Phase transition and entropy spectrum of the BTZ black hole with torsion

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In this paper, we study the phase transition and the entropy spectrum of the Bañados-Teitelboim-Zanelli (BTZ) black hole with torsion obtained in $(1+2)$-dimensional Poincaré gauge theory. By calculating the heat capacity, we find that the BTZ black hole with torsion we consider will experience phase transition at some critical point. This indicates that the critical behaviors of black holes do not only depend on the geometry of spacetime, but also have to do with the theory of gravity under consideration. In addition, we derive the entropy spectrum of the BTZ black hole according to the quasinormal modes and the adiabatic invariance. It shows that the area or entropy spectrum will also rely on the concrete gravitational action.

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I. INTRODUCTION

Black holes as a prediction of general relativity (GR) have played a great role in the development of modern theoretical physics. Specifically, the discovery of Hawking radiation [1] places black hole physics as a subject at the intersection of theories of gravity, quantum mechanics, statistical physics, and field theory. Since then, the thermodynamics of black holes has received a lot of attention, and the thermodynamic properties of many black holes have been studied extensively [2–6]. Among them, the BTZ black hole in three-dimensional gravity is of particular importance [7,8]. In lower spacetimes, many physical problems can be quite simpler.

In general relativity and many other theories of gravity, curvature plays an essential role, while torsion has received less attention. However, torsion also has its geometrical meaning and plays some role in gravitation theory. Since the 1970s, many theories of gravity with torsion have been proposed, such as Poincaré gauge gravity, de Sitter gauge gravity, teleparallel gravity, $f(T)$ gravity, etc. In particular, Mielke and Baekler proposed a model of three-dimensional gravity with torsion (the MB model), which also has a BTZ-like black hole as a solution except for torsion [9,10]. This model aroused the following research on the thermodynamics of the BTZ black hole with torsion and AdS/CFT with torsion [11–13].

Phase transitions and critical phenomena are important characteristics of ordinary thermodynamics. Thus, the natural question to ask is whether there also exists a phase transition in black hole thermodynamics. Because black holes also have the standard thermodynamic quantities, such as temperature, entropy, and even pressure and volume [14–16], their critical phenomena may be similar to ones in the ordinary thermodynamic system. Until now, there have mainly been two different views about the phase transition of black holes in equilibrium. The first one, which is originated from Davies [17], insists that the phase transition of black holes turns up at the point where the heat capacities diverge. This characteristic is also present in ordinary thermodynamic systems which exhibit second-order phase transitions. Many researches on this topic have been carried out [18–21]. In this sense, there is no phase transition for the BTZ black hole obtained in GR, because the heat capacity of the ordinary BTZ black hole is always positive. Recently, based on Hawking-Page phase transition [22], some interesting works on asymptotically anti–de Sitter black holes have been done, which show that there exist phase transitions similar to the van der Waals–Maxwell vapor-liquid phase transition [16,23–27]. Another point of view thinks the phase transition takes place in the extremal limit of black holes. According to the fluctuation theory of equilibrium thermodynamics, some second moments of relevant quantities for Schwarzschild black holes, Kerr black holes, RN black holes, and (charged) dilaton black holes have been calculated [28–31]. It shows that nothing special happens for them at the Davies points. But the second moments will diverge when nonextremal black holes become extremal ones. Moreover, this kind of phase transition may be relevant to the quasinormal modes of black holes [32–35].

In this paper, we first study the phase transition and critical behavior of the BTZ black hole obtained in the MB model. Although the metric part of the BTZ black hole with torsion is the same as that obtained in GR, the different actions will make their thermodynamics very different. The modified action in the MB model will correct the conserved charges such as mass and angular momentum. Correspondingly, the entropy of the BTZ black hole and the first law of black hole thermodynamics will also be changed. Based on these results, one can reanalyze the phase transitions of the BTZ black hole in the two senses mentioned above. It shows that the heat capacity of the BTZ black hole in the MB model is not always positive

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anymore, but changes signs at some points and may diverge at the critical point. Thus, for the BTZ black hole in the MB model, phase transition exists. Next, according to the fluctuation theory of thermodynamics, we recalculate some second moments. They are still finite in the nonextremal case and diverge in the extremal case. This indicates that the second viewpoint, which insists that the phase transition takes place in the extremal limits of black holes, still holds. Moreover, according to the quasinormal modes (QNMs) and the adiabatic invariance, we study the area spectrum and entropy spectrum of the nonrotating BTZ black hole and rotating BTZ black hole in the MB model.

The paper is arranged as follows: in the next section, we simply introduce the MB model and its BTZ black hole solution and the corresponding thermodynamic quantities. In Sec. III, we calculate the heat capacity at constant angular momentum and some second moments, by which we discuss the phase structure and critical phenomena of the BTZ black holes. In Sec. IV, the area spectrum and entropy spectrum of the BTZ black hole are given. We make some concluding remarks in Sec. V.

II. MIELKE-BAEKLER MODEL AND THERMODYNAMIC QUANTITIES OF THE BTZ BLACK HOLE

To express theories of gravity, one can employ two dynamically independent 1-forms: the connection $\omega^a{}_b$ and coframe e^a. In GR, these two fields are assumed to be linked by the torsion-free condition $T = 0$. We can define curvature and torsion 2-forms out of $\omega^a{}_b$ and coframe e^a by

$$T^a = de^a + \omega^a{}_b \wedge e^b, \tag{2.1}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{2.2}$$

In geometry, $\omega^a{}_b$ and e^a reflect the affine and metric properties of the manifold, while in physics, the torsion and curvature may be related to the energy momentum tensor and spin current, respectively.

Now we should introduce the topological three-dimensional gravity model with torsion proposed by Mielke and Baekler [9,10], which is a natural generalization of Riemannian GR with a cosmological constant. The gravitational action is usually written as

$$\begin{aligned} I = \int & 2ae^a \wedge R_a - \frac{\Lambda}{3}\varepsilon_{abc} e^a \wedge e^b \wedge e^c \\ & + \alpha_3 \left(\omega^a \wedge d\omega_a + \frac{1}{3}\varepsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c \right) + \alpha_4 e^a \wedge T_a, \end{aligned} \tag{2.3}$$

where the dual expressions R_a and ω_a are defined by $R^{ab} = \epsilon^{abc} R_c$ and $\omega^{ab} = \epsilon^{abc}\omega_c$. In Eq. (2.3), the first term corresponds to the Einstein-Cartan action, with $a = \frac{1}{16\pi G}$. The second term is the cosmological term, which is a constant. The last two terms are the Chern-Simons term and the Nieh-Yan term, which should be given particular attention.

Using curvature, we can construct characteristic classes, integrations of which will be topological invariants of the manifold, which reflect the global properties of the manifold. The well-known characteristic classes for a four-dimensional real manifold are the Pontrjagin and Euler classes [36]:

$$P = \frac{1}{8\pi^2} R^{ab} \wedge R_{ab}, \tag{2.4}$$

$$E = \frac{1}{32\pi^2}\varepsilon_{abcd} R^{ab} \wedge R^{cd}. \tag{2.5}$$

The integrations of P and E on the manifold take integer numbers, which distinguish topologically different manifolds.

It is well known that locally

$$\begin{aligned} P_4 &= dQ, \\ Q &= \omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega \end{aligned} \tag{2.6}$$

is the local Chern-Simons (CS) form, which has many applications in physics. One of the important feature of CS modified gravity is its emergence within predictive frameworks of more fundamental theories. For example, the low-energy limit of string theory comprises general relativity with a parity-violating correction term that is nothing but the Pontryagin density. This term is crucial for canceling gravitational anomaly in string theory through the Green-Schwartz mechanism.

The Nieh-Yan (N-Y) form is a special 2-form only for the Riemann-Cartan manifold [37–39]. On the four-dimensional manifold M it can be written as

$$\begin{aligned} N &= T^a \wedge T_a + R_{ab} \wedge e^a \wedge e^b = dQ_{\mathrm{NY}}, \\ Q_{\mathrm{NY}} &= e^a \wedge T_a. \end{aligned} \tag{2.7}$$

The N-Y form is a kind of Chern-Simons form and will have its application to manifolds with boundaries and reflect the role of torsion in geometry.

After variation to the connection $\omega^a{}_b$ and vielbein e^a, two vacuum equations can be obtained from the MB action:

$$T^a = \frac{p}{2}\varepsilon^a{}_{bc} e^b \wedge e^c, \tag{2.8}$$

$$R^a = \frac{q}{2}\epsilon^a_{bc} e^b \wedge e^c, \tag{2.9}$$

with the two constant coefficients p, q defined by $p = \frac{\alpha_3\Lambda+\alpha_4 a}{\alpha_3\alpha_4-a^2}$ and $q = -\frac{\alpha_4^2+a\Lambda}{\alpha_3\alpha_4-a^2}$.

The curvatures in Riemann-Cartan geometry can be connected to their counterparts in Riemannian geometry. In particular, in three-dimensional spacetime, the equations above can be simplified to equations without torsion

$$\tilde{R}^a = \frac{\Lambda_{\rm eff}}{2}\varepsilon^a{}_{bc}e^b\wedge e^c, \tag{2.10}$$

where $\tilde{R}^a$ is the curvature without torsion and $\Lambda_{\rm eff} = q - \frac{1}{4}p^2$ is the effective cosmological constant. One can let $\Lambda_{\rm eff} = -\frac{1}{l^2} < 0$ to construct an asymptotically anti–de Sitter space.

As in the three-dimensional Einstein equation, Eq. (2.10) has the well-known BTZ solution. But in this case, torsion is contained in the gravitational action. The metric is

$$ds^2 = -N(r)^2dt^2 + \frac{1}{N(r)^2}dr^2 + r^2(d\phi + N_\phi dt)^2, \tag{2.11}$$

where

$$N(r)^2 = \frac{r^2}{l^2} - M_0 + \frac{J_0^2}{4r^2}, \qquad N_\phi(r) = \frac{J_0}{2r^2}. \tag{2.12}$$

Here we have considered $8G = 1$. This metric is the same as the one in GR, except that $l = 1/\sqrt{-\Lambda_{\rm eff}}$ here. For this metric, there are two horizons: the outer one, r_+, and the inner one, r_-. From $N^2(r) = 0$, one can obtain the expressions of both horizons:

$$r_\pm^2 = \frac{M_0 l^2}{2}(1\pm\Delta), \qquad \Delta = [1-(J_0/M_0 l)^2]^{1/2}. \tag{2.13}$$

Conversely, M_0 and J_0 can be expressed as follows:

$$M_0 = \frac{r_+^2 + r_-^2}{l^2}, \qquad J_0 = \frac{2r_+r_-}{l}, \tag{2.14}$$

because Hawking radiation is just a kinematic effect, which only depends on the event horizon and is irrelevant to the dynamical equations and the gravitational theories. Therefore, the temperature of BTZ black hole in the MB model has a similar form to that in GR, which is

$$T = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}. \tag{2.15}$$

Certainly, because of the existence of l, the temperature is relevant to the coefficients α_3, α_4, Λ of the MB Lagrangian. We define

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r_+} = \frac{J_0}{2r_+^2}, \tag{2.16}$$

which can be regarded as the angular velocity of the BTZ black hole. The entropy of the BTZ black hole in GR can be obtained by means of the Euclidean action method, which is

$$S_0 = 4\pi r_+, \tag{2.17}$$

which is twice the perimeter of the horizon. In fact, the entropy can also be derived by employing the first law of black hole thermodynamics. Because black holes are also thermodynamic systems, they must obey the first law of thermodynamics. For the BTZ black hole in GR, it is

$$dM_0 = TdS_0 + \Omega_H dJ_0. \tag{2.18}$$

Thus, the entropy can be derived according to T and M_0, namely

$$S_0 = \int T^{-1}dM_0 + C_0 = \int T^{-1}\left(\frac{\partial M_0}{\partial r_+}\right)\bigg|_{J_0} dr_+ + C_0. \tag{2.19}$$

It should be noted that the J_0 in the integration should be taken as constant. One can express M_0 and T as functions of r_+ and J_0, which are

$$M_0 = \frac{r_+^2}{l^2} + \frac{J_0^2}{4r_+^2}, \qquad T = \frac{r_+^2 - \frac{l^2J_0^2}{4r_+^2}}{2\pi l^2 r_+}. \tag{2.20}$$

Substituting them into Eq. (2.19), one can easily obtain the result in Eq. (2.17).

For the MB model, we can also employ this method to derive the entropy of the BTZ black hole with torsion. But things are slightly different. Blagojevic *et al.* have proved that the gravitational conserved charges in the MB model should be [40]

$$\begin{aligned} M &= M_0 + 2\pi\alpha_3\left(\frac{pM_0}{2} - \frac{J_0}{l^2}\right), \\ J &= J_0 + 2\pi\alpha_3\left(\frac{pJ_0}{2} - M_0\right). \end{aligned} \tag{2.21}$$

Obviously, when $\alpha_3 = 0$ they will return to their conventional interpretations as energy and angular momentum, as with the BTZ metric in general relativity. In this case, if the first law of black hole thermodynamics still holds, the relation should be

$$dM = TdS + \Omega_H dJ. \tag{2.22}$$

Therefore, M and J should be used in Eq. (2.19):

$$S = \int T^{-1} dM + C = \int T^{-1} \left(\frac{\partial M}{\partial r_+} \right) \bigg|_J dr_+ + C. \tag{2.23}$$

One thing to note is that in the integral, r_- cannot be regarded as constant but depends on r_+ and J. The entropy can be obtained easily:

$$S = 4\pi r_+ + 4\pi^2 \alpha_3 \left(p r_+ - \frac{2r_-}{l} \right). \tag{2.24}$$

This result agrees with the one Blagojevic *et al.* obtained using the partition function with the aid of the Euclidean action method [12]. It differs from the Bekenstein-Hawking result [41,42] by an additional term but coincides with Solodukhin's result if $p = 0$ [43]. Black hole entropy is not always equated with one quarter of the event horizon area. In fact, it is related to the gravitational theory under consideration. It can be easily verified that in the MB model the entropy, temperature, and the conserved charges not only satisfy the first law of thermodynamics but also fulfill the Smarr-like formula

$$M = \frac{1}{2} T S + \Omega_H J. \tag{2.25}$$

This further implicates that with torsion, the BTZ black hole can still be seen as a thermodynamic system and the thermodynamic laws still hold. It should be noted that in the expression of the entropy of the BTZ black hole, no torsion exists explicitly; only α_4 in p implicitly. In particular, when $\alpha_3 = 0$, the entropy in Eq. (2.24) returns to the usual BTZ black hole entropy. It means that the N-Y term $\alpha_4 e^a \wedge T_a$ influences the BTZ metric and the exact form of entropy only when the CS term exists.

III. PHASE TRANSITION OF THE BTZ BLACK HOLE

It is well known that the heat capacity of BTZ black holes in GR is always positive. This point makes it different from its counterpart, Kerr black holes in 2 + 1 dimensions, the heat capacity of which exhibits a discontinuous jump from negative to positive. In the MB model, the heat capacity at constant angular momentum J of BTZ black holes can be expressed as

$$C_J = \frac{\partial M}{\partial T} \bigg|_J, \tag{3.1}$$

where M, J, and T are given in Eqs. (2.15) and (2.21). Combining Eq. (2.15) with Eq. (2.13), one can derive a relation

$$2\pi^2 l^2 T^2 (1 + \Delta) = M_0 \Delta^2. \tag{3.2}$$

From Eq. (2.21), one can express M_0, J_0 as functions of M, J:

$$M_0 = \frac{(1 + \pi\alpha_3 p) M + 2\pi\alpha_3 J / l^2}{A},$$
$$J_0 = \frac{(1 + \pi\alpha_3 p) J + 2\pi\alpha_3 M}{A}, \tag{3.3}$$

where $A = (1 + \pi\alpha_3 p)^2 - 4\pi^2 \alpha_3^2 / l^2$ is a constant.

One can easily obtain the heat capacity at constant angular momentum:

$$C_J = \frac{4\pi r_+ \Delta}{B(2 - \Delta) - D(2 + \Delta)(1 - \Delta)/\sqrt{1 - \Delta^2}}, \tag{3.4}$$

where $B = \frac{1 + \pi\alpha_3 p}{A}$, $D = \frac{2\pi\alpha_3}{Al}$ are both constants. Obviously, when $\alpha_3 = 0$, it will return to the usual heat capacity of BTZ black holes in GR, which is $C_{J_0} = \frac{4\pi r_+ \Delta}{2 - \Delta}$ [44]. When $J_0 = 0$, namely nonrotating BTZ black holes, $\Delta = 1$; thus $C_J = C = 4\pi r_+ / B = 4\pi l \sqrt{M_0} / B$. When $J_0 = M_0 l$, namely the extremal case, $\Delta = 0$; thus $C_J = 0$ ($B \neq D$), which corresponds to the vanishing Hawking temperature.

Several comments are needed. First, although when $\alpha_3 = 0$, the conserved charges, entropy, the heat capacity of BTZ black holes in the MB model all have expressions similar to those of BTZ black holes in GR, this will not make the theory the same as the GR because of the different actions. Do not forget that l is just an effective cosmological radius, which is dependent on the parameters α_3, α_4, Λ. In fact, they will still be different so long as α_4 is not equal to zero. Moreover, the existence of the Nieh-Yan term in the action reminds us that the theory under consideration is a gravitational theory with torsion. Correspondingly, the geometry on which the theory is based is Einstein-Cartan geometry. Second, the heat capacity at constant angular momentum for the usual BTZ black hole in GR is always positive because $\Delta \in [0, 1]$. But the heat capacity in the MB model can be negative or positive, which indicates the existence of a second-order phase transition in the sense of Davies [17], who first studied the heat capacity of Kerr-Newman black holes and claimed that a second-order phase transition will emerge when black holes get across the discontinuity point of heat capacity.

Below, we will discuss the heat capacity and analyze the critical phenomena of BTZ black holes in the MB model. From Eq. (3.4), in order to have a divergence in C_J, the following condition must be satisfied:

$$B(2 - \Delta) - D(2 + \Delta)(1 - \Delta)/\sqrt{1 - \Delta^2} = 0. \tag{3.5}$$

It can be easily verified that $(2 - \Delta) \geq (2 + \Delta)(1 - \Delta)/\sqrt{1 - \Delta^2}$ always succeeds because of $0 \leq \Delta \leq 1$, thus $B \leq D$. This means that the conserved charge M can be negative. Because of the parameters M_0, l, B, D the analytical solution for the heat capacity is a little problematic. Thus, we try to give some numerical solutions by assigning some values to the parameters. We can take B, D, l or α_3, α_4, Λ as the free parameters. Here we use the former set of parameters for simplicity. Firstly, we set $B = 0.5$, $D = 0.6$. The critical value of Δ_c can be easily obtained, $\Delta_c = 0.748048$. According to Eq. (2.13), the corresponding critical value of r_+ is dependent on M_0 and l. We should express C_J as a function of r_+ with the parameters M_0, l, B, D:

$$C_J = \frac{4\sqrt{2}\pi r_+^2/l\sqrt{M_0}}{\sqrt{2}B(3 - 2r_+^2/M_0 l^2)r_+/l\sqrt{M_0} - \sqrt{2}D(1 + 2r_+^2/M_0 l^2)\sqrt{1 - r_+^2/M_0 l^2}}. \tag{3.6}$$

