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Establishing a universal relation between gravitational waves and black hole lensing

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Black hole lensing and gravitational waves are, respectively, closely dependent on the property of the lens and radiation source. In this paper, a universal relation between them is established for a rotating, asymptotically flat black hole acting simultaneously as a lens and a gravitational wave source. The relation only relies on the lens geometry and observable; it does not depend on the specific nature of the black hole. Therefore, the possible gravitational wave sources could be located with modern astronomical instruments from the side of the lensing without knowing the specific nature of the black hole lens. Moreover, the low bound of the gravitational wave frequency can also be well determined for a gravitational wave source.

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I. INTRODUCTION

The theoretical prediction of the black holes in general relativity arouses great interest in studying strong gravitational physics. Black holes are also generally believed to be good test beds for both gravity and quantum mechanics. The existence of many candidates for black holes in our Universe, especially in the center of galaxies, is implied through indirect astrophysical observation.

In general, each black hole is endowed with a Hawking temperature [1,2], determined by its unique nature. However, for a black hole with the same mass as the Sun, its Hawking temperature is 10^{-8} K, which is very difficult to directly observe using the thermodynamic method. Fortunately, the existence of a black hole can sometimes be inferred by observing its gravitational interactions with its surroundings. Ignoring the impact of the small mass particle on the background, the gravitational effects can be well understood with the geodesics of the test particle. Therefore, the geodesic plays a significant role in studying the gravitational effects.

In particular, the unstable circular null geodesic is closely related to the astronomical observation. The corresponding physics includes the quasinormal mode (QNM) frequencies, lensing in strong gravitational fields, and so on. The QNM frequencies [3–5] were considered to be the resonances of a classical scattering problem, i.e., the scattering of particles in the vicinity of a black hole with the boundary conditions of only purely outgoing waves at infinity and ingoing waves at the event horizon. Analytically solving the QNM frequencies through the scattering equation is not easy work. However, in the eikonal limit, Mashhoon [6] presented an analytical method to obtain them. Following this method, the QNM frequencies can be understood with the massless particles propagating along the unstable circular orbit and slowly leaking out to infinity [7–10]. This was thoroughly studied by Cardoso *et al.* in Ref. [11]. The real part of the QNMs is related to the angular velocity of the unstable circular null orbit, and the imaginary part is related to the Lyapunov exponent that determines the instability time scale of the orbit [12–14],

$$\omega_{\mathrm{QNM}} = \Omega_c m - i(n + 1/2)|\lambda|, \tag{1}$$

where m, n, Ω, and λ are the quantum overtone number, angular momentum, angular velocity at the unstable null geodesics, and Lyapunov exponent, respectively. This result was proved to be valid not only for the static, spherical spacetime, but also for the equatorial orbits in the geometry of rotating black hole solutions. A few years ago, the gravitational waves (GWs) were closely linked to the QNM frequencies. In particular, the lowest-lying modes of QNM were found to give a better understanding of the properties of the gravitational wave signal. Thus, the QNMs dominate the gravitational waves radiating from a black hole. On the other hand, GW detection is still a grand challenge due to the lack of strong GW sources with proper frequency.

On the other hand, black hole lensing is more easily observed by modern astronomical instruments, such as the very long baseline interferometry with a reachable resolution of $\sim 10\,\mu$arcsec. In strong gravitational fields, the black hole lensing is related to the unstable circular orbit (for a recent review see [15]). This, to some extent, implies that there is an elegant relation between the black hole lensing and GWs. This conjecture was first made by Decanini and Folacci [16]. And subsequently, the relation was found by Stefanov *et al.* [17] for a static, spherically symmetric black hole in an asymptotically flat spacetime. Considering that a black hole acts simultaneously as a gravitational lens and a source of GWs, one could have an opportunity to obtain the characteristic frequencies of the GWs by examining the property of the black hole lensing.

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This will help us solve the difficulty problem of determining the location of the GW sources [17].

However, a nonrotating object is more an exception than a rule in our Universe; in particular, observation shows that rotating black holes most likely exist [18], and much research illustrates that the dimensionless spin may adopt a larger value, or even approach its practical limitation $a/M = 0.998$ [19,20]. Thus, in order to determine the nature of the black holes or explore the GWs using such a relation [17], one faces the problem that the background should be generalized to a stationary, axisymmetric spacetime, and this is the focus of the current paper. The relation is also applied to the Kerr-Newman (KN) black hole as an example, which helps us enormously in determining the nature of a black hole and detecting the GWs if the rotating black hole acts simultaneously as a gravitational lens and a source of GWs. The low bound of the frequency of the GWs can also be well determined.

This paper is organized as follows. In Sec. II, we briefly review the black hole lensing and quasinormal mode frequency, then a universal relation between them is established for a rotating, asymptotically flat black hole acting simultaneously as a lens and a gravitational wave source. The relation is applied to the KN black hole in Sec. III, and the low bound of the gravitational wave frequency is determined in Sec. IV. The final section is devoted to a brief summary.

II. BLACK HOLE LENSING AND QUASINORMAL MODE FREQUENCY

Generally, the Petrov type-D metric can be written as

$$ds^2 = -\frac{\Delta(r)}{\rho(r,\theta)^2}(dt + P(\theta)d\phi)^2 + \frac{\rho(r,\theta)^2}{\Delta(r)}dr^2 + \rho(r,\theta)^2 d\theta^2 + \frac{\sin^2\theta}{\rho(r,\theta)^2}(a dt + R(r)d\phi)^2, \quad (2)$$

where $\Delta(r)$ and $R(r)$ are the function of r, and $P(\theta)$ is the function of θ, while $\rho(r,\theta)$ is the function of r and θ. Imposing a proper boundary condition, this metric can describe an asymptotically flat black hole with or without the spin parameter. In such a spacetime, there are two Killing fields, $\xi_{t,\phi} = \partial_{t,\phi}$, related to two conserved constants along the geodesics,

$$E = -g_{\mu\nu}\xi_t^\mu p^\nu, \quad (3)$$

$$L = g_{\mu\nu}\xi_\phi^\mu p^\nu, \quad (4)$$

with p^μ the four-momentum of a test particle. E and L represent the energy and angular momentum of the particle. In this paper, we only focus on the equatorial plane ($\theta = \pi/2$), on which the reduced metric can be expressed as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\phi^2 - D(r)dtd\phi. \quad (5)$$

Comparing with (2), the metric functions can be obtained. Then the motion of a photon could be derived following Chandrasekhar [21] with all the coordinates and parameters adimensionalized by the black hole mass M,

$$\dot{t} = 2\frac{2CE - DL}{4AC + D^2},$$
$$\dot{\phi} = 2\frac{DE + 2AL}{4AC + D^2},$$
$$\dot{r}^2 = 4\frac{CE^2 - DEL - AL^2}{B(4AC + D^2)}. \quad (6)$$

The radial motion of a photon can be rewritten as

$$\dot{r}^2 + V_{\mathrm{eff}} = 0, \quad (7)$$

where the effective potential is $V_{\mathrm{eff}} = -4\frac{CE^2 - DEL - AL^2}{B(4AC + D^2)}$. Note that Eq. (7) is very similar to the classical equation of motion with a kinetic energy plus a potential energy. However, the dot here denotes the ordinary differentiation with respect to an affine parameter rather than the coordinate t.

Next, it is worthwhile to introduce an important concept, the unstable circular null geodesic, which is located at the maximum of the effective potential V_{eff}, and satisfies the following conditions,

$$V_{\mathrm{eff}}|_{r=r_c} = 0, \qquad V'_{\mathrm{eff}}|_{r=r_c} = 0, \qquad V''_{\mathrm{eff}}|_{r=r_c} < 0, \quad (8)$$

with r_c the radius of the unstable circular null geodesics. Solving the first condition, we have the minimum angular momentum

$$\tilde{L}_c = \frac{L_c}{E} = \frac{-D_c + \sqrt{4A_cC_c + D_c^2}}{2A_c}, \quad (9)$$

where all the functions are evaluated at r_c. The second condition leads to

$$A_cC'_c - A'_cC_c + \tilde{L}_c(A'_cD_c - A_cD'_c) = 0. \quad (10)$$

Considering the third condition, r_c is found to be the largest root of Eq. (10). For the case that the photon coming from infinity, approaching the nearest distant r_0, and then returning back to infinity, the deflection angle can be expressed as

$$\alpha(r_0) = 2\int_{r_0}^{\infty}\frac{\dot{\phi}}{\dot{r}}dr - \pi. \quad (11)$$

In an asymptotically flat black hole background, the deflection angle increases with the decreasing of r_0 and gets infinity when r_0 approaches r_c. Near r_c, Bozza [22] showed an approximate deflection angle,

$$\alpha(u) = -\bar{a}\ \ln\left(\frac{u}{u_c} - 1\right) + \bar{b} + \mathcal{O}(u - u_c), \quad (12)$$

where the minimum impact parameter u_c and the strong deflection limit coefficients $\bar{a}$, $\bar{b}$ are parametrized with the radius r_c. For our study, we only show the expressions of the parameters u_c and $\bar{a}$:

$$u_c = \tilde{L}_c, \quad (13)$$

$$\bar{a}^2 = \frac{2A_c B_c}{A_c C_c'' - A_c'' C_c + u_c(A_c'' D_c - A_c D_c'')}. \quad (14)$$

Since we have adimensionalized the black hole parameters with the mass M, u_c is a dimensionless quantity. According to the lens geometry presented in Ref. [23], there would be two infinite series of images close to the black hole. Assuming that the source, lens, and observer are highly aligned, there would be two observables. The first one is the minimum angular position θ_∞ of the images, and the second one is the ratio $\tilde{r} = e^{2\pi/\bar{a}}$ between the flux of the first image and the sum of the others, which is related to the relative magnitudes as $r_m = 2.5\ \lg\tilde{r}$. The two observables can be constructed from u_c and $\bar{a}$,

$$\theta_\infty = \frac{u_c}{D_{\mathrm{OL}}}, \qquad r_m = \frac{6.8219}{\bar{a}}. \quad (15)$$

Here D_{OL} denotes the distance between the observer and the lens. Since u_c and $\bar{a}$ depend on the property of the black hole lens, this relation provides an opportunity to check the consistency between the theoretical model and astronomical observations.

Now, let us turn to the QNM frequencies. As pointed out above, in the eikonal limit, the frequencies are in the form of Eq. (1), and the parameters Ω_c and λ are given by [11],

$$\Omega_c = \frac{\dot{\phi}}{\dot{t}}\bigg|_{r_c}, \qquad \lambda = \sqrt{-\frac{V_{\mathrm{eff}}''}{2\dot{t}^2}}\bigg|_{r_c}. \quad (16)$$

Supposing that the equatorial plane of the spacetime can be described by the metric (5), and using Eqs. (6) and (7), the parameters Ω_c and λ can be expressed as [24]

$$\Omega_c = \frac{1}{\tilde{L}_c}, \quad (17)$$

$$\lambda^2 = \frac{A_c C_c'' - A_c'' C_c + \tilde{L}_c(A_c'' D_c - A_c D_c'')}{2A_c B_c \tilde{L}_c^2}. \quad (18)$$

Combining with the gravitational lensing, we can express Ω_c and λ in terms of u_c and $\bar{a}$,

$$\Omega_c = \frac{c}{u_c R_s}, \qquad \lambda = \frac{c}{\bar{a} u_c R_s}, \quad (19)$$

where $R_s = \frac{GM}{c^2}$ with G, c, and M the Newton gravitational constant, speed of light, and mass of the black hole, respectively. Note that here we have restored the dimension. This result implies that the angular velocity at the unstable circular null geodesics is just the inverse of the minimum impact parameter. Further, using the observables from the lensing, we have

$$\Omega_c = \frac{c}{\theta_\infty D_{\mathrm{OL}}},$$
$$\lambda = 0.1468c\,\frac{r_m}{\theta_\infty D_{\mathrm{OL}}}. \quad (20)$$

This is our primary result. It is worthwhile to point out that this relation is dependent on the lens geometry, i.e., the distance D_{OL}, while it is independent of the nature of the black hole. So with predetermined D_{OL}, Ω_c and λ can be accurately determined by the observables θ_∞ and r_m without knowing the detail of the black hole lens.

III. KERR-NEWMAN BLACK HOLES

The KN black hole is the most general vacuum solution of the Einstein-Maxwell equations that can describe the spacetime geometry in the region surrounding a charged, rotating mass. The metric corresponding to it is in the form of (2), and the metric functions are given by

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \qquad P = -a\sin^2\theta, \quad (21)$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad R = -(r^2 + a^2), \quad (22)$$

where a, Q, and M are the spin, charge, and mass of the black hole. So a KN black hole can be characterized by a pair of parameters $(a/M, Q/M)$. Adopting the method described above, the observables θ_∞ and r_m could be obtained for a KN black hole lens. In order to observe a notable lensing, the black hole lens must be a supermassive one. In general, at the center of each galaxy, there is supposed to be a supermassive black hole with mass $M \sim 10^5 - 10^9 M_\odot$, where $M_\odot$ is the solar mass, so the gravitational lensing by it is most easily observed. For example, the supermassive black hole Sgr A* at the center of our Milky Way is estimated to be $M = 2.8 \times 10^6 M_\odot$, and the distance between the observer and the black hole is supposed to be $D_{\mathrm{OL}} = 8.5$ kpc. Supposing that the black hole can be described by the KN metric, the minimum angular position θ_∞ for this case can be calculated as

$$\theta_\infty \approx 9.8557 \times 10^{-6} u_c \left(\frac{M_{BH}/M_\odot}{D_{\mathrm{OL}}/1\ kpc}\right)\mu \times \mathrm{arcsec}. \quad (23)$$

For each pair of parameters $(a/M, Q/M)$, the observables θ_∞ and r_m are uniquely determined. Conversely, the black hole parameters will be determined by θ_∞ and r_m.

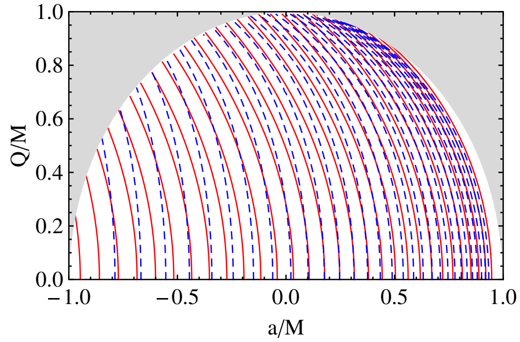

FIG. 1 (color online). Contour plots of the observables θ_∞ and r_m in the $(a/M, Q/M)$ plane. θ_∞ is described by the full lines with values from $16.88 \sim 44.88$ μarcsec from right to left. r_m is described by the dashed lines with values $2.2 \sim 8.8$ from right to left. The light gray zone represents naked singularities.

Therefore, in order to determine the parameters of the black hole, we show in Fig. 1 the contour curves of constant θ_∞ (full lines) and r_m (dashed lines) in the plane $(a/M, Q/M)$. It is clear that each point in the plane is characterized by the values of both θ_∞ and r_m, so this provides an accurate way to determine the black hole parameters using the data from astronomical observations.

Next, we wish to determine the QNM frequencies from the observables θ_∞ and r_m. With the help of Eqs. (19) and (20), we plot the contour curves of the constant θ_∞ (full lines) and r_m (dashed lines) in the plane (Ω_c, λ) in Fig. 2. It is interesting to note that in the figure, the contour curves of the constant θ_∞ and r_m are limited to a belt representing a black hole lens rather than a naked singularity one. The curves of the constant θ_∞ are vertical lines in the figure due to the fact that $\Omega_c \propto 1/\theta_\infty$. Each point in this belt is characterized by the values of θ_∞ and r_m. Thus, with the data form astronomical observations, we can get the expected value of the angular velocity and Lyapunov exponent of the unstable circular null geodesics.

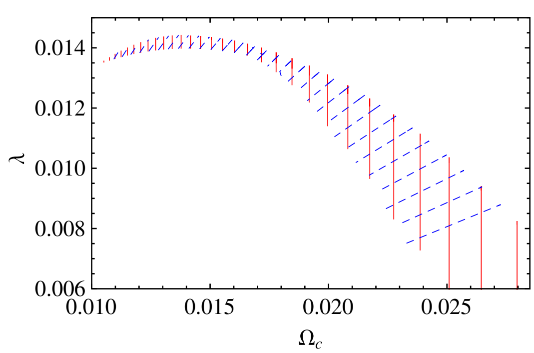

FIG. 2 (color online). Contour plots of the observables θ_∞ and r_m in the (Ω_c, λ) plane. θ_∞ is described by the full lines with values from $16.88 \sim 44.88$ μarcsec from right to left. r_m is described by the dashed lines with values $2.2 \sim 8.8$ from right to left.

IV. GRAVITATIONAL WAVES FROM ASTRONOMICAL OBSERVATION

The QNM frequency is a complex number, and its imaginary part will decay when the wave travels out far from the black hole. Therefore, for an observer located at infinity, only the real part of the QNM frequency is meaningful, which should be related to the GWs. Based on this conjecture for Sgr A*, the possible range of the GWs can be seen in Fig. 2 and is approximately

$$f \sim (1 \sim 3)m \times 10^{-2}\ \text{Hz}. \tag{24}$$

Here m is an integer. Since the relation (20) is independent of the nature of the black hole lens, we see that this is a universal result for the black hole located at the center of the Milky Way.

For $m \sim 10^2$, we look forward to obtaining the GWs in the range of several Hz. With the further increase of m, one can get the GWs of ~kHz. If this is true, we can use the tunable resonant sensor proposed in Ref. [25] to directly detect GWs, which bases on the optically trapped and cooled dielectric microspheres or micro-discs, and is believed to be effective in the range of 50–300 kHz. On the other hand, Eq. (24) also gives a low bound f_{lb} $(m = 1)$ for the GWs

$$f_{\text{lb}} \sim 10^{-2}\ \text{Hz}. \tag{25}$$

Thus, there is no hope of observing the GWs with $f < 10^{-2}$ Hz from the black hole at the center of the Milky Way.

This low bound of frequency for the GWs can also be extended to other black holes. From the relation (20), we find the low bound f_{lb} depends on the distance D_{OL} and the observable θ_∞. In Fig. 3, we show the low bound f_{lb}.

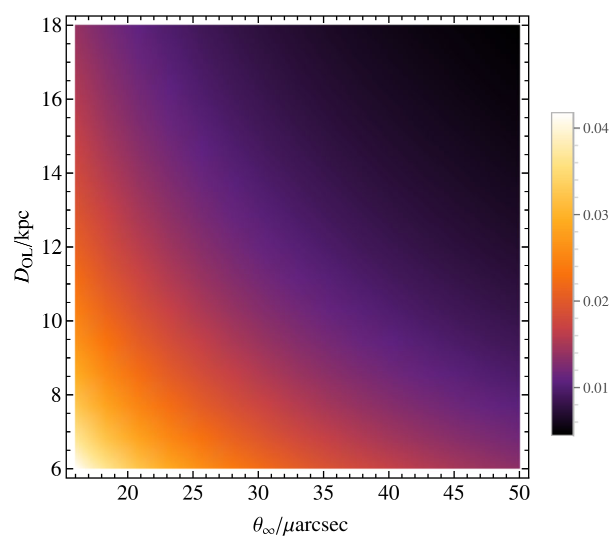

FIG. 3 (color online). Behavior of the low bound f_{lb} for different values of D_{OL} and θ_∞.

With D_{OL} varying from 6 to 18 kpc, and θ_∞ from 10 to 50 μarcsec, f_{lb} decreases from 0.04 to 0.005 Hz. Thus given different values of D_{OL} and θ_∞, we get the value of the low bound f_{lb} from this figure immediately.

V. CONCLUSION

In this paper, we have shown an elegant relation between the GWs and black hole lensing with the assumption that the black hole acts simultaneously as a GW source and a gravitational lens. This relation is effective for the black hole with or without the spin parameter in an asymptotically flat spacetime. It is also found to be independent of the nature of the black hole lens. Thus we could obtain the information on the GW emitted by the black hole through its lensing geometry and observable, which are more likely to be observed by modern astronomical instrument. The low bound f_{lb} of the frequency of the GWs is given. Although directly detecting GWs is a challenge, the result provides a possible way to test it, or at least to locate the GW sources that are most likely to be observed by optical or radio telescopes with the proper frequency range.

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