes the same form as in four-dimensional Einstein gravity, then derive the required mass in the unified first law. This is different from the Einstein case in which we first define Misner-Sharp mass and then prove it satisfies the unified first law. We call this method the "inverse unified first law" method.

III. MISNER-SHARP MASS FOR n-DIMENSIONAL $f(R)$ GRAVITY

In this section, we explore the Misner-Sharp mass for $f(R)$ gravity in an n-dimensional spacetime via the unified first law method and the conserved charge method, respectively. We shall demonstrate that the two methods are equivalent.

The action of the $f(R)$ gravity in an n-dimensional spacetime reads

$$S=\frac{1}{16\pi G}\left(\int_{\mathcal{M}} d^n x\sqrt{-\det(g)}f(R)+\int_{\partial M} d^{n-1}x\sqrt{-\det(h)}2B_o\right)+S_m. \tag{12}$$

Here G denotes the n-dimensional Newtonian constant, B_o represents the corresponding boundary term for the $f(R)$ term, g is the n-dimensional metric, h denotes the induced metric on the boundary, and S_m labels the action of matter fields. The field equation that follows the action (12) can be written as

$$f_R R_{\mu\nu}-\frac{1}{2}fg_{\mu\nu}-\nabla_\mu\nabla_\nu f_R+g_{\mu\nu}\Box f_R=8\pi G T_{\mu\nu}, \tag{13}$$

where $T_{\mu\nu}$ denotes the stress energy for matter fields and $f_R=\frac{\partial f}{\partial R}$. We work on the manifold $(\mathcal{M},g)$ defined in (6). The components of the field equation (48) in the coordinates (6) read

$$\begin{aligned}
-8\pi G T_{uu}&=(n-2)f_R\frac{\varphi_{,u}r_{,u}+r_{,uu}}{r}+f_{R,uu}+\varphi_{,u}f_{R,u},\\
-8\pi G T_{vv}&=(n-2)f_R\frac{\varphi_{,v}r_{,v}+r_{,vv}}{r}+f_{R,vv}+\varphi_{,v}f_{R,v},\\
8\pi G T_{uv}&=f_R\varphi_{,uv}-(n-2)f_R\frac{r_{,uv}}{r}+\frac{1}{2}fe^{-\varphi}\\
&\quad+f_{R,uv}+\frac{n-2}{r}(r_{,u}f_{R,v}+r_{,v}f_{R,u}).
\end{aligned} \tag{14}$$

Clearly, the components of the stress-energy tensor inherit the symmetry between u and v in the metric. We note that the sectional curvature of the submanifold $\mathcal{K}$ does not appear in these components, as shown above. This property is a result of $\mathcal{M}=\mathcal{T}\times\mathcal{K}$, and k shows itself in the Ricci scalar R of the whole manifold $\mathcal{M}$:

$$R=\frac{k(n-3)(n-2)}{r}+2e^{\varphi}\left(\frac{(n-3)(n-2)r_{,u}r_{,v}}{r^2}-\varphi_{,uv}+\frac{2(n-2)r_{,uv}}{r}\right). \tag{15}$$

A well-defined M_{ms} in (11) requires F to be a closed form $dF=0$, which means

$$A_{,v}dv\wedge du+B_{,u}du\wedge dv=0. \tag{16}$$

Then we obtain the constraint for a well-defined M_{ms},

$$A_{,v}=B_{,u}. \tag{17}$$

Substituting (9) and (10) into (17), where the stress energy is shown in (14), we obtain the constraint

$$\begin{aligned}
&(\varphi_{,u}r_{,u}+r_{,uu})(f_{R,vv}+\varphi_{,v}f_{R,v})\\
&\quad=(\varphi_{,v}r_{,v}+r_{,vv})(f_{R,uu}+\varphi_{,u}f_{R,u}),
\end{aligned} \tag{18}$$

which is independent on dimension of the spacetime and the sectional curvature of $\mathcal{K}$. The derivation of the above equation is rather complicated, where we have used

$$f_{,u}=f_R R_{,u}, \tag{19}$$

$$f_{,v}=f_R R_{,v}, \tag{20}$$

and R is given by (15). Under this constraint, F is closed—and thus M_{ms} is well defined. M_{ms} can be calculated by

$$M_{ms}=\int Bdv \tag{21}$$

or

$$M_{ms}=\int Adu. \tag{22}$$

It is easy to show that the above two equations are equivalent under the constraint (17). The latter one yields

$$\begin{aligned}
M_{ms}&=\frac{V_{n-2}^k r^{n-3}}{8\pi G}\left[\frac{n-2}{2}(k-I^{ab}\partial_a r\partial_b r)f_R+\frac{1}{2(n-1)}r^2(f-f_R R)-rI^{ab}\partial_a f_R\partial_b r\right]\\
&\quad-\frac{V_{n-2}^k}{8\pi G}\int du\left[f_{R,u}e^{\varphi}(r^{n-2}r_{,v})_{,u}+f_{R,u}\left(\frac{(n-2)k}{2}r^{n-3}-\frac{1}{2(n-1)}r^{n-1}R\right)+f_{R,v}r^{n-2}(r_{,u}e^{\varphi})_{,u}\right].
\end{aligned} \tag{23}$$

Thus we complete the generalization of Misner-Sharp mass to a higher dimension case for $f(R)$ gravity by using one of its most important properties: It satisfies the unified first law. Then arises a question: Is this a reasonable generalization of Misner-Sharp mass in n-dimensional $f(R)$ gravity? To determine the answer is an arduous task, but this generalization can be reintroduced in another way. The other important property of Misner-Sharp mass is that it is associated with the conserved charge corresponding Kodama vector in a spherically symmetric spacetime. One can extend the Misner-Sharp mass to Gauss-Bonnet gravity by using this property [17], which is equivalent to the inverse first law method [7]. Here we check to see whether the two methods are equivalent in the n-dimensional $f(R)$ gravity case.

In Ref. [18], the Kodama vector is defined in a spherically symmetric spacetime. Now we generalize the concept of the Kodama vector for the spherically symmetric case to one with the maximally symmetric submanifold. The generalized Kodama vector is defined as

$$K^\mu = -\epsilon^{\mu\nu}\nabla_\nu r \tag{24}$$

on $(\mathcal{M}, g)$, where $\epsilon_{\mu\nu} = \epsilon_{ab}(dx^a)_\mu (dx^b)_\nu$ and ϵ_{ab} is the compatible volume element to the metric I on $\mathcal{T}$. The Greek indices μ and ν run from 0 to $n-1$. This is a straightforward generalization of the Kodama tensor. From direct calculation, Eq. (24) can be rewritten as

$$K^\mu = e^\varphi \left[r_{,v} \left(\frac{\partial}{\partial u}\right)^\mu - r_{,u} \left(\frac{\partial}{\partial v}\right)^\mu \right]. \tag{25}$$

Just like in the case of a Killing vector, the Kodama vector induces a conserved current in Einstein gravity. Mimicking the case of Einstein gravity, we define

$$J^\mu = -T^\mu{}_\nu K^\nu. \tag{26}$$

However, we must make sure that J^μ is conserved in n-dimensional $f(R)$ gravity. First, Bianchi identity ensures that the stress energy of matter field $T_{\mu\nu}$ is still conserved:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{27}$$

Then J^μ is conserved,

$$\nabla_\mu J^\mu = 0, \tag{28}$$

only if

$$\nabla_\mu \nabla_\nu (f_R \nabla^\mu K^\nu) = 0. \tag{29}$$

With the aid of J, we define a charge,

$$Q_J = \int_S {}^* J, \tag{30}$$

where a star denotes the Hodge dual of the vector J. S is an $(n-1)$-dimensional hypersurface in $\mathcal{M}$. Q_J is really a conserved charge under the constraint (29). Slicing $\mathcal{M}$ with constant v, we reach

$$\begin{aligned} Q_J = {} & \frac{V_{n-2}^k r^{n-3}}{8\pi G}\left[\frac{n-2}{2}(k - I^{ab}\partial_a r \partial_b r) f_R + \frac{1}{2(n-1)} r^2 (f - f_R R) - r I^{ab} \partial_a f_R \partial_b r\right] \\ & - \frac{V_{n-2}^k}{8\pi G}\int du \left[f_{R,u}\, e^\varphi (r^{n-2} r_{,v})_{,u} + f_{R,u}\left(\frac{(n-2)k}{2} r^{n-3} - \frac{1}{2(n-1)} r^{n-1} R\right) + f_{R,v}\, r^{n-2} (r_{,u}\, e^\varphi)_{,u}\right]. \end{aligned} \tag{31}$$

One can check that the conserved charge Q_J is just M_{ms} in (23). We achieve the same result through different routes. Therefore, Eqs. (31) and (23) should be a reasonable generalization of Misner-Sharp mass in n-dimensional $f(R)$ gravity. At the least, it inherits two significant properties of Misner-Sharp mass in Einstein gravity: It satisfies the unified first law and corresponds to a conserved charge associating to the Kodama vector. For $n = 4$ and $k = 1$, our result reduces the previous result in [7].

To display a deeper relation between the two routes, we prove the equivalence of the constraints in Eqs. (17) and (29). We clearly calculate the first constraint (17):

$$B_{,u} - A_{,v} = \frac{(n-2) V_{n-2}^k r^{n-2} e^\varphi}{16\pi G}[(\varphi_{,u} r_{,u} + r_{,uu})(f_{R,vv} + \varphi_{,v} f_{R,v}) - (\varphi_{,v} r_{,v} + r_{,vv})(f_{R,uu} + \varphi_{,u} f_{R,u})]. \tag{32}$$

On the other hand, the lhs of (29) yields

$$\nabla_\mu \nabla_\nu (f_R \nabla^\mu K^\nu) = e^{2\varphi}[(\varphi_{,u} r_{,u} + r_{,uu})(f_{R,vv} + \varphi_{,v} f_{R,v}) - (\varphi_{,v} r_{,v} + r_{,vv})(f_{R,uu} + \varphi_{,u} f_{R,u})]. \tag{33}$$

It is clear that both of the constraints are equivalent to (18).

IV. STATIC CASE

The spacetime $(\mathcal{M}, g)$ becomes a stationary one if there exists a timelike Killing vector. Moreover, if it permits an $(n-2)$-dimensional maximally symmetric submanifold $(\mathcal{K}, \gamma)$, it is a static one since the metric can be written in a time orthogonal coordinate system:

$$ds^2 = -l(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\gamma_{ij}dz^i dz^j. \tag{34}$$

In principle, for any concrete metric form, one can always make coordinate transformations to write it in double null form (6). Thus one can use all the results from the last section. Then one makes an inverse coordinate transformation to get the results in the original coordinates. For the special status of a static spacetime, we also explicitly present the Misner-Sharp mass in the static coordinates.

To obtain the Misner-Sharp mass, we only need T_{tt}, T_{rr}, and T_{rt}. The components of the field equation in this coordinate system read

$$8\pi G T_{tt} = \frac{1}{2}fl + \frac{h}{r}\left[\left(\frac{n}{2}-1\right)f_R l' - (n-2)lf_R{}'\right] + \frac{1}{4}f_R\left(h'l' - \frac{hl'^2}{l} + 2hl''\right) - \frac{1}{2}f_R{}'lh' - hlf''_R, \tag{35}$$

$$8\pi G T_{rr} = -\frac{f}{2h} + \frac{1}{r}\left[(n-2)f'_R - \left(\frac{n}{2}-1\right)\frac{f_R h'}{h}\right] + \frac{f_R}{4l}\left(\frac{l'^2}{l} - \frac{h'l'}{h} - 2l''\right) + \frac{f_R{}'l'}{2l}, \tag{36}$$

$$T_{tr} = 0, \tag{37}$$

where a prime denotes the derivative with respect to r. Again, the sectional curvature of $\mathcal{K}$ does not appear since $\mathcal{M} = \mathcal{K} \times \mathcal{I}$. The Ricci scalar reads

$$R = \frac{(n-3)(n-2)}{r^2}(k-h) - \frac{n-2}{r}\left(h' + \frac{hl'}{l}\right) + \frac{1}{2l}\left(\frac{hl'^2}{l} - h'l' - 2hl''\right). \tag{38}$$