Obviously, in order to have real roots, $1 - r_+^2/M_0 l^2 \geq 0$ must be satisfied, namely $0 \leq r_+ \leq l\sqrt{M_0}$.

Numerical analysis shows that there are six roots for r_+: four imaginary roots, one negative real root and one positive real root r_e, which is just the critical one. In Table I, we give the critical value of r_+ where the phase transition occurs. According to the table, it is evident that the value of r_e depends on the choices of the parameters M_0, l, B, D. For fixed B, D, l, a greater M_0 gives a greater r_+. Similarly, for fixed B, D, M_0, the bigger the values of l are, the greater the r_e is. Moreover, the choices of B, D have no significant influence on the values of r_+. These values of r_+ in Table I all satisfy the condition $r_+ \leq l\sqrt{M_0}$, obviously.

Figure 1 shows the relations between C_J and r_+, which reveal that the phase transition occurs at the critical point r_e. In the figures, one can see that besides the transition point r_e, there exists another point r_0 where the heat capacity will change from positive to negative. The heat capacity is positive when $r_+ < r_0$ and $r_+ > r_e$, while it is negative for $r_0 < r_+ < r_e$. This characteristic is a little like the critical behavior of Born-Infeld AdS black holes [45]. Nevertheless, the point r_0 is not the transition point, because no discontinuity of heat capacity turns up.

Besides the phase transition in the sense of Davies above, there is another viewpoint that the phase transition will take place from the extremal to nonextremal black holes [28–31,44]. According to the fluctuation theory of equilibrium thermodynamics, employing the Massieu function Φ to characterize the equilibrium states of a thermodynamic system,

TABLE I. Numerical solutions for r_+ for given values of M_0, l, B, D based on the conditions $B \leq D$ and $0 \leq \Delta \leq 1$.

B	D	M_0	l	r_e
0.5	0.6	1	1	0.934893
0.5	0.6	0.5	1	0.661069
0.5	0.6	2	1	1.32214
0.5	0.6	1	0.5	0.467446
0.5	0.6	1	2	1.86979
0.4	0.6	1	1	0.967353
0.2	0.6	1	1	0.993451

$$d\Phi = \sum_{i=1}^{n} X_i dx_i, \tag{3.7}$$

where the variables x_i are called intrinsic variables and are determined by the thermodynamic environment. The X_i are the conjugate variables of x_i [46,47]. Below, we will analyze the equilibrium fluctuations of BTZ black holes in the MB model in the microcanonical ensemble. In the microcanonical ensemble, nothing can be exchanged with the environment. The proper Massieu function is the entropy of the system. For the BTZ black hole under consideration, it will be

$$d\Phi = dS = \beta dM - \mu dJ, \tag{3.8}$$

where $\beta = 1/T$ and $\mu = \beta\Omega_H$. Comparing the above equation with Eq. (3.7), the intrinsic variables are $\{M, J\}$, and the corresponding conjugate variables are $\{\beta, -\mu\}$. One can easily derive the eigenvalues for the fluctuation mode

$$\lambda_M = \frac{\partial M}{\partial \beta}\bigg|_J = -T^2 C_J, \qquad \lambda_J = -\frac{\partial J}{\partial \mu}\bigg|_M = -T I_M, \tag{3.9}$$

where C_J is the heat capacity and

$$I_M \equiv \beta \frac{\partial J}{\partial \mu}\bigg|_M = \left[\frac{B}{2r_+^2} - \frac{M_0 \mu \Delta}{2\pi r_+^2}\frac{\partial r_+}{\partial J}\bigg|_M - \frac{D\mu\Delta}{2\pi l r_+} - \frac{M_0\mu}{2\pi r_+}\frac{\partial \Delta}{\partial J}\bigg|_M\right]^{-1} \tag{3.10}$$

is the moment of inertia of BTZ black holes, where

$$\frac{\partial r_+}{\partial J}\bigg|_M = \frac{lD(1+\Delta) + M_0 l^2 \frac{\partial \Delta}{\partial J}|_M}{4r_+}, \qquad \frac{\partial \Delta}{\partial J}\bigg|_M = \frac{D}{l}\frac{1-\Delta^2}{\Delta M_0} - \frac{B(1-\Delta^2)}{\Delta J_0}. \tag{3.11}$$

Some second moments can be easily obtained:

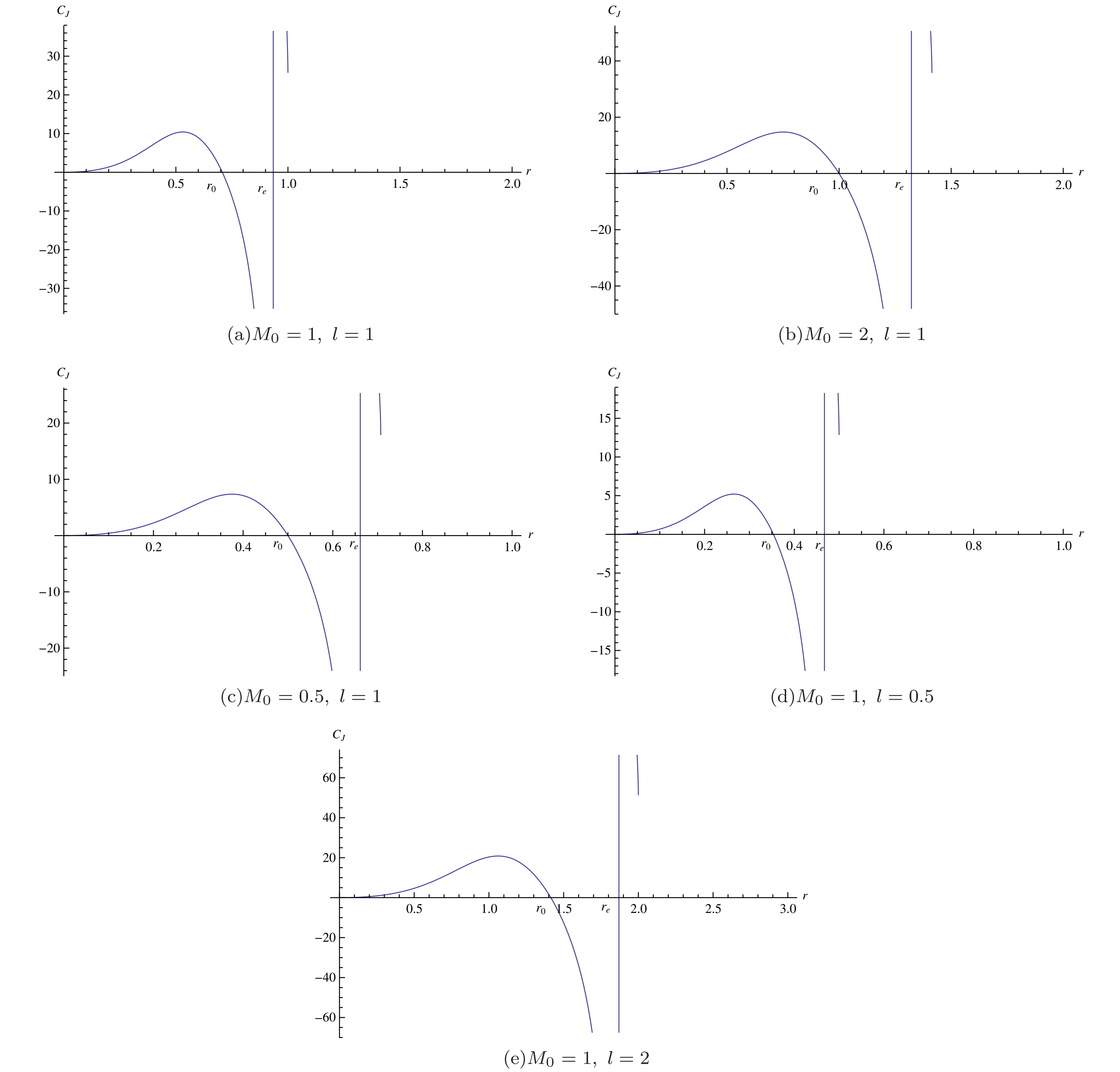

FIG. 1 (color online). Discontinuity of heat capacity C_J for the BTZ black hole with torsion in the MB model with $B = 0.5$, $D = 0.6$ at $r_+ = 0.934893$, 1.32214, 0.661069, 0.467446, and 1.86979 [(a)–(e), respectively].

$$\langle \delta\beta\delta\beta \rangle = -\frac{k_B \beta^2}{C_J}, \qquad \langle \delta\mu\delta\mu \rangle = -\frac{k_B \beta}{I_M},$$
$$\langle \delta\Omega_H \delta\Omega_H \rangle = -k_B \left[\frac{T}{I_M} + \frac{\Omega_H^2}{C_J} \right], \qquad \langle \delta\Omega_H \delta\beta \rangle = \frac{k_B \beta \Omega_H}{C_J}. \tag{3.12}$$

Thus, for nonextremal black holes, the second moments are all finite. When the extremal limit $\Delta = 0$ is approached, because of $C_J = 0$, $I_M = 0$, the second moments above will diverge. The divergence means the extremal BTZ black hole is a critical point. This conclusion is the same as the one for the BTZ black hole in GR. In particular, for the BTZ black hole in the MB model, the entropy is also a homogeneous function of M, J. Therefore, the scaling laws at the critical point are still expected to be satisfied.

At last, we mention the special case $B = D$. According to the heat capacity [Eq. (3.4)], in the extremal case it is

$$C_J = \lim_{\Delta \to 0} \frac{4\pi r_+ \Delta}{B(2-\Delta) - D(2+\Delta)(1-\Delta)/\sqrt{1-\Delta^2}}. \tag{3.13}$$

If $B \neq D$, it will give $C_J = 0$. When the condition $B = D =$ finite value is satisfied, it shows that $C_J \to \infty$. Thus, in this case, the divergence point of heat capacity is just the extremal point. But unfortunately, $B = D$ cannot be satisfied. According to their definition, $B = D$ means $A = 0$. Thus, $B = D = \infty$, which is irrelevant. Therefore, the two kinds of phase transition cannot be unified in one form.

IV. ENTROPY SPECTRUM AND THE AREA SPECTRUM OF THE BTZ BLACK HOLE

In what follows, we firstly concentrate on analyzing a nonrotating BTZ black hole. When requiring $J_0 = 0$, the inner horizon r_- vanishes. Thus, the angular velocity of BTZ black hole $\Omega_H = 0$. The first law of thermodynamics turns out to be

$$dM = TdS. \tag{4.1}$$

Based on these conditions, we calculate the area spectrum of the BTZ black hole in the MB model according to QNMs.

Recently, the Dirac quasinormal modes of a nonrotating BTZ black hole with torsion in the MB model have been obtained [48]:

$$\omega = \frac{\xi}{l} - i\sqrt{M_0}\left(\frac{1}{2l} + \frac{2}{l}n + m + \frac{3p}{4}\right),$$
$$\omega = -\frac{\xi}{l} - i\sqrt{M_0}\left(\frac{3}{2l} + \frac{2}{l}n + m + \frac{3p}{4}\right), \quad n = 0, 1, 2\ldots \tag{4.2}$$

where ξ corresponds to the eigenvalue of the Dirac operator.

According to Kunstatter [49], there exists an adiabatic invariant

$$I = \int \frac{dE}{\omega(E)}. \tag{4.3}$$

In general, dE in the nominator can be set equal to dM_0, but for the BTZ black hole with torsion, this does not hold anymore. From Eq. (2.21),

$$dE = dM = (1 + \pi\alpha_3 p)dM_0. \tag{4.4}$$

Fortunately, although the energy E is not equal to the mass M_0, the first law of black hole thermodynamics still holds. Moreover, in the large-n limit, via Bohr-Sommerfeld quantization, the adiabatic invariant can be expressed as

$$I = n\hbar. \tag{4.5}$$

If considering the real part of the quasinormal modes as the fundamental vibrational frequency for the BTZ black hole with torsion. one can obtain $\omega(E) \approx \omega_R = \pm\xi/l$. Thus,

$$I = \int \frac{(1 + \pi\alpha_3 p)dM_0}{\omega_R} = \pm\frac{l(1 + \pi\alpha_3 p)M_0}{\xi} + C, \tag{4.6}$$

where C is a integration constant. The mass spectrum is still equally spaced with the modified coefficient

$$M_0 = \pm\frac{\xi\hbar n}{l(1 + \pi\alpha_3 p)}. \tag{4.7}$$

The mass spacing is $\Delta M_0 = \pm\frac{\xi\hbar}{l(1+\pi\alpha_3 p)}$. According to Eq. (2.13) and $A = 2\pi r_+$, the area spectrum is not equally spaced. This case is like a generalization of the result in Ref. [50].

In the viewpoint of Maggiore, one can treat the perturbed black holes as a damped harmonic oscillator [51]. Thus, the physically relevant vibrational frequency should be

$$\omega(E) = \sqrt{\omega_R^2 + |\omega_I^2|}, \tag{4.8}$$

where ω_R and ω_I are, respectively, the real and imaginary parts of the quasinormal mode frequency. In the large damping case, $\omega_R \ll \omega_I$, therefore,

$$\omega(E) \approx |\omega_I|. \tag{4.9}$$

In the semiclassical limit, the characteristic frequency can be identified with a transition between adjacent quasinormal frequencies; namely, the $\omega(E)$ in Eq. (4.3) should be replaced by

$$\Delta\omega = (|\omega_I|)_n - (|\omega_I|)_{n-1} = 2\sqrt{M_0}/l. \tag{4.10}$$

Using the $\Delta\omega$, we can obtain the adiabatic invariant

$$I = l(1 + \pi\alpha_3 p)\sqrt{M_0} = n\hbar \tag{4.11}$$

or

$$\sqrt{M_0} = \frac{n\hbar}{l(1 + \pi\alpha_3 p)}. \tag{4.12}$$

According to $r_+ = l\sqrt{M_0}$, one can easily find that the horizon area is quantized and equally spaced as follows:

$$A = 2\pi r_+ = \frac{2\pi n\hbar}{1 + \pi\alpha_3 p}. \tag{4.13}$$

For a nonrotating BTZ black hole in the MB model, the black hole entropy is quantized and can be expressed as

$$S = (1 + \pi\alpha_3 p)4\pi r_+ = 4\pi\hbar n. \tag{4.14}$$

Without using the QNMs, one can also calculate the entropy spectrum of black holes via adiabatic invariance [52]. The adiabatic covariant action is

$$I_{\text{adia}} = \oint p_i dq_i = 2\pi nh. \tag{4.15}$$

Employing the Hamiltonian H of the system, which is the total energy of the black hole, and the corresponding

Hamiltonian equations, for an ordinary rotating BTZ black hole, the adiabatic covariant action can be expressed as [53,54]

$$I_{\text{adia}} = \oint p_i dq_i = \hbar \int_{(0,0)}^{(M_0,J_0)} \frac{dM_0' - \Omega_H dJ_0'}{T} = \hbar S_0. \tag{4.16}$$

For the BTZ black hole in the MB model, the conserved charges are modified, and the Hamiltonian of the system should be M but not M_0. According to Eq. (2.22), Eq. (4.16) still holds, except that M_0, J_0, S_0 change into M, J, S. Thus, the entropy spectrum of the rotating BTZ black hole in the MB model is

$$S = 2\pi n, \quad n = 1, 2, 3\ldots \tag{4.17}$$

According to Eq. (2.24), the area spectrum is not quantized because of the existence of r_-. But for the nonrotating case, the entropy spectrum and area spectrum are both quantized and equally spaced.

V. DISCUSSION AND CONCLUSIONS

In this paper, we studied the phase transition of the BTZ black hole obtained in the MB topological gravitational model. Although the gravitational action contains torsion, the metric part of the BTZ solution with torsion looks like the usual BTZ solution. Because of the existence of the Chern-Simons term and the Nieh-Yan term, the conserved charges for the BTZ black hole should be corrected. It is these corrections that lead to the different thermodynamic properties and critical behavior.

By calculating the heat capacity at constant angular momentum, we find that it is not always positive anymore, but can change signs. Thus, at the points where the heat capacity diverges, the phase transition happens. This means that the phase transition of black holes depends on the concrete theory of gravity. For the same BTZ black hole in different theories of gravity, there are different critical behaviors. Assigning some values to the free parameters B, D, l, M_0, we obtain some phase-transition points from a numerical analysis. From the figures, one can see that two points exist where the heat capacity changes signs. At the point r_0, the heat capacity changes from positive to negative, but it does not diverge. At another point, r_e, however the heat capacity will diverge and change from negative to positive. Therefore, r_e is the critical point where the phase transition turns up. Moreover, we calculate some second moments of the thermodynamic quantities for the BTZ black hole in the microcanonical ensemble. It shows that they are finite in the nonextremal case and diverge in the extremal limit of BTZ black hole. It means the phase transition happens from the extremal to the nonextremal BTZ black hole. This conclusion is the same as the one for the ordinary BTZ black hole obtained in GR.

It should be noted that concerning the critical phenomena of the BTZ black hole, we only studied the phase transition. In fact, the critical exponents are also very important. Whether the critical exponents are universal or depend on the details of the gravitational theory will be considered in the next work. Furthermore, we will further study whether the behavior of the BTZ black hole obtained in the Chern-Simons modified gravity or in the MB model is analogous to the Van der Waals fluid when treating the effective cosmological constant as a thermodynamic pressure.

In this paper, we also discussed the area spectrum of a nonrotating BTZ black hole in the MB model according to quasinormal modes. In the given QNM frequency, no torsion exists. Thus, this cannot reflect the influence of torsion on the area spectrum directly. But the corrected mass M will influence the quantized area. According to Maggiore's viewpoint, one can still obtain the quantization of the horizon area; however, the quantized entropy of the BTZ black hole is independent of the parameters α_3, p. In addition, we also simply discussed the method of adiabatic covariant action to obtain the entropy spectrum. Because the conserved charges and the first law of black hole thermodynamics in the MB model are corrected, the result is different. But the quantization of black hole entropy is not changed. This indicates that the area spectrum and the entropy spectrum of black holes are not only dependent on the spacetime metric, but also relevant to the theories of gravity.

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