Gravitating binaries at the fifth post-Newtonian order in the post-Minkowskