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# Strong constraints on magnetized white dwarfs surpassing the Chandrasekhar mass limit

Rajaram Nityananda*
*Indian Institute of Science Education and Research, Pune 411008, India*

Sushan Konar†
*National Centre for Radio Astronomy, Tata Institute of Fundamental Research, Pune 411007, India*
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We show that recently proposed white dwarf models with masses well in excess of the Chandrasekhar limit, based on modifying the equation of state by a superstrong magnetic field in the center, are very far from equilibrium because of the neglect of Lorentz forces. An upper bound on the central magnetic fields, from a spherically averaged hydrostatic equation, is much smaller than the values assumed. Robust estimates of the Lorentz forces are also made without assuming spherical averaging. These again bear out the results obtained from a spherically averaged model. In our assessment, these estimates rule out the possibility that magnetic tension could change the situation in favor of larger magnetic fields. We conclude that such super-Chandrasekhar models are unphysical, and exploration of their astrophysical consequences is premature.

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## I. INTRODUCTION

Models for "white dwarf" -like stars (i.e., stars supported against gravity by electron degeneracy pressure) with masses significantly exceeding the Chandrasekhar limit (e.g., $2.3-2.6\ M_\odot$) and radii significantly smaller than hitherto considered possible ($\sim 70-600$ km) have been proposed [1,2] and their astrophysical consequences explored [3,4]. These models are based on the altered equation of state coming from the quantization of electron motion in superstrong ($\gtrsim 10^{15}$ G) magnetic fields. We show below that these models are not in hydrostatic equilibrium, a fact missed in the original and subsequent work, which ignores the unavoidable gradient of magnetic pressure. A brief comment to this effect has been submitted [5], and the present paper gives more details and, in particular, lifts the assumption of spherical symmetry.

The main concern of this note is a counterintuitive feature of these models—the magnetic pressure ($P_m$) is not included in the equation of hydrostatic equilibrium even though its value at the center exceeds the electron pressure ($P_e$) by nearly 2 orders of magnitude. Given that the models balance gravity solely against electron pressure, this implies that the field pressure/energy density greatly exceeds the traditional dimensional estimate of central pressure, given by $GM^2/R^4$, where $M$ and $R$ denote the mass and the radius of the spherical star and $G$ is the gravitational constant. The justification given in [1] and most recently in [6] is that the field is uniform in the central region of interest, and therefore, there is no force coming from the gradient of $P_m$. The field is then presumed to taper off to much smaller values at the surface (consistent with observations) without affecting the analysis, which relies solely on electron pressure. In the next section, we show in a spherically averaged model that this is not possible—the equation of equilibrium is violated by a large factor in the transition region, where the strong uniform central field reduces to much smaller values. In Sec. II B, we relax the assumption of spherical averaging and use the magnetic virial theorem to derive general bounds on the central fields, which are still far less that the proposed values. In Sec. III, we comment on the need to include general relativity and other effects while dealing with highly relativistic electrons in extremely strong magnetic fields.

## II. BOUNDS ON THE CENTRAL MAGNETIC FIELD

### A. Spherical symmetry

We initially restrict to an averaged spherically symmetric model, as in [1], in which the stress tensor of the magnetic field can be replaced by an equivalent isotropic pressure. We take this $P_m$ to be $B^2/24\pi$, one third the trace of the Maxwell stress tensor (also one third the energy density), where $B$ is the magnitude of the magnetic field. Then at a radius $r$ inside the star, the equation of hydrostatic equilibrium reads

$$\frac{dP_e}{dr} + \frac{dP_m}{dr} = -\rho(r)g(r), \tag{1}$$

*Present address: Azim Premji University, Bangalore 560100, India.
rajaram.nityananda@iiserpune.ac.in
†sushan@ncra.tifr.res.in;
http://www.ncra.tifr.res.in:8081/~sushan/

 

