

Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics

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We study the thermodynamics of D -dimensional Born-Infeld AdS black holes in the extended phase space. We find that the usual small-large black hole phase transition, which exhibits analogy with the Van de Waals liquid-gas system, holds in all dimensions greater than three. However, different from the four-dimensional case, in the system of higher dimensional Born-Infeld AdS black holes there is no reentrant phase transition. For the three-dimensional Born-Infeld AS black hole, there does not exist critical phenomena.

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I. INTRODUCTION

Black hole thermodynamics has been an intriguing subject of discussions for many years. It was found that in the black hole spacetimes, we can define standard thermodynamic variables such as temperature and entropy etc., and more strikingly we observe that black hole spacetimes contain rich phase structures and admit critical phenomena which are in analogy with the thermodynamic systems in nature.

Since the discovery of the AdS/CFT correspondence [1–3], the study of thermodynamics in AdS black holes has become more attractive nowadays. In view of the AdS/CFT correspondence, the bulk AdS black hole spacetimes admit a gauge duality description by thermal conformal field theory living on the AdS boundary. The first investigation of the thermodynamic properties in AdS black holes was reported in [4], where it was demonstrated that a certain phase transition in the phase space of the Schwarzschild AdS black hole exists. This phase transition can be interpreted as a confinement/deconfinement phase transition in the dual quark gluon plasma in the language of AdS/CFT correspondence [5]. More interesting discoveries were obtained for the charged AdS black holes [6,7], where it was found that a first order small black hole and big black hole phase transition is allowed which is superficially analogous to a liquid-gas phase transition of the van der Waals fluid. This superficial reminiscence was also observed in other AdS backgrounds [8–20].

Recently the study of thermodynamics in AdS black holes has been generalized to the extended phase space, where the cosmological constant is identified with thermodynamic pressure and its variations are included in the first law of black hole thermodynamics [21–25]. Including the

variable cosmological constant in the first law was motivated by the geometric derivations of the Smarr formula [26]. In so doing, the AdS black hole mass is identified with enthalpy and there exists a natural conjugate thermodynamic volume to the cosmological constant. In the extended phase space with cosmological constant and volume as thermodynamic variables, it was interestingly found that the system admits a more direct and precise coincidence between the first order small-large black hole phase transition and the liquid-gas change of phase occurring in fluids [27]. Considering the extended phase space, and hence treating the cosmological constant as a dynamical quantity, is a very interesting theoretical idea in disclosing possible phase transitions in AdS black holes [28]. More discussions in this direction can be found as well in [29–37].

Generalizing the discussion of the extended phase space thermodynamics to higher dimensional singly spinning AdS black holes, it was uncovered that besides the usual small-large black hole phase transitions, there exists a new phenomenon of reentrant phase transitions for all $d \geq 6$ dimensions, in which a monotonic variation of the temperature yields two phase transitions from large to small and back to large black holes [38]. This new reentrant phase transition was also observed in the multispinning $D = 6$ Kerr–anti-de Sitter black hole with fixed angular momenta J_1 and J_2 [39]. It was argued in [38,39] that the new phase structure in the thermodynamics of rotating black holes resembles to binary fluids seen in superfluidity and superconductivity. In addition to including the rotation, especially in higher dimensions where one has more rotation parameters, which can make the phase transitions more interesting, the impact of possible nonlinear electrodynamics extensions of the Reissner–Nordström (RN)-AdS black hole on the extended phase space has also been examined. But this study was restricted in the four-dimensional Born-Infeld AdS black holes [28]. To be consistent with the corresponding Smarr relation [27], it was argued that further extension of the phase space is needed by including

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the variation of the maximal field strength b in the first law of thermodynamics. The thermodynamic quantity conjugate to the Born-Infeld parameter b is called Born-Infeld vacuum polarization. In four-dimensional Born-Infeld AdS black holes [28], it was found that the impact of the nonlinearity can bring the new phenomenon of reentrant phase transition which was observed in rotating AdS holes [38,39].

It is interesting to generalize the discussion of the impact of nonlinear electrodynamics on the extended phase space thermodynamics to other spacetime dimensions and examine whether the phase structures observed in four-dimensional Born-Infeld AdS black holes are general. This is the motivation of the present paper. We will show that the four-dimensional Born-Infeld AdS black hole is very special. When the spacetime dimension is greater than four, there only exists a usual small-large black hole phase transition but no new reentrant phase transition. In addition, in three-dimensional Born-Infeld AdS black holes, we cannot find any critical phenomena. Without loss of generality, we will discuss the D -dimensional Born-Infeld AdS black holes with different topologies, such as Ricci flat, hyperbolic, or spherical hypersurfaces.

This paper is organized as follows. In Sec. II, we will discuss the solutions of D -dimensional Born-Infeld AdS black holes. In Sec. III, we will investigate the phase transitions of D -dimensional Born-Infeld AdS black holes. We will summarize our results in Sec. IV.

II. SOLUTIONS OF D -DIMENSIONAL BORN-INFELD ADS BLACK HOLES

The action of Einstein gravity in the presence of Born-Infeld field reads [20]

$$\mathcal{I} = \frac{1}{16\pi} \int d^D x \sqrt{-g} [-2\Lambda + R + \mathcal{L}(\mathcal{F})], \quad (1)$$

where $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$ is the negative cosmological constant, b is the Born-Infeld parameter, and $\mathcal{L}(\mathcal{F})$ is given by

$$\mathcal{L}(\mathcal{F}) = 4b^2 \left(1 - \sqrt{1 + \frac{F^{\mu\nu} F_{\mu\nu}}{2b^2}} \right). \quad (2)$$

In the limit $b \rightarrow \infty$, it reduces to the standard Maxwell field $\mathcal{L}(\mathcal{F}) = -F^{\mu\nu} F_{\mu\nu} + \mathcal{O}(F^4)$. If we take $b = 0$, $\mathcal{L}(\mathcal{F})$ disappears.

We assume the nonrotating metric in the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij} dx^i dx^j, \quad (3)$$

where the coordinates are labeled as $x^\mu = (t, r, x^i)$, ($i = 1, \dots, D-2$). h_{ij} describes the $(D-2)$ -dimensional hypersurface with constant scalar curvature $(D-2)(D-3)k$. The constant k characterizes the topology which can be $k = 0$ (flat), $k = -1$ (negative curvature), and $k = 1$ (positive curvature). The solution is given by [40,41]

$$\begin{aligned} f(r) = & k + \frac{r^2}{l^2} - \frac{m}{r^{D-3}} + \frac{4b^2 r^2}{(D-1)(D-2)} \\ & \times \left(1 - \sqrt{1 + \frac{(D-2)(D-3)q^2}{2b^2 r^{2D-4}}} \right) + \frac{2(D-2)q^2}{(D-1)r^{2n-4}} {}_2F_1 \\ & \times \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{(D-2)(D-3)q^2}{2b^2 r^{2D-4}} \right], \end{aligned} \quad (4)$$

$$F = dA, \quad A = -\sqrt{\frac{D-2}{2(D-3)}} \frac{q}{r^{D-3}} {}_2F_1 \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{(D-2)(D-3)q^2}{2b^2 r^{2D-4}} \right] dt, \quad (5)$$

where m and q are related to the mass M and charge Q of black holes as

$$Q = \frac{q \Sigma_k}{4\pi} \sqrt{\frac{(D-2)(D-3)}{2}}, \quad M = \frac{(D-2) \Sigma_k}{16\pi} m. \quad (6)$$

Here Σ_k represents the volume of constant curvature hypersurface described by $h_{ij} dx^i dx^j$. The electromagnetic potential difference (Φ) between the horizon and infinity is

$$\Phi = \frac{4\pi Q}{(D-3) \Sigma_k r_+^{D-3}} {}_2F_1 \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}} \right]. \quad (7)$$