In this case, the definition of Misner-Sharp mass (11) becomes

$$F \equiv dM_{ms} = A(t,r)dt + B(t,r)dr, \tag{39}$$

where

$$A(t,r) = V^k_{n-2}r^{n-2}h(r_{,r}T_{tr} - r_{,t}T_{rr}), \tag{40}$$

$$B(t,r) = V^k_{n-2}r^{n-2}\frac{(r_{,r}T_{tt} - r_{,t}T_{tr})}{l}. \tag{41}$$

One immediately sees that F in (39) is a closed form since

$$A_{,r} = B_{,t} = 0. \tag{42}$$

Hence, the Misner-Sharp mass is naturally well defined for a static spacetime in n-dimensional $f(R)$ gravity.

Now we check the constraint in the conserved charge method. The Kodama vector reduces to a Killing one in a static spacetime. In the static coordinates (34), the timelike Killing vector reads

$$K = \sqrt{\frac{h(r)}{l(r)}}\frac{\partial}{\partial t}. \tag{43}$$

It is easy to show that the constraint (29) is also satisfied automatically. The Misner-Sharp mass, or the conserved charge associated with K, reads

$$M_{ms} = \int_\Sigma B dr \tag{44}$$

$$= \frac{V^k_{n-2}r^{n-3}}{8\pi G}\left[\frac{n-2}{2}(k - I^{ab}\partial_a r\partial_b r)f_R + \frac{1}{2(n-1)}r^2(f - f_R R) - rI^{ab}\partial_a f_R \partial_b r\right] + \frac{V^k_{n-2}}{16\pi G}\int dr\left[r^{n-2}h' + (n-2)r^{n-3}(h-k) + \frac{r^{n-1}}{n-1}R\right]f_{R,r}, \tag{45}$$

where Σ marks a slice with constant t. For Einstein gravity with a cosmological constant, Eq. (45) degenerates to

$$M_{ms} = \frac{V^k_{n-2}r^{n-3}}{8\pi G}\left[\frac{n-2}{2}(k - I^{ab}\partial_a r\partial_b r)f_R + \frac{r^2}{2(n-1)}(f - f_R R)\right]. \tag{46}$$

In a general $f(R)$ theory, the uniqueness theorem of the spherically symmetric space becomes invalid. R may not vanish even for vacuum solutions. In that case, the integration term cannot be omitted. Let us consider the four-dimensional action in the following form:

$$S = \frac{1}{16\pi G}\left(\int_{\mathcal{M}} d^4x\sqrt{-\det(g)}R^{d+1} + \int_{\partial M} d^3x\sqrt{-\det(h)}2B_o\right), \tag{47}$$

where d is a constant. The corresponding field equation reads

$$(1+d)R^d R_{\mu\nu} - \frac{1}{2}R^{d+1}g_{\mu\nu} - (1+d)\nabla_\mu\nabla_\nu R^d + g_{\mu\nu}(1+d)\Box R^d = 0. \tag{48}$$

It is easy to see that the Schwarzschild metric is a solution for the above equation. Obviously, the Misner-Sharp mass for the Schwarzschild metric in $f(R)$ gravity is the same as in Einstein gravity, since the Schwarzschild metric has a vanishing Ricci scalar R. The $f(R)$ gravity permits a nontrivial spherically symmetric solution aside from the Schwarzschild metric. The following metric also solves the field equation [19]:

$$ds^2 = -l(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{49}$$

where

$$l(r) = r^{2d\frac{(1+2d)}{(1-d)}} + \frac{c}{r^{\frac{(1-4d)}{(1-d)}}}, \tag{50}$$

$$h(r) = \frac{(1-d)^2}{(1-2d+4d^2)(1-2d(1+d))}\left[1 + \frac{c}{r^{\frac{(1-2d+4d^2)}{(1-d)}}}\right]. \tag{51}$$

Here c is a constant, which reduces to the Schwarzschild mass parameter $c = -2M$ in Einstein gravity. This solution is called the Clifton-Barrow solution. Now we check the physical sense of this solution by calculating its Misner-Sharp mass. With this metric, the first part (the non-integrated part) in (45) becomes

$$\frac{6^d d(1+d)(2d-1)\left(\frac{d(1+d)}{(2d-1+2d^2)r^2}\right)^d r^{\frac{1+d}{1-d}}\left(\frac{2d(2d-1+2d^2)r^{\frac{2d}{-1+d}}}{1-2d} + \frac{c(-1+d)^2 r^{\frac{1+4d^2}{d-1}}}{d}\right)}{2(4d-1-6d^2+4d^3+8d^4)}, \tag{52}$$

and the integrated part reads

$$\frac{6^d(1+d)\left(\frac{d(1+d)}{(2d-1+2d^2)r^2}\right)^d r^{\frac{1+d}{1-d}}\left(4d^4 r^{\frac{2d}{d-1}} + cr^{\frac{1+4d^2}{d-1}} - 4cdr^{\frac{1+4d^2}{d-1}} + d^3\left(4r^{\frac{2d}{d-1}} - 2cr^{\frac{1+4d^2}{d-1}}\right) + d^2\left(5cr^{\frac{1+4d^2}{d-1}} - 2r^{\frac{2d}{d-1}}\right)\right)}{2(4d-1-6d^2+4d^3+8d^4)} + c_1, \tag{53}$$

where c_1 is an integration constant. The condition that (45) reduces to the Misner-Sharp mass in Einstein gravity requires that

$$c_1 = -\frac{c}{2}. \tag{54}$$

Finally, we reach the Misner-Sharp mass for the Clifton-Barrow solution,

$$2GM_{ms} = (-c)^{(1-d)/(1-2d+4d^2)}, \tag{55}$$

which is independent on r (just as in the Schwarzschild case).

V. CONCLUSION

The unified first law is a significant approach in gravithermodynamics. Different from traditional black hole thermodynamics, in which the quantities depend on the asymptomatic behavior of the whole manifold, it concentrates on a finite region of the spacetime. Since there is no well-defined energy density of a gravity field, the quasi-local mass plays a critical role in the unified first law. The Misner-Sharp mass is the proper mass form which guarantees the success of the unified first law. Moreover, we can derive the Schwarzschild solution from the Misner-Sharp mass form, in which we need not solve the field equation. It is appropriate to explore the generalization of Minser-Sharp mass in n-dimensional modified gravity. $f(R)$ gravity has the essential properties of higher order derivative theory and is also easy to handle. In this article, we obtain the Misner-Sharp mass for $f(R)$ gravity in n-dimensional spacetime with $(n-2)$-dimensional maximally symmetric submanifold by using the inverse first law method in a double-null coordinate system. In this method, we first assume the unified first law still takes the same form as it does in Einstein gravity. Then we identify the required mass form as the generalized Misner-Sharp mass in this case. We find a constraint for a well-defined Misner-Sharp mass. Also, we derive the Misner-Sharp mass by the conserved charge method. This method presents exactly the same mass form and an equivalent constraint for the existence of a well-defined Misner-Sharp mass, so it seems to be a reasonable generalization of the Misner-Sharp mass in n-dimensional $f(R)$ gravity.

A static spacetime is an important special case in gravity theory. We also present the implicit form of the Misner-Sharp mass in a static spacetime in n-dimensional $f(R)$ gravity. The Clifton-Barrow solution is an interesting one in $f(R)$ theory. As an example, we calculate the Misner-Sharp mass for this solution. We find that it is a constant which is independent of the radius r, just as in the case of the Schwarzschild solution.

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