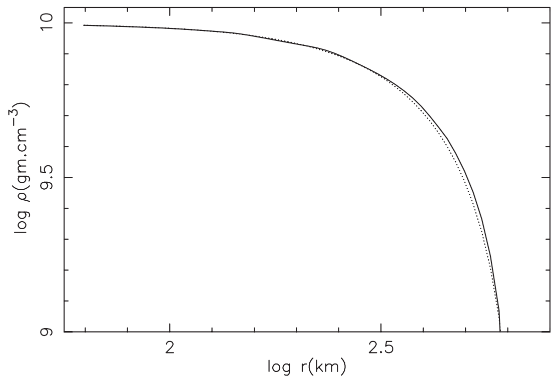


FIG. 1. Radial density profiles. The solid line denotes the actual profile as calculated by Das and Mukhopadhyay [1] for a star with $B_c = 8.81 \times 10^{15}$ G and $E_{\mathrm{F}\times\mathrm{max}} = 20 m_e c^2$, and the dotted line is a $\sin(x)/x$ curve, where $x$ is equal to $\pi r/r_{\mathrm{max}}$.

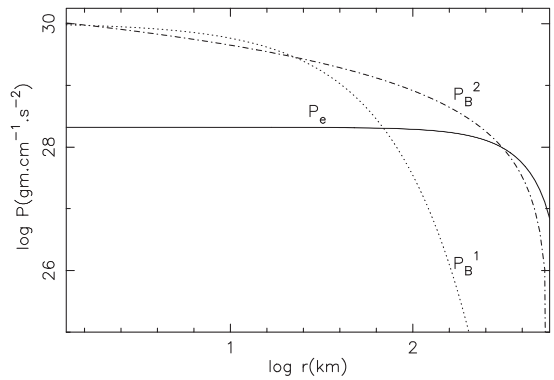


FIG. 2. Radial pressure profiles. The solid line ($P_e$) denotes the pressure due to the electrons in presence of a quantizing magnetic field. The dotted ($P_B^1$) and the dash-dotted ($P_B^2$) lines represent the pressure due to the magnetic field. A central field of $8.8 \times 10^{15}$ G has been assumed to fall off to $10^9$ G at the surface. While the dash-dotted curve represents a linear falloff, the dotted curve shows the case of a power-law falloff for the magnetic field.

where $g(r)(= GM(r)/r^2)$ is the radially inward gravitational force on a unit mass, $\rho(r)$ and $M(r)$ being the mass density at and the total mass contained within a radius $r$. In the proposed models, the second term on the left-hand side is assumed to be negligible compared to the first. Integrating both sides of Eq. (1), from the center ($r = 0$) to the surface of the star ($r = R$), denoted by suffixes $c$ and $s$, we obtain

$$(P_{\mathrm{ec}} - P_{\mathrm{es}}) + (P_{\mathrm{mc}} - P_{\mathrm{ms}}) = \int_0^R \rho(r) g(r) dr. \qquad (2)$$

Clearly, the second bracket on the right-hand side should be smaller than the first if the neglect of the magnetic contribution to the hydrostatic equation is to be valid, but (equating the surface pressures to zero) the exact opposite is true of the proposed models, the magnetic pressure being very much greater than that of the degenerate electrons, which is an obvious contradiction.

We consider a particular case explored in [1] to illustrate this point. Figure 1 shows the radial density profile of a proposed stellar model, which has a central magnetic field equal to $8.81 \times 10^{15}$ G. The maximum Fermi energy of the constituent electrons is assumed to be $20 m_e c^2$, where $m_e$ is the mass of the electrons and $c$ is the velocity of light. We note that this density profile is well approximated by a function of the form $\sin(x)/x$. This, of course, is the form of the radial density profile for a star that is a $n = 1$ polytrope. For a polytrope, one assumes the gas pressure to be given by

$$P = K\rho^{\gamma} = K\rho^{(n+1)/n}, \qquad (3)$$

where $\gamma$ is the adiabatic index and $n$ is called the polytropic index. Here, $K$ is a dimensional constant characterizing the gas. Physically, the $n = 1$ polytrope corresponds to the case of an extreme relativistic gas with unfilled lowest Landau level. It should be noted that nonmagnetic white dwarfs are described by $n = 1.5$ in the region where electrons are nonrelativistic, rising to $n = 3$ as the electrons become relativistic.

In Fig. 2, we compare the radial variations of $P_e$ and $P_m$. To calculate $P_m$, we assume two different radial profiles of the magnetic field, in both cases matching to a value of $10^9$ G at the surface, the maximum observed surface field for white dwarfs.

This figure confirms our earlier assertion. No attempt to taper off a large, uniform magnetic field in the center to essentially zero ($B_s/B_c << 1$) at the surface can avoid a gradient of $P_m$ to be much larger than that of $P_e$ (at least in certain locations), which balances gravity in the proposed models.

### B. Beyond spherical symmetry

Our constraints on the equilibrium of models supported by electron pressure alone have so far been derived in the spirit of spherical averaging. This would be strictly applicable only if the field was sufficiently disordered to result in an average isotropic pressure within a region smaller than the scale length over which pressure and density vary significantly. It has been pointed out [6] that magnetic tension in an ordered field has been left out of such a picture. According to this view, the magnetic tension could actually act in an opposite way to magnetic pressure and possibly play a significant role in stabilizing a nonspherical configuration with a suitably ordered field.

We first examine this possibility by deriving a constraint on the field strength in the case of a poloidal field. Assume the field in the center to be along the $z$ axis. The model is now axisymmetric rather than spherical, with the shorter dimension along the $z$ axis, as expected from the steeper density gradient needed to balance gravity aided by magnetic tension. However, the tension along $z$ axis is

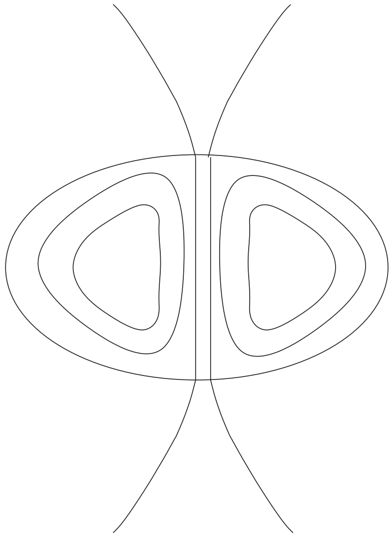

FIG. 3. Schematic representation of a poloidal magnetic field inside a self-gravitating oblate spheroid. Notice that most of the field lines close inside the star. This is expected in a situation where there is a strong field in the central region tapering down to a very small value near the surface.

now accompanied by a lateral pressure $P_{m\perp}\,(=B^2/8\pi)$ in the $xy$ plane. We assume the fraction of the central flux leaking out of the star to be very small since the maximum surface fields observed in white dwarfs are 6 orders of magnitude smaller than the central fields being envisaged. In this case, most of the field lines would necessarily have to return with an opposite sign and cross the equatorial plane within the star. The situation is shown in Fig. 3 for a poloidal field configuration.

We can now apply our earlier argument in the equatorial plane, with a three times stronger magnetic pressure at the center ($P_{m\perp}$), and an appropriately weaker gravity term. The gravitational potential gradient term in the hydrostatic equation gets reduced in the equatorial plane due to oblateness. Therefore, it appears that in the poloidal case equilibrium in the $z$ direction would have to be bought at the price of even greater disequilibrium in the equatorial plane.

However, one has to consider the possibility of a toroidal field whose tension could help maintain equilibrium in the equatorial plane (and stabilize the poloidal configuration as well), but now this would be attained at the cost of outward forces away from the equatorial plane. We constrain this more general situation below using the magnetic virial theorem.

The Lorentz force density $\mathbf{f^L}$ inside a continuous medium with current density $\mathbf{J}$ and magnetic field $\mathbf{B}$ is given by

$$\mathbf{f^L} = \frac{\mathbf{J}\times\mathbf{B}}{c} = \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{4}$$

By the virial identity for the Lorentz force [for details see [7], p. 158, Eq. (78)] we have

$$\int_0^R \mathbf{r}.\mathbf{f^L}\,d^3\mathbf{r} = \int_0^R \frac{\mathbf{B}^2}{8\pi}\,d^3\mathbf{r} = E_B, \tag{5}$$

where $E_B$ is the total magnetic energy of the system and $\mathbf{r}$ is the radius vector. In writing this, we have neglected the surface terms at the upper limit of integration $R$, the stellar radius. This is justified in view of the surface fields being much smaller than the postulated central fields. One immediate conclusion from this identity is that the average value of $\mathbf{r}.\mathbf{f^L}$ is positive. This implies that the average Lorentz force is outward, tension notwithstanding (this is in conformity with the spherically averaged model, which has a positive isotropic pressure).

Since the outward magnetic force cannot exceed gravity anywhere, we proceed as follows. We use the virial identity to give a lower bound on the maximum value of the Lorentz force density in terms of the central magnetic field. This has to be less than the maximum value of the gravitational force density. The resulting upper bound on the central field is conservative but will be enough to rule out the kind of central fields being postulated. The method is general but is illustrated for the polytropic models below.

To obtain the maximum allowed magnitude of $\mathbf{B}$, we make an underestimate of the maximum value of the Lorentz force density $\mathbf{f^L}$ (as a function of radius) within the star. To this end, we use Eq. (5) in the following form:

$$f^L_{\max}\int_0^R \frac{\mathbf{r}.\mathbf{f^L}}{f^L_{\max}}\,d^3\mathbf{r} = E_B, \qquad \Rightarrow f^L_{\max} = \frac{E_B}{I}, \tag{6}$$

where $f^L_{\max}$ is the maximum value of the Lorentz force density and $I$ is the integral defined as $\int_0^R \frac{\mathbf{r}.\mathbf{f^L}}{f^L_{\max}}\,d^3\mathbf{r}$. To obtain a conservative lower limit of $f^L_{\max}$, corresponding to the conservative lower limit of the maximum magnetic field, we need to underestimate $E_B$ and overestimate $I$. A lower bound on $f^L_{\max}$ is therefore given by

$$f^L_{\max} > \frac{E_-}{I_+}, \tag{7}$$

where $E_-$ is an underestimate of $E_B$ and $I_+$ is an overestimate of $I$. For an equilibrium model, $f^L_{\max}$ is less than the maximum value of the gravity term.

We now examine the maximum value of the gravity term in the hydrostatic equation by considering the inward gravitational force per unit volume, $f^g(r)(=\rho(r)g(r))$. For a given total mass and radius, the value and the location of $f^g_{\max}(r)$ are strongly dependent on the central concentration of the mass distribution and less so on the flattening so long as it is modest. We therefore illustrate this by means of polytropic spherical models, though the method is general. Figure 4 shows the radial variation of

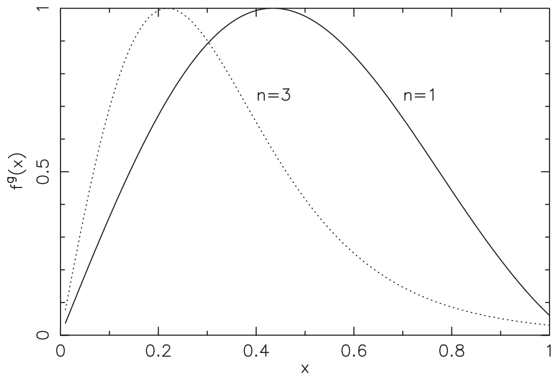


FIG. 4. $f^g(=\rho g)$ vs fractional radius $x(=r/R_*)$ for different polytropic ($n=1$, 3) models; $f^g(r)$ for each $n$ being scaled to unity at the maximum for ease of comparison. Note that the $n=3$ case corresponds to a star supported by highly relativistic particles.

$f^g(r)$ for different polytropic models. It is observed that the gravitational force density increases from zero at the center reaches a maximum value at a certain radius ($R_m^g$) and then gradually falls to zero again at the surface. For a centrally concentrated $n=3$ polytropic model, it peaks early ($R_m^g \sim 0.2R_*$), while for the $n=1$ model, it peaks at $R_m^g \sim 0.5R_*$, where $R_*$ is the corresponding stellar radius.

For a star to be in equilibrium it is necessary that $f_{\max}^L$ should be smaller than $f_{\max}^g$. We have seen that Lorentz force scales as the gradient of the field. Therefore, $f_{\max}^L < f_{\max}^g$ implies that the uniform magnetic field at the center, which has zero Lorentz force, would also have to drop to smaller values around the location of maximum gravity. This is indeed the most favorable situation for equilibrium. If Lorentz force exceeds gravity there, it would do so even more if the falloff were to occur at some other $r$, greater or less than $R_m^g$. Using this physical idea, we set $B(r)=B_c$ for $r<R_m^g$. Then the magnetic energy is given by

$$E_B = \int_0^{R_m^g} \frac{B_c^2}{8\pi} d^3\mathbf{r} + \int_{R_m^g}^{R} \frac{B_{nc}^2}{8\pi} d^3\mathbf{r} \geq \frac{4\pi}{3}(R_m^g)^3 \frac{B_c^2}{8\pi}, \quad (8)$$

since $B_{nc}$, the "noncentral" magnetic field for $r<R_m^g$, is always smaller than the uniform central field $B_c$. Therefore, we take the underestimate $E_-$ of the field energy, $E_B$, to be,

$$E_- = \frac{1}{6}(R_m^g)^3 B_c^2. \quad (9)$$

To obtain an overestimate of $I$, we note that the integrand is $r(f^L/f_{\max}^L)\cos\theta$, where both $f^L/f_{\max}^L$ and $\cos\theta$ are always less than unity. Then an overestimate is obtained by replacing these two factors by a unity, and we have

$$I = \int_0^R \frac{\mathbf{r}.\mathbf{f}^{\mathbf{L}}}{f_{\max}^L} d^3\mathbf{r} \leq f \int^R \cos\theta r d^3\mathbf{r} = \pi R^4 = I_+. \quad (10)$$

Requiring $f_{\max}^L$ to be less than $f_{\max}^g$, we then obtain a constraint on $B_c$ by using Eqs. (7), (9), and (10). Expressed in terms of the equivalent isotropic magnetic pressure at the center, $P_{\rm cm}$, the result is

$$P_{\rm cm} < \frac{1}{4}(\rho g)_{\max}\left(\frac{R_m^g}{R}\right)^{-3} R, \quad (11)$$

taking $P_m = B^2/24\pi$ as before. In order to express the right-hand side in terms of average stellar quantities, we now use a spherically symmetric polytropic model purely as a convenient parametrization of a family of models with increasing central concentration.

To facilitate easy understanding of what follows, we lay down the basic structure of the polytropic models here [8,9]. The solution for the structure of a gravitationally bound object with a polytropic equation of state depends on the coupling of Eq. (3) to the condition of hydrostatic equilibrium given by Eq. (1). From this, it follows that

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho, \quad (12)$$

where $P$ denotes the total pressure at a radius $r$ inside the star. Motivated by the fact that the density is proportional to $T^n$ ($T$ is the system temperature) in a polytropic gas of index $n$, a convenient definition of $\rho$ is

$$\rho = \lambda\theta^n, \quad (13)$$

where $\lambda$ is a constant. Substitution of the values of pressure and density for a polytrope of index $n$ into Eq. (12) gives us

$$(n+1)K\lambda^{1/n}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -4\pi G\lambda\theta^n. \quad (14)$$

This reduces to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \quad (15)$$

by defining a dimensionless distance variable $\xi = r/a$, where

$$a = \left[\frac{(n+1)K\lambda^{(1-n)/n}}{4\pi G}\right]^{\frac{1}{2}}. \quad (16)$$

Equation (15) is called the Lane-Emden equation for the structure of a polytrope of index $n$.

Let us now consider the central gravitational pressure, $P_c$, which is the pressure implied by the equation of equilibrium, and the mass and radius, for a given degree of central concentration which increases with the polytropic index $n$. It is precisely $P_c$ that is exceeded by the central magnetic pressure in the models under discussion.

Using the standard solutions for various quantities inside a polytropic star, we obtain the following relation between $P_{\rm cm}$ and $P_c$:

$$P_{\rm cm} < -\frac{1}{4}(n+1)(\xi\theta^n\theta')_{R_m^g}\left(\frac{R_m^g}{R}\right)^{-4} \times P_c. \quad (17)$$

To obtain the above condition, we have used the following standard relations, generic to any polytropic model:

$$\rho_c = -\frac{1}{4\pi}\frac{\xi(R)}{\theta'(R)}\frac{M}{R^3}, \quad (18)$$

$$P_c = \frac{1}{4\pi(n+1)(\theta'(R))^2}\frac{GM^2}{R^4}, \quad (19)$$

$$M(r) = \frac{r^2\theta'(r)}{R^2\theta'(R)}M, \quad (20)$$

where $\theta$ is the Lane-Emden function of polytropic order $n$ and $\xi$ is the dimensionless radial parameter defined above. Here $\theta'$ denotes the derivative of $\theta$ with respect to $\xi$. $\rho_c$, $M$, and $R$ stand for the central density, the total mass, and the radius of the star, respectively.

Evidently, the ratio between $P_{\rm cm}$ and $P_c$, given by

$$\mathcal{Q}(n) = -\frac{1}{4}(n+1)(\xi\theta^n\theta')_{R_m^g}\left(\frac{R_m^g}{R}\right)^{-4}, \quad (21)$$

now depends solely on the polytropic index $n$.

It is seen from Table I that for a range of polytropic indices the proportionality between the conservative upper limit on $P_m$ and $P_c$ is dependent on the location of $f^g_{\rm max}$, apart from a small factor (which in polytropic models depends only on $n$). This result itself is remarkably insensitive to the polytropic index, thanks to the choice of $R_m^g$ as the independent variable though the actual region of high gravity and magnetic field gradient naturally moves inward as $n$ increases.

There is some scope for weakening the bound by invoking factors that we have not included. These are nonsphericity and a nonpolytropic running of pressure and density. The upper limits on the integrals in $E_-$ and $I_+$ could differ from $R_m^g$, though not by a large factor. It should now be amply clear, with due allowance for magnetic tension, that the central magnetic pressure cannot exceed that inferred from the mass distribution by a factor of 100 or larger.

TABLE I. Variation of the numerical factor $\mathcal{Q}$ with $n$.

| n | 1 | 1.5 | 2 | 3 |
|---|---|---|---|---|
| $\mathcal{Q}(\frac{R_m^g}{R})^4$ | 2.10 | 1.89 | 1.76 | 1.62 |

Consider, for example, the extreme model, which has $M = 2.58\ M_\odot$, $R = 69.5$ KM with a central magnetic field of $B_c = 8.8 \times 10^{17}$ G [3]. Since the equation of state matches the $n = 1$ polytropic case very closely, using the above table, we can obtain the upper bound to the central magnetic field. It can be seen from Fig. 4 above that $R_m^g/R$ is close to 0.5 for this case. Using this, we find the maximum central field to be given by $B_{\text{upper-bound}} \simeq 10^{16}$ G. This is almost 2 orders of magnitude smaller than the field claimed to be present in the center of such an object.

Although we have used spherically symmetric polytropes to illustrate the trend with varying central concentration, the argument is more general, which can be used as a reality check given the running of density and magnetic field in any proposed model, even an anisotropic one. We conclude that ordered fields in an anisotropic model cannot qualitatively change the conclusions drawn from the average spherical model.

## III. THE EXTREME RELATIVISTIC LIMIT

The electrons, in the models under consideration, have Fermi energies significantly above their rest energy. We point out certain generic features of such extreme relativistic systems. Consider the case when the magnetic field is such that the lowest Landau level (LLL) is just full. Then the relation between the electron density ($n_e$) and the cyclotron frequency ($\omega_c = eB/m_e c$) is [10] (using $\hbar = m_e = c = 1$ from here onward),

$$n_e = \omega_c^{3/2}/\sqrt{2}\pi^2, \quad (22)$$

where $p_F$ is the Fermi momentum of the electrons. Therefore, the electron pressure is given by

$$P_e = \frac{1}{2}n_e E_F = \frac{1}{\sqrt{2}\pi}\omega_c^2, \quad (23)$$

where $E_F$ is the Fermi energy of the electrons and is proportional to $n_e$ for ultrarelativistic particles. From these two expressions, we note the following results.

(1) At the center of the star, with an exact LLL filling, $P_{\rm c-LL}$ is proportional to the four-third power of the electron density, just as $P_{\rm eC}$, the gas pressure in the usual (Chandrasekhar) case. However, it has a different numerical coefficient; that is, $P_{\rm e-LL} = 2^{-1/3}\pi^{2/3}n_e^{4/3}$ as opposed to $P_{\rm eC} = 3^{1/3}4^{-1}\pi^{2/3}n_e^{4/3}$. Therefore, $P_{\rm e-LL}$ is a factor of $4 \times 6^{-1/3}$ or approximately 2.2 times greater than $P_{\rm eC}$, but at lower densities, as we move away from the center, the pressure varies as $n_e^2$, so the running of density with radius is very close to the n = 1 polytrope, which is a stiffer equation of state. A combination of the enhanced numerical factor and a stiffer polytropic index (for the equation of state) is then responsible for the super-Chandrasekhar mass

obtained in the proposed models [1–4], of course, with neglect of magnetic pressure.

(2) Plasma beta ($\beta_p$), the ratio of gas pressure to magnetic pressure (taken, as before, to be $B^2/24\pi$), at the LLL condition is independent of the field strength in the extreme relativistic limit and is given by $12\alpha/\pi$, i.e., around $2.8 \times 10^{-2}$ ($\alpha = e^2/c\hbar$ is the fine structure constant). Generically, the magnetic pressure is 2 orders of magnitude greater than the electron pressure in a relativistic model with only a few Landau levels filled. This incidentally implies that if the electrons are already relativistic then the rest energy of the magnetic field is rather significant because $100\,(\times)\,20\,m_e c^2$ is not very far from $4000 m_e c^2$, which is the rest energy of the nuclei, per electron. This factor goes in the direction of softening the equation of state, toward $p \propto \rho$, and points to the need for a general relativistic treatment since pressure gravitates in general relativity. It should be noted that for the particular model mentioned above [3] a field of $\sim 10^{17}$ G would have a rest mass density comparable to the average density of the star. Since higher fields are being postulated in the central regions, it is clear that general relativistic effects would play a major role. Another important factor that would modify the equation of state under such conditions is neutronization (inverse $\beta$ decay), the absorption of electrons by protons to produce neutrons, which becomes favorable with increasing electron energy [10]. As a result of neutronization, the electron number decreases, and the ionic component of the pressure (which has been completely neglected so far) begins to become important. Moreover, at higher densities, pycnonuclear reactions would modify the composition of the matter, making it even more difficult for the star to be treated as an ordinary white dwarf (with standard compositions). The proposed formalism does not provide for these energetically favorable processes either, the effect of which has been investigated recently [11]. This particular issue (and a number of other concerns) is now being addressed by other groups as well [12]. The results from a self-consistent investigation into the structures of strongly magnetized white dwarfs has just become available [13], and it is seen that while the masses of such objects do exceed the traditional Chandrasekhar limit, neither do the structures deviate too far from those of the nonmagnetized white dwarfs nor do the maximum field strengths ($B \sim 10^{14}$ G) differ significantly from those expected from simple stability arguments presented here.

## IV. CONCLUSIONS

We would like to reiterate that the effects we are considering here arise due to a rather basic physical reason—the currents that generate the magnetic field flow somewhere inside the star (one is not considering stars in external fields) and they must experience a $\mathbf{J} \times \mathbf{B}$ force. There exist *force free* configurations with current flowing parallel to $\mathbf{B}$, which have been extensively studied in the context of solar physics, for example, but it is a known characteristic of these configurations that the forces are redistributed to the boundaries rather than vanishing everywhere. These are the words of one of the founders of the subject: “While we may be able to cancel the stresses inside a given region, we cannot arrange for its cancellation everywhere” ([7], p. 159). Our basic point is that it is critical to account for the transition from the force free, uniform field in the center to much smaller fields outside, via a region of strong average outward forces that carries the current. The transition zone could be narrow; for example, if we had a spherical region with a uniform field and a dipole field outside, both have zero curl and are in current free and hence force free regions [6]. However, all of the current is then carried in the boundary in between these regions, and the integral of the force density continues to be finite from the virial theorem. Models that do not account for these considerations are not in equilibrium and are clearly unphysical. To conclude then, in our view, it is quite premature to construct astrophysical scenarios until at least one equilibrium model of a star has been obtained with such extreme conditions.

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