In terms of the horizon radius r_+ , the Arnowitt-Deser-Misner (ADM) mass M of the Born-Infeld AdS black holes is obtained:

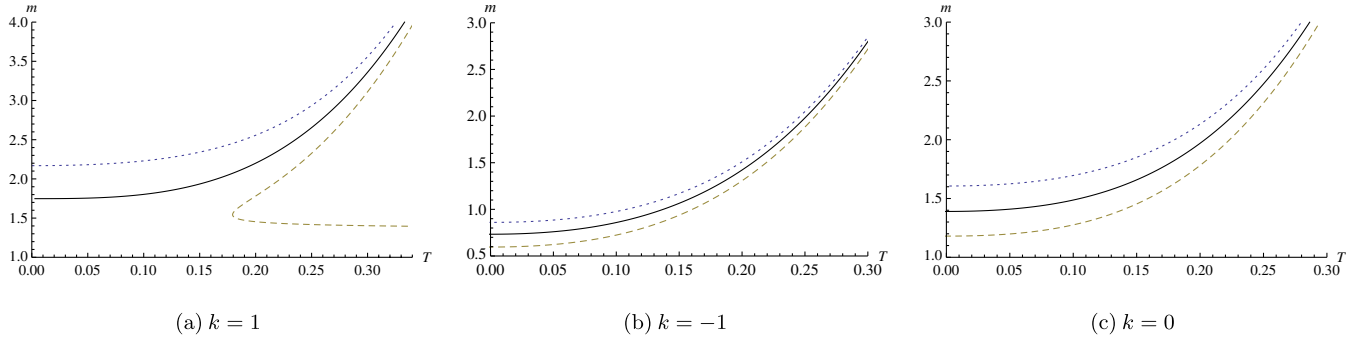


FIG. 1 (color online). The behaviors of temperatures of black holes with different topologies versus mass parameters m . We fix $Q = 1$, $l = 1$, $D = 4$. The lines from above to below correspond to $b = 1, 0.5$ and 0.3 , respectively.

$$M = \frac{(D-2)\Sigma_k r_+^{D-3}}{16\pi} \left[k + \frac{r_+^2}{l^2} + \frac{4b^2 r_+^2}{(D-1)(D-2)} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}}} \right) + \frac{64\pi^2 Q^2}{(D-1)(D-3)\Sigma_k^2 r_+^{2D-6}} {}_2F_1 \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}} \right] \right]. \quad (8)$$

The Hawking temperature and entropy of the Born-Infeld AdS black holes are

$$T = \frac{1}{4\pi} \left[\frac{(D-1)r_+}{l^2} + \frac{(D-3)k}{r_+} + \frac{4b^2 r_+}{D-2} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}}} \right) \right], \quad (9)$$

$$S = \frac{\Sigma_k}{4} r_+^{D-2}. \quad (10)$$

From the temperature expression, we can derive the condition to accommodate the extremal black holes with $T = 0$. For $D = 4$, the vanishing of T requires

$$k + \left(\frac{3}{l^2} + 2b^2 \right) r_{\text{ext}}^2 - 2b \sqrt{b^2 r_{\text{ext}}^4 + Q^2} = 0, \quad (11)$$

which leads to two possible solutions of the horizon radius for the extremal black holes,

$$r_{\text{ext}1,2}^2 = \frac{l^2(1 + \frac{3}{2b^2 l^2})}{6(1 + \frac{3}{4b^2 l^2})} \times \left[-k \pm \sqrt{k^2 + \frac{12(1 + \frac{3}{4b^2 l^2})}{b^2 l^2(1 + \frac{3}{2b^2 l^2})^2} \left(b^2 Q^2 - \frac{k^2}{4} \right)} \right]. \quad (12)$$

For the spherical case with $k = 1$, one can see that the solution with a minus sign in front of the square root is always unphysically negative. For the solution with a plus sign in front of the square root, when $bQ \geq 1/2$ it is real;

when $0 < bQ < 1/2$ this solution is imaginary. This shows that for $k = 1$, only when $bQ \geq 1/2$, it is possible to have two horizons enveloping the central singularity and these two horizons degenerate for the extreme case with $T = 0$ for the black hole. When $0 < bQ < 1/2$, the black hole can never become extreme. For the hyperbolic topology $k = -1$, although the solution of r_{ext}^2 with a minus sign in front of the square root is unphysically negative, r_{ext}^2 with a plus sign in front of the square root is always real and positive no matter what value of bQ we take, since $1 + \frac{12(1 + \frac{3}{4b^2 l^2})}{b^2 l^2(1 + \frac{3}{2b^2 l^2})^2} (b^2 Q^2 - \frac{1}{4}) = \frac{4b^2(9Q^2 + b^2 l^2(l^2 + 12Q^2))}{(3 + 2b^2 l^2)^2} > 0$. Thus

when $k = -1$, the Born-Infeld AdS black hole exists for all values of bQ and this black hole can become extreme to allow $T = 0$. For the black hole with flat topology $k = 0$, there exists one real positive root $r_{\text{ext}}^2 = \frac{2bQl^2}{[3(3 + 4b^2 l^2)]^{1/2}}$ to ensure $T = 0$, which shows that for the flat case, no matter what value of bQ we choose, the Born-Infeld AdS black hole can become extreme. The behaviors of the black hole temperature T versus the mass m are plotted for fixed $Q = 1, l = 1$ and different values of b in four dimensions in Fig. 1, where it shows that there does not exist any extremal spherical Born-Infeld AdS black holes with $T = 0$ when $0 < bQ < 1/2$.

TABLE I. The numerical results of these parameters r_{ext} and m of five-dimensional Born-Infeld AdS black holes with different topologies. We take $Q = 1$ and $l = 1$.

Parameters		$k = 1$					$k = -1$				
b		0.01	0.3	0.5	1	10	0.01	0.3	0.5	1	10
r_{ext}		0.0042	0.1234	0.1960	0.3296	0.5360	0.7092	0.7543	0.7723	0.7937	0.8101
m		0.0357	0.3282	0.4406	0.6124	0.8353	-0.2204	-0.0840	-0.0581	-0.0343	-0.0200

When the spacetime dimension is higher than four, to accommodate $T = 0$, we require

$$\frac{(D-1)r_{\text{ext}}^2}{l^2} + (D-3)k + \frac{4b^2 r_{\text{ext}}^2}{D-2} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_{\text{ext}}^{2D-4}}} \right) = 0. \quad (13)$$

For the black hole with flat topology, there always exists one real positive root $r_{\text{ext}} = \left[\frac{256l^4 \pi^2 b^2 Q^2}{(D-1)(D-2)((D-1)(D-2)+8b^2 l^2) \Sigma_k^2} \right]^{2D-4}$ and positive mass parameter m of extremal higher dimensional black hole to ensure $T = 0$, which is same as the case in four-dimensional spacetimes. However, it is difficult to obtain the analytic solution of this equation for the cases of $k = 1$ and $k = -1$. We can only present numerical results of the mass parameter m when $T = 0$ for $D = 5$ in the Table I. Note that for spherical topology, no matter what value of bQ , we can always have extreme black holes with $T = 0$. This implies that the five-dimensional case is different from the four-dimensional case, there is no limit for the value of bQ to allow the existence of the Born-Infeld AdS black hole even for spherical topology. For the five-dimensional hyperbolic Born-Infeld AdS black hole with $k = -1$, however, $T = 0$ can only be achieved for negative m . Since m is associated with the mass M , according to Eq. (6), this implies a negative mass of the spacetime, which is not physical. Thus, we restrict our consideration to $m > 0$ so that the black hole can never become extreme. In Fig. 2 we show the temperatures T of black holes with m for different topologies for $D = 5$.

To see more closely the behaviors of the Born-Infeld AdS black hole, we examine the function $f(r)$. The black hole possesses a singularity at $r = 0$, which can be a naked singularity or enveloped by one or two horizons, depending on the values of the parameters. The expansion of the function $f(r)$ around $r = 0$ takes the form [41]

$$f(r) = k - \frac{m-A}{r^{D-3}} - \left(\frac{2Cb}{D-1} - B \right) \times \sqrt{\frac{2}{(D-2)(D-3)}} \frac{4\pi Q}{\Sigma_k r^{D-4}} + \mathcal{O}(r), \quad (14)$$

where

$$\begin{aligned} A &= \frac{64\pi^2 Q^2}{(D-1)(D-3)\sqrt{\pi}\Sigma_k^2} \left(\frac{b^2 \Sigma_k^2}{16\pi^2 Q^2} \right)^{\frac{D-3}{2D-4}} \Gamma\left[\frac{3D-7}{2D-4}\right] \\ &\quad \times \Gamma\left[\frac{1}{2D-4}\right], \\ B &= \frac{4b}{(D-1)C} \frac{\Gamma\left[\frac{3D-7}{2D-4}\right] \Gamma\left[\frac{-1}{2D-4}\right]}{\Gamma\left[\frac{D-3}{2D-4}\right] \Gamma\left[\frac{2D-5}{2D-4}\right]}, \\ C &= \sqrt{\frac{2D-6}{D-2}}. \end{aligned} \quad (15)$$

For $D = 4$, Eq. (14) can be simplified as [8,42]

$$\begin{aligned} f(r) &= k - \frac{m-A}{r} - 2bQ + \mathcal{O}(r), \\ A &= \frac{1}{3} \sqrt{\frac{b}{\pi}} Q^{3/2} \Gamma\left(\frac{1}{4}\right)^2. \end{aligned} \quad (16)$$

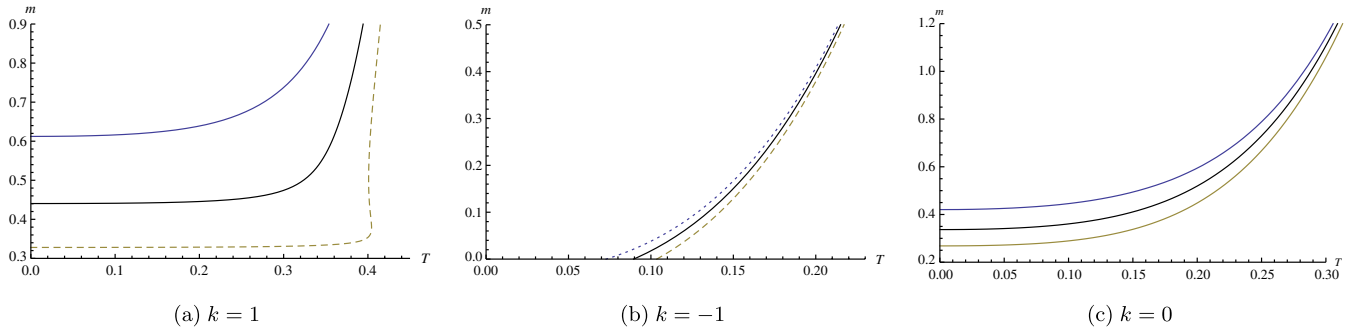


FIG. 2 (color online). The behaviors of mass parameters m of black holes with different topologies versus temperatures T . We fix $Q = 1$, $l = 1$, $D = 5$. The lines from up to below correspond to $b = 1, 0.5$, and 0.3 , respectively.

For the spherical case with $k = 1$, the behavior of the solution was discussed in detail in [28]. If we take $m > A$, the function $f(r) = 1 - \frac{m-A}{r} - 2bQ + \mathcal{O}(r)$ approaches $-\infty$ near the origin $r = 0$ and approaches $+\infty$ when $r \rightarrow +\infty$, which is independent of the value of bQ . The behavior of the Born-Infeld AdS resembles the “Schwarzschild-like” (S) type with a spacelike singularity and one horizon enveloping it. If we take $m < A$, the function $f(r) = 1 - \frac{m-A}{r} - 2bQ + \mathcal{O}(r)$ approaches $+\infty$ near the origin $r = 0$. In this case, only when $bQ > 1/2$ we can find the solutions of r to satisfy $f(r) = 1 - \frac{m-A}{r} - 2bQ + \mathcal{O}(r) = 0$, which describe the horizons of the black hole. The behavior of $f(r)$ likes the “Reissner-Nördstrom” (RN)-type, which can have two horizons if $m_{\text{ext}} < m < A$ and these two horizons degenerate when $m_{\text{ext}} = A$, while the RN type describes a naked singularity when $m < m_{\text{ext}} < A$. When $bQ < 1/2$, the $f(r)$ only possesses the S-type black holes when $m \geq A$. This is consistent with the analysis of the temperature. These properties are shown in Fig. 3a. It is interesting to note that $m = A$ describes the “marginal” case, where the function $f(r)$ Eq. (16) reduces to $f_A = 1 - 2bQ + \mathcal{O}(r)$. Obviously the function f_A possesses one horizon for $bQ > 1/2$; but when $bQ < 1/2$, f_A is positive and describes a naked singularity, see Fig. 4a.

For the topologies with $k = 0$ and -1 , when $m > A$, the function $f(r) = k - \frac{m-A}{r} - 2bQ + \mathcal{O}(r)$ tends to $-\infty$ near $r = 0$ and approaches $+\infty$ when $r \rightarrow +\infty$, so that the

Born-Infeld AdS black hole possesses the property of the S-type black hole. When $m < A$, $f(r)$ attains $+\infty$ near $r = 0$ and $+\infty$ as $r \rightarrow +\infty$. No matter what positive values of bQ we take, the black hole can always have the RN type when $m < A$ for $k = 0$ and -1 . This is different from the spherical case and agrees with the observations we got in the temperature expression. The result is shown in Figs. 3(b) and 3(c). As to the marginal case $m = A$, the function $f(r)$ can be written as $f_A = k - 2bQ + \mathcal{O}(r)$, which is always negative at $r = 0$ and takes $+\infty$ as $r \rightarrow +\infty$. This shows that for the marginal case the black hole always possesses a horizon for positive values of bQ , see Figs. 4(b) and 4(c). Finally, these different types for the black hole solution $f(r)$ [Eq. (4)] in four-dimensional spacetimes are shown in Fig. 5.

Now we turn our discussion to higher dimensional Born-Infeld AdS black holes. Taking $D = 5$, Eq. (14) becomes

$$f(r) = k - \frac{m - A'}{r^2} - \frac{b'Q}{r} + \mathcal{O}(r), \quad b' = \frac{58\sqrt{2}b}{25\pi},$$

$$A' = \frac{(2b^2Q^4)^{1/3}}{\pi^{11/6}} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{1}{6}\right). \quad (17)$$

For all topological cases when $m > A'$, we see that the function $f(r)$ tends to $-\infty$ near $r = 0$ and $+\infty$ when $r \rightarrow +\infty$. Therefore the behavior of $f(r)$ describes the Schwarzschild-type black hole. If we take $m < A'$, the

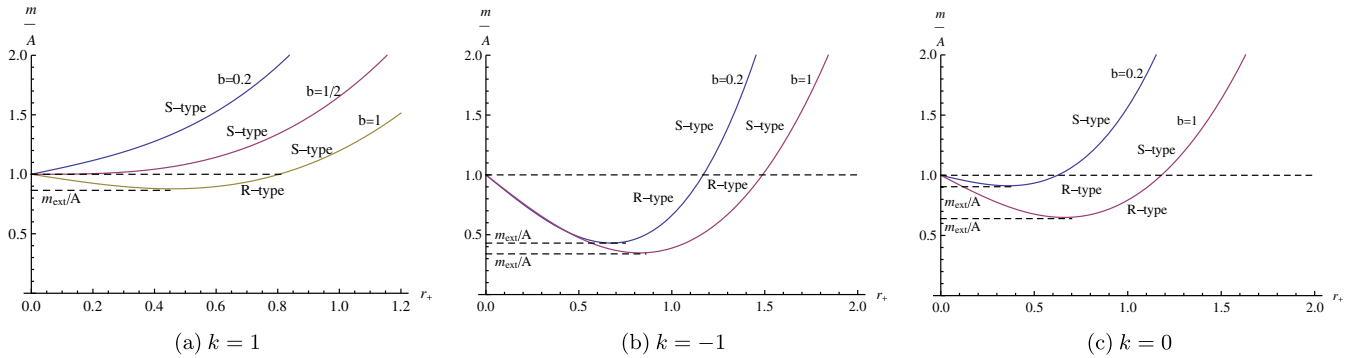


FIG. 3 (color online). Solutions of black hole horizons in four dimensions as functions of m/A . We take $Q = 1$, $l = 1$.

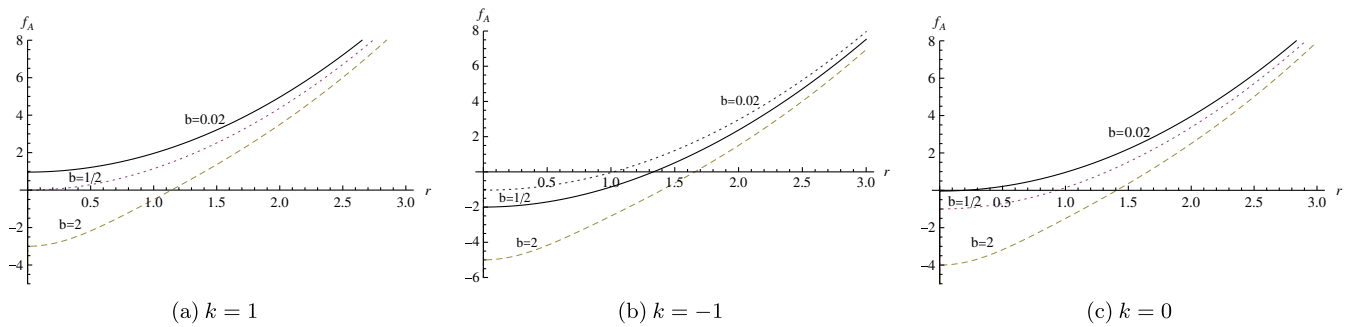


FIG. 4 (color online). Marginal case in four dimensions. We fix $Q = 1$, $l = 1$.

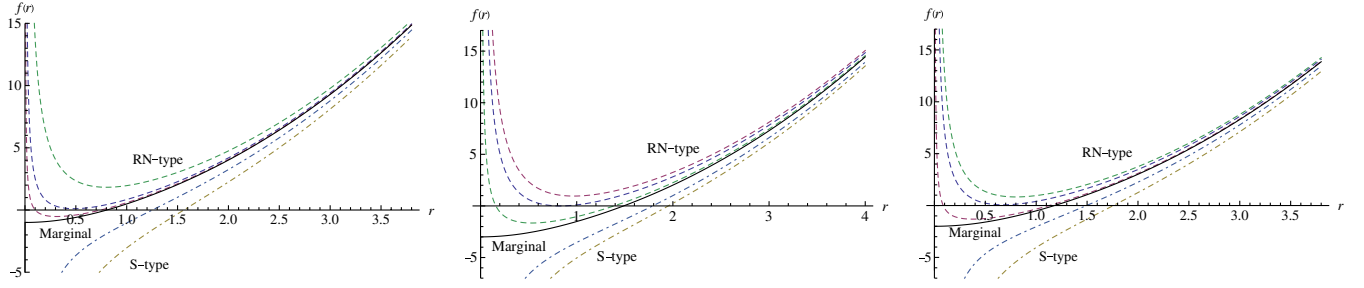


FIG. 5 (color online). Types of the four-dimensional Born-Infeld AdS black holes with different topologies. The marginal type of black hole corresponds to $m = A$, the S-type black hole corresponds to $m > A$, and the RN type of black hole is for $m < A$. For the RN-type black hole, the three dashed lines from top to below correspond to the naked singularity taking $m < m_{\text{ext}} < A$, extremal black hole (BH) $m = m_{\text{ext}} < A$ and two horizon solutions $m_{\text{ext}} < m < A$, respectively. In plotting the figure, we have used $Q = 1, b = 1$.

function $f(r)$ approaches $+\infty$ when $r \rightarrow 0$ and $r \rightarrow +\infty$. This asymptotical property keeps for all values of k . For small r , k is negligible compared with $b'Q/r$, thus for all positive values of $b'Q$ the black hole can possess the RN type which may have two horizons when $m_{\text{ext}} < m < A'$ and only one degenerated horizon when $m_{\text{ext}} = A'$. For the spherical case, when the spacetime dimension is higher than four, the restriction of bQ to accommodate the RN-type Born-Infeld AdS black hole is washed out. This is in agreement with the analysis of the temperature expression. When $k = -1$, there does not exist an extreme black hole and when the mass parameter m is within the range

$0 < m < A'$, it describes the RN-type black hole. The numerical picture is shown in Fig. 6.

When $m = A'$, Eq. (17) reduces to

$$f_{A'} = f(m = A') = k - \frac{b'Q}{r} + \mathcal{O}(r). \quad (18)$$

Different from the case of $D = 4$, the term $\frac{b'Q}{r}$ is very large for small r , so that $f_{A'} \rightarrow -\infty$ near $r = 0$ and $f_{A'} \rightarrow +\infty$ as $r \rightarrow +\infty$, provided that $b'Q > 0$. The marginal case $m = A'$ is characterized by the existence of a spacelike singularity enveloped by a horizon for all different

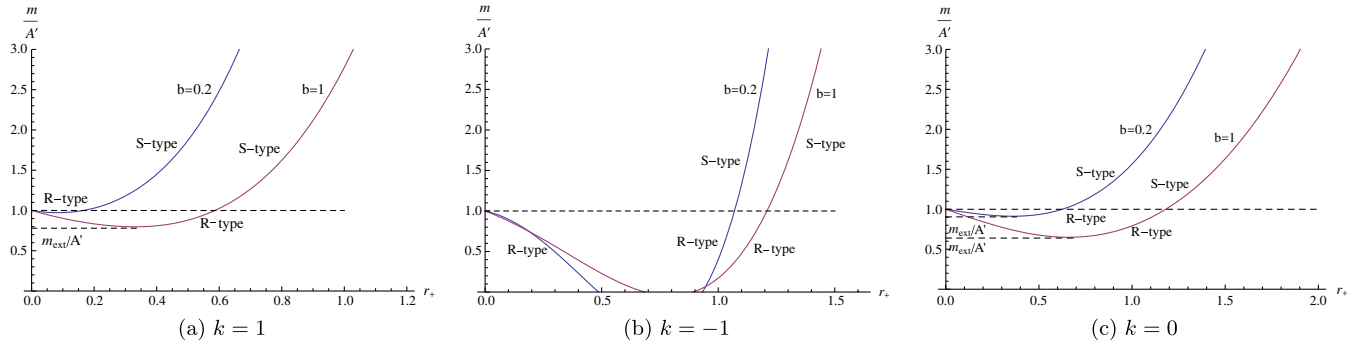


FIG. 6 (color online). Solutions of black hole horizons in five dimensions as functions of m/A' . We take $Q = 1, l = 1$.

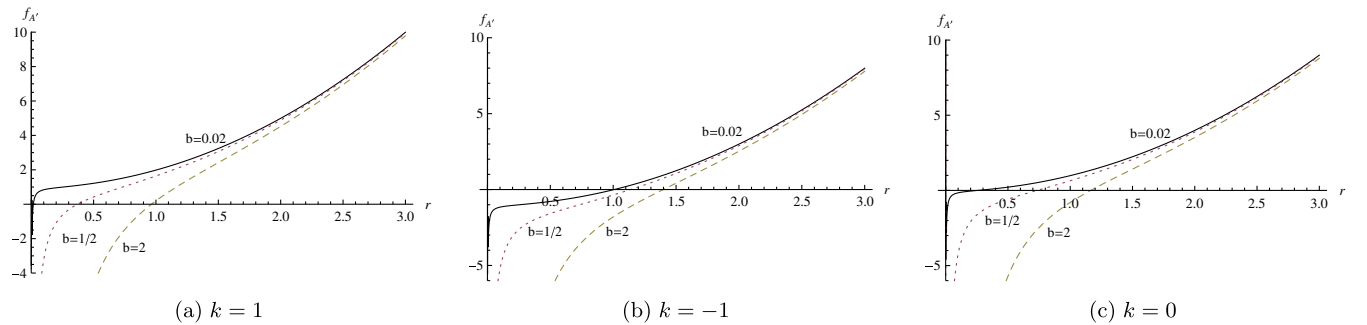


FIG. 7 (color online). Marginal case in five dimensions. We fix $Q = 1, l = 1$.

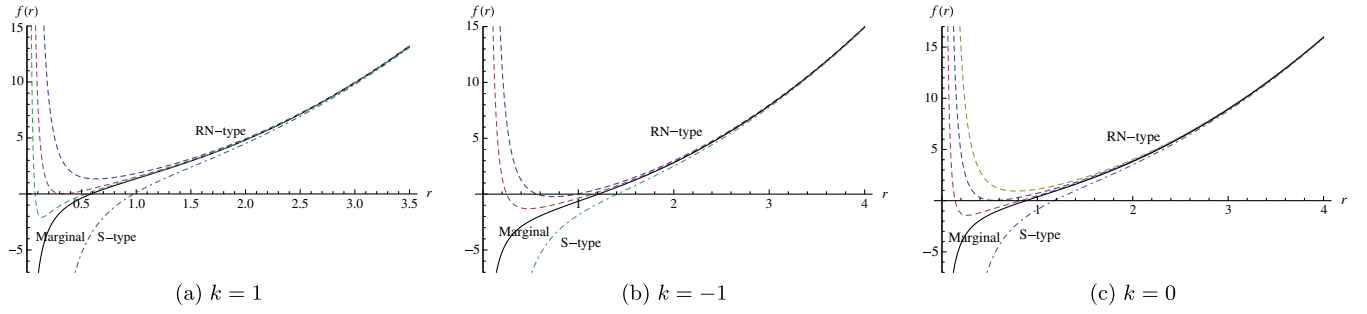


FIG. 8 (color online). Types of the five-dimensional Born-Infeld AdS black holes with different topologies. Here the marginal case ($m = A'$) belongs to the S-type. For the RN-type black hole ($m < A'$) with $k = 1$ and 0 , the three dashed lines from top to below correspond to the naked singularity taking $m < m_{\text{ext}} < A'$, extremal BH $m = m_{\text{ext}} < A'$ and two horizon solutions $m_{\text{ext}} < m < A'$, respectively. However, the RN type of Born-Infeld AdS black hole with $k = -1$ always possesses two horizons. We take $Q = 1, b = 1$.

topologies, see Fig. 7. Then these different types for the black hole solution $f(r)$ [Eq. (4)] in five-dimensional spacetimes are shown in Fig. 8, where the marginal case ($m = A'$) also belongs to the S-type, which is drawn with a black line. For the RN-type black hole ($m < A'$) with $k = 1$ and 0 , the three dashed lines from top to below correspond to the naked singularity taking $m < m_{\text{ext}} < A'$, extremal BH $m = m_{\text{ext}} < A'$ and two horizon solutions $m_{\text{ext}} < m < A'$, respectively. It is interesting to note that since there does not exist an extremal black hole for five-dimensional hyperbolic spacetime, the Born-Infeld AdS black hole always possesses two horizons in the form of RN type. When $D > 5$, $(\frac{2Cb}{D-1} - B)\sqrt{\frac{2}{(D-2)(D-3)}}\frac{4\pi Q}{\Sigma_k}$ always maintains positive for $k = 0, 1$ and -1 , one can see that the behavior of $f(r)$ is similar to the case of $D = 5$.

III. PHASE TRANSITIONS OF D -DIMENSIONAL BORN-INFELD ADS BLACK HOLES

A. Phase transitions

In the geometric units $G_N = \hbar = c = k = 1$, we interpret the cosmological constant $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$ as a positive thermodynamic pressure,

$$P = -\frac{1}{8\pi}\Lambda = \frac{(D-1)(D-2)}{16\pi l^2}. \quad (19)$$

The black hole mass M can be considered as the enthalpy rather than the internal energy of the gravitational system [26]. In the Born-Infeld case, M should be the function of entropy, pressure, charge, and Born-Infeld coupling coefficient [28]. Moreover, those thermodynamic quantities satisfy the following differential form:

$$dM = TdS + \Phi dQ + VdP + \mathfrak{B}db, \quad (20)$$

where the thermodynamic volume V conjugate to P equals to $\frac{\Sigma_k r_+^{D-1}}{D-1}$. $\mathfrak{B}$ is a quantity conjugate to b and is called the “Born-Infeld vacuum polarization”

$$\begin{aligned} \mathfrak{B} = & \frac{\Sigma_k b r_+^{D-1}}{2(D-1)\pi} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}}} \right) \\ & + \frac{4Q^2\pi}{(D-1)b\Sigma_k r_+^{D-3}} {}_2F_1 \\ & \times \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}} \right]. \end{aligned} \quad (21)$$

For $D = 4$, Eq. (21) reduces to [28]

$$\begin{aligned} \mathfrak{B} = & \frac{2br_+^3}{3} \left(1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right) \\ & + \frac{Q^2}{3br_+} {}_2F_1 \left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{Q^2}{b^2 r_+^4} \right]. \end{aligned} \quad (22)$$

By scaling argument, we can obtain the generalized Smarr relation for the Born-Infeld AdS black holes in the extended phase space,

$$M = \frac{D-2}{D-3}TS + \Phi Q - \frac{2}{D-3}VP - \frac{1}{D-3}\mathfrak{B}b. \quad (23)$$

In four-dimensional spacetime, this expression reduces to that described in [28].

Using Eqs. (9) and (19), the equation of state $P(V, T)$ can be obtained

$$\begin{aligned} P = & \frac{(D-2)T}{4r_+} - \frac{(D-2)(D-3)k}{16\pi r_+^2} \\ & - \frac{b^2}{4\pi} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}}} \right). \end{aligned} \quad (24)$$

To compare with the van der Waals fluid equation in D dimensions, we can translate the “geometric” equation of state to a physical one by identifying the specific volume v of the fluid with the horizon radius of the black hole r_+ as $v = \frac{4r_+}{D-2}$ [27,28].

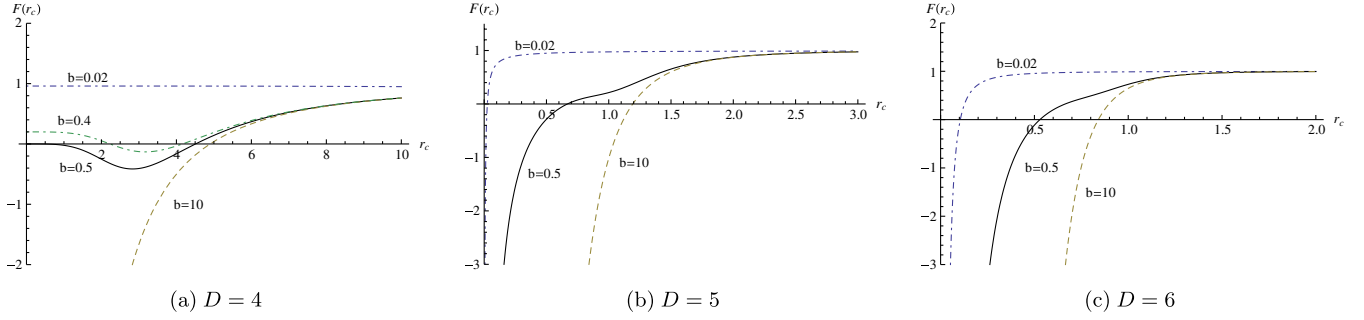


FIG. 9 (color online). The behaviors of $F(r_c)$ for various values of b in different spacetime dimensions. We fix $Q = 1$, $l = 1$, and $k = 1$.

We know that the critical point occurs when $P = P(v)$ has an inflection point

$$\left. \frac{\partial P}{\partial r_+} \right|_{T=T_c, r_+=r_c} = 0, \quad \left. \frac{\partial^2 P}{\partial r_+^2} \right|_{T=T_c, r_+=r_c} = 0. \quad (25)$$

Then we can obtain the critical temperature,

$$T_c = \frac{(D-3)k}{2\pi r_c} - \frac{16\pi Q^2}{\Sigma_k^2 r_c^{2D-5}} \left(1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_c^{2D-4}} \right)^{-1/2}, \quad (26)$$

and the equation for the critical horizon radius r_c (specific volume $v_c = \frac{4r_c}{D-2}$),

$$F(r_c) = k - \frac{32(2D-5)\pi^2 Q^2}{(D-3)\Sigma_k^2 r_c^{2D-6}} \left(1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_c^{2D-4}} \right)^{-1/2} + \frac{512(D-2)\pi^4 Q^4}{(D-3)b^2 \Sigma_k^4 r_c^{4D-10}} \left(1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_c^{2D-4}} \right)^{-3/2} = 0, \quad (27)$$

where r_c denotes the critical value of r_+ .

For different topological spacetimes, we can investigate the phase structure and $P-V$ criticality in the extended phase space. Obviously the critical temperature T_c [Eq. (26)] is negative for hyperbolic and flat black hole horizons, so that no phase transition happens there. Thus, we only need to explore the phase transition and $P-V$ criticality in the spherical Born-Infeld AdS black hole. The behaviors of $F(r_c)$ for different values of b in different spacetime dimensions can be seen in Fig. 9.

When $D = 4$, the critical point was discussed in detail in [28]. Rewriting Eq. (27) with $v_c = 2r_c$, we have

$$x^3 - \frac{3b^2}{32Q^2}x + \frac{b^2}{256Q^4} = 0, \quad x = \left(v_c^4 + \frac{16Q^2}{b^2} \right)^{-1/2}, \quad (28)$$

where it has one or more positive real roots for r_c and displays different phase transitions of four-dimensional

Born-Infeld AdS black hole. For four dimensions, these results can also be obtained by analyzing the function $\frac{\partial F(r_c)}{\partial r_c}$,

$$\frac{\partial F(r_c)}{\partial r_c} = \frac{12Q^2}{r_c^3} \left(1 + \frac{Q^2}{b^2 r_c^4} \right)^{-5/2} \left(1 - \frac{Q^2}{b^2 r_c^4} \right), \quad (29)$$

which can take positive, zero, or negative values depending on the values of b and Q . We summarize real roots of Eq. (29) and the critical points in the Table II [28].

When the spacetime dimension $D \geq 5$, it is difficult to give analytical roots of $F(r_c) = 0$. The number of real roots of Eq. (27) can be decided by examining

$$\frac{\partial F(r_c)}{\partial r_c} = \frac{64(2D-5)\pi^2 Q^2}{\Sigma_k^2 r_c^{2D-5}} \left(1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_c^{2D-4}} \right)^{-5/2} \times \left[1 + \frac{8(D-6)\pi^2 Q^2}{(D-3)b^2 \Sigma_k^2 r_c^{2D-4}} + \frac{128(D-4)\pi^4 Q^4}{(2D-5)b^4 \Sigma_k^4 r_c^{4D-8}} \right]. \quad (30)$$

Taking $D = 5$, we have

TABLE II. The behaviors of critical points for different values of b in four dimensions by setting $Q = 1$.

Born-Infeld AdS black holes				
Coupling coefficient	$b > \frac{1}{2Q}$	$b < \frac{1}{\sqrt{8}Q}$	$\frac{1}{\sqrt{8}Q} < b < \frac{\sqrt{3+2\sqrt{3}}}{6Q}$	$\frac{\sqrt{3+2\sqrt{3}}}{6Q} < b < \frac{1}{2Q}$
Number of real root x	One	One	Three	Three
Number of critical points	One	None	Two	One
Types of BI-AdS BH	RN	S	S	S

TABLE III. Phase space parameters at critical points for different values of b when spacetime dimension $D = 5$ and 6. We fix $Q = 1$. As $b \rightarrow \infty$, the value of $\frac{P_c v_c}{T_c}$ approaches the critical value of 5D and 6D Reissner-Nördstrom black holes [28].

Parameters	$D = 5$					$D = 6$				
b	0.02	0.1	1	10	100	0.02	0.1	1	10	100
v_c	0.0340	0.1698	1.5196	1.5902	1.5908	0.1050	0.2350	0.7813	0.8409	0.8413
T_c	6.2500	1.2500	0.2177	0.2135	0.2134	3.0313	1.3560	0.5017	0.4866	0.4864
P_c	61.3592	2.4536	0.0584	0.0565	0.0559	10.8253	2.1643	0.2704	0.2531	0.2530
$\frac{P_c v_c}{T_c}$	0.3333	0.3333	0.4076	0.4147	0.4167	0.3750	0.3750	0.4212	0.4374	0.4375

$$\begin{aligned}
& 1 + \frac{8(D-6)\pi^2 Q^2}{(D-3)b^2 \Sigma_k^2 r_c^{2D-4}} + \frac{128(D-4)\pi^4 Q^4}{(2D-5)b^4 \Sigma_k^4 r_c^{4D-8}} \\
& = \left(1 + \frac{4(D-6)\pi^2 Q^2}{(D-3)b^2 \Sigma_k^2 r_c^{2D-4}} \right)^2 \\
& + \frac{48(2D-9)(D-2)^2 \pi^4 Q^4}{(2D-5)(D-3)^2 b^4 \Sigma_k^4 r_c^{4D-8}} > 0,
\end{aligned} \quad (31)$$

so that $\frac{\partial F(r_c)}{\partial r_c}$ is always positive for $D = 5$ and there is only one real root for $F(r_c) = 0$. Thus, the equation of state [Eq. (24)] allows only one critical point for each value of b when $D = 5$. This property keeps for $D \geq 6$, since the function $\frac{\partial F(r_c)}{\partial r_c}$ is always positive, so that only one critical point exists for $D \geq 6$. Compared with the

four-dimensional case, the phase structure for the higher dimensional case is much simpler.

We can numerically solve Eq. (27). We list the phase space parameters at critical points for different values of b when spacetime dimension $D = 5$ and 6 in Table III. We plot the $P - V$ diagrams for $D = 5$ and 6 in Fig. 10. The two upper dashed lines correspond to the “ideal gas” phase behavior when $T > T_c$. For $T < T_c$, there is a small-large black hole phase transition in the system. Different from the four-dimensional Born-Infeld AdS black hole case, we have not observed the reentrant phase transition when the spacetime dimension is higher.

To get more information about the phase transition, we can examine the free energy. For the canonical ensemble, the Gibbs free energy is $G = M - TS$, which reads

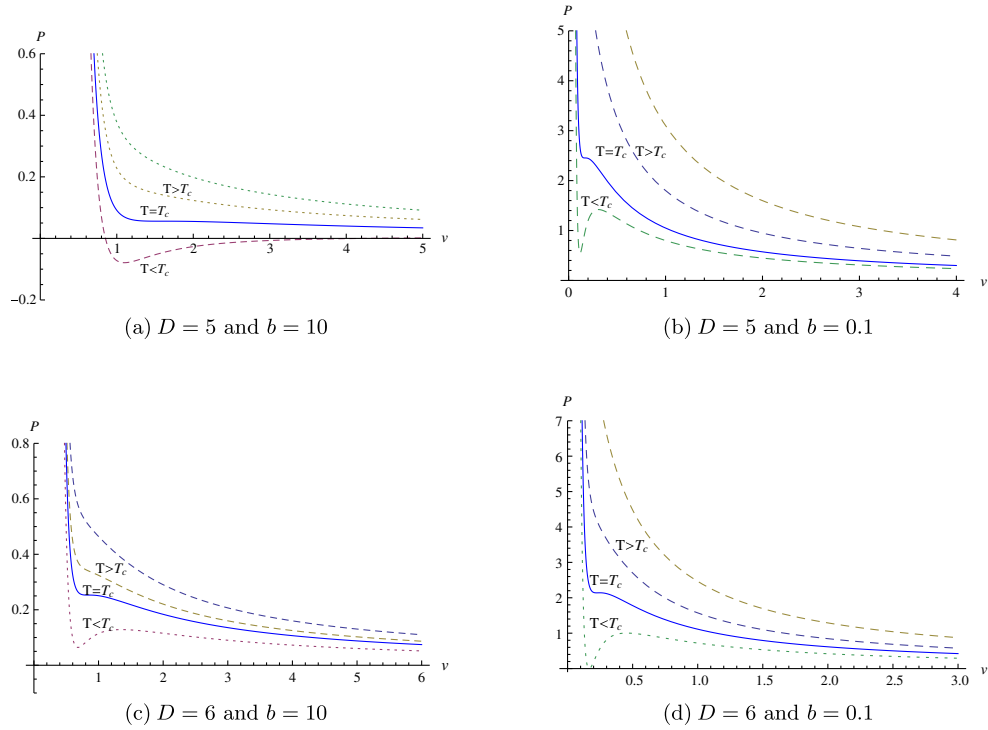


FIG. 10 (color online). The $P - V$ diagram of Born-Infeld AdS black holes when the spacetime dimensions $D = 5, 6$. We fix $Q = 1$. The two upper dashed lines correspond to the idea gas phase behavior for $T > T_c$. The critical temperature case is denoted by the solid line. The lines below are with temperatures smaller than the critical temperature.

$$G(T, P) = \frac{\Sigma_k}{16\pi} \left[k r_+^{D-3} - \frac{16\pi P r_+^{D-1}}{(D-1)(D-2)} - \frac{4b^2 r_+^{D-1}}{(D-1)(D-2)} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}}} \right) \right. \\ \left. + \frac{64(D-2)^2 \pi^2 Q^2}{(D-1)(D-3) \Sigma_k^2 r_+^{D-3}} {}_2F_1 \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{16\pi^2 Q^2}{b^2 \Sigma_k^2 r_+^{2D-4}} \right] \right]. \quad (32)$$

Here r_+ is understood as the function of pressure and temperature, $r_+ = r_+(P, T)$, via the equation of state [Eq. (24)]. We plot the change of the free energy G with T for fixed Q and different spacetime dimensions in Fig. 11, which reveals the existence of “swallow tail” behavior of the free energy G . The presence of the characteristic swallow tail behavior means that the small-large black hole phase transition which occurs in the system is of the first order. The coexistence line in the (P, T) plane describes that two phases share the same Gibbs free energy and temperature during the phase transition, which is plotted in Fig. 12. Note that this coexistence line looks very similar to the van de Waals liquid-gas system.

Now we turn to compute the critical exponents characterizing the behavior of physical quantities in the vicinity of the critical point in the Born-Infeld AdS black hole system. Although the equation of state $P(V, T)$ [Eq. (24)] is dimensional dependence, we will show this does not affect the behavior of the critical exponents. Near the critical point, the critical behavior of the van de Waals liquid-gas system can be characterized by the following critical exponents [27]:

$$C_v = T \left. \frac{\partial S}{\partial T} \right|_v \propto \left(-\frac{T - T_c}{T_c} \right)^{-\alpha}, \\ \eta = \frac{v_s - v_l}{v_c} \propto \left(-\frac{T - T_c}{T_c} \right)^\beta, \\ \kappa_T = -\frac{1}{v} \left. \frac{\partial v}{\partial P} \right|_T \propto \left(-\frac{T - T_c}{T_c} \right)^{-\gamma}, \\ P - P_c \propto (v - v_c)^\delta. \quad (33)$$

In order to compute the critical exponent α , the entropy S of the Born-Infeld AdS black hole [Eq. (10)] can be rewritten as $\frac{\Sigma_k^{\frac{D-1}{4}}}{4} ((D-1)V)^{\frac{D-2}{D-1}}$. Obviously this entropy S is independent of T and hence we have the critical exponent $\alpha = 0$. To obtain other exponents, define

$$p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c} \quad (34)$$

and introduce the expansion parameters

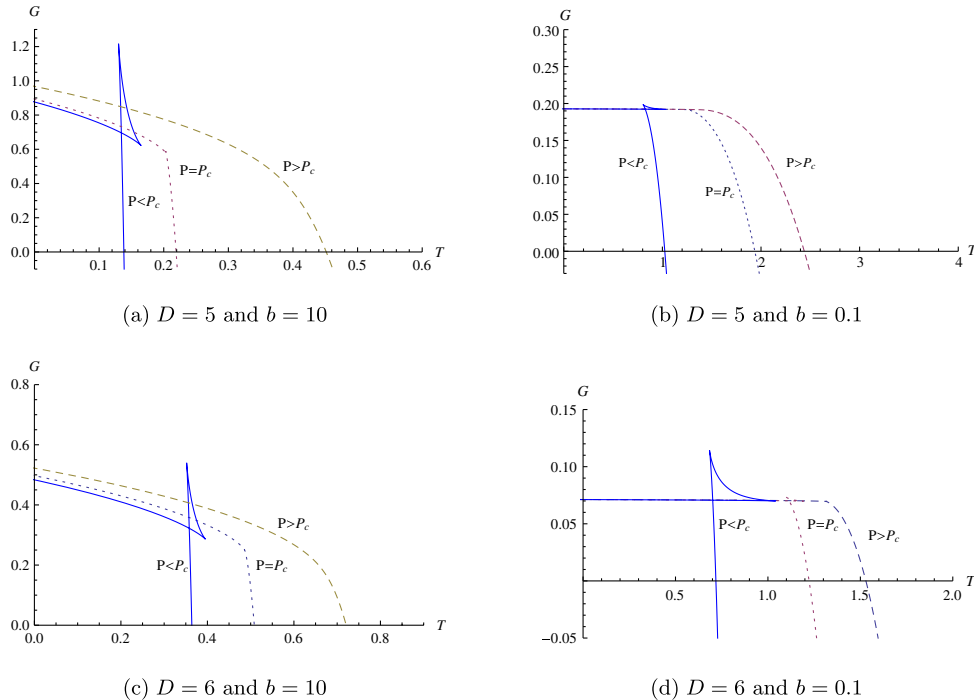


FIG. 11 (color online). The Gibbs free energy as a function of temperature different spacetime dimensions $D = 5, 6$. We fix $Q = 1$.

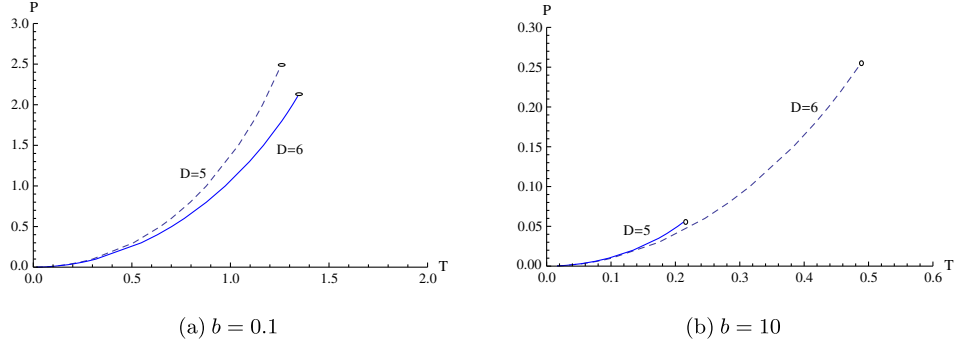


FIG. 12 (color online). The coexistence line of the small-large Born-Infeld AdS black hole phase transition for different dimensions in the (P, T) plane. The critical point is shown by a small circle at the end of the coexistence line.

$$\tau = t + 1, \quad \nu = \omega + 1, \quad (35)$$

then the expansion for this equation of state near the critical point is given by

$$p = 1 + a_{10}t + a_{11}t\omega + a_{03}\omega^3 + \mathcal{O}(t\epsilon^2, \epsilon^4). \quad (36)$$

During the phase transition, the pressure remains constant

$$p = 1 + a_{10}t + a_{11}t\omega_s + a_{03}\omega_s^3 = 1 + a_{10}t + a_{11}t\omega_l + a_{03}\omega_l^3, \\ \Rightarrow a_{11}t(\omega_s - \omega_l) + a_{03}(\omega_s^3 - \omega_l^3) = 0, \quad (37)$$

where ω_s and ω_l denote the “volume” of small and large black holes. Using Maxwell’s area law, we can obtain

$$\int_{\omega_l}^{\omega_s} \omega \frac{dp}{d\omega} d\omega = 0 \Rightarrow a_{11}(\omega_s^2 - \omega_l^2) + \frac{3}{2}a_{03}(\omega_s^4 - \omega_l^4) = 0. \quad (38)$$

The nontrivial solution of Eqs. (37) and (38) appears only when $a_{11}a_{03}t < 0$, and reads as

$$\omega_s = \frac{\sqrt{-a_{11}a_{03}t}}{3|a_{03}|}, \quad \omega_l = -\frac{\sqrt{-a_{11}a_{03}t}}{3|a_{03}|}. \quad (39)$$

In general, it seems impossible to give an analytical expression for a_{11} and a_{03} . The numerical results of a_{10} , a_{11} , and a_{03} for different b are shown in Table IV. Therefore we have

$$\eta = \omega_s - \omega_l = 2\omega_s \propto \sqrt{-t} \Rightarrow \beta = 1/2. \quad (40)$$

The isothermal compressibility can be computed as

$$\kappa_T = -\frac{1}{v} \frac{\partial v}{\partial P} \Big|_{v_c} \propto -\frac{1}{\frac{\partial p}{\partial \omega} \Big|_{\omega=0}} = -\frac{1}{a_{11}t}, \quad (41)$$

which indicates that the critical exponent $\gamma = 1$. In addition, we obtain the shape of the critical isotherm $t = 0$,

$$p - 1 = a_{03}\omega^3 \Rightarrow \delta = 3. \quad (42)$$

It is easy to check that these critical exponents satisfy the following thermodynamic scaling laws:

$$\alpha + 2\beta + \gamma = 2, \quad \alpha + \beta(1 + \delta) = 2, \\ \gamma(1 + \delta) = (2 - \alpha)(\delta - 1), \quad \gamma = \beta(\delta - 1). \quad (43)$$

Therefore these thermodynamic exponents for the higher dimensional Born-Infeld AdS black holes coincide with those of the van der Waals fluid [27], and also take the same values in the (higher dimensional) Maxwell case [28].

B. Phase transition of three-dimensional Born-Infeld AdS black hole

We extend our investigate to the three-dimensional Born-Infeld black holes. The metric is given by [42,43]

TABLE IV. The values of a_{10} , a_{11} , and a_{03} with different values of b for $D = 5, 6$ and $Q = 1$.

Parameters		$D = 5$					$D = 6$				
b		0.02	0.1	1	10	100	0.02	0.1	1	10	100
a_{10}		2.9994	3.0003	2.4534	2.4004	2.4004	2.6667	2.6667	2.3742	2.2862	2.2857
a_{11}		-2.9994	-3.0003	-2.4534	-2.4004	-2.4004	-2.6667	-2.6667	-2.3742	-2.2862	-2.2857
a_{03}		-0.9911	-0.9885	-1.5791	-1.9967	-2.0012	-1.3341	-1.3240	-1.5787	-2.6550	-2.6672

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\theta^2,$$

$$F = E(r)dt \wedge dr, \quad E(r) = \frac{Q}{\sqrt{r^2 + Q^2/b^2}} \quad (44)$$

with

$$f(r) = -m + \frac{r^2}{l^2} + 2b^2r \left(r - \sqrt{r^2 + Q^2/b^2} \right) - 2Q^2 \ln \left(r + \sqrt{r^2 + Q^2/b^2} \right) + 2Q^2 \ln \left(l + \sqrt{l^2 + Q^2/b^2} \right) - 2b^2l \left[l - \sqrt{l^2 + Q^2/b^2} \right]. \quad (45)$$

The temperature of the black hole is $T = \frac{2r_+}{l^2} + 4b^2(r_+ - \sqrt{r_+^2 + Q^2/b^2})$. Taking $\Lambda = -1/l^2 = -8\pi P$, we can have the equation of state,

$$P = \frac{T}{16\pi r_+} - \frac{b^2}{4\pi} \left(1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^2}} \right). \quad (46)$$

According to Eq. (25), one can find that such an equation does not admit any inflection point. Hence, the three-dimensional Born-Infeld black hole does not exhibit any critical behavior. This situation agrees with the finding in

the three-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole [28] and the three-dimensional black hole with scalar field [31].

IV. CLOSING REMARKS

In this paper we have studied the thermodynamic behaviors of D -dimensional Born-Infeld AdS black holes in an extended phase space by treating the cosmological constant and the maximal field strength b and their conjugate quantities as thermodynamic variables, respectively. We have written out the equations of state and examined the phase structures by using the standard thermodynamic techniques. We have shown that systems with dimensions higher than four admit a first order small-large black hole phase transition which resembles the liquid-gas phase transition in fluids. We have not found the reentrant phase transition, which was observed in four-dimensional Born-Infeld AdS black holes, for higher-dimensional spacetimes. In addition, in the three-dimensional Born-Infeld AdS black hole, we have shown that there does not exist any critical phenomena. This agrees with the findings in other three-dimensional AdS black holes.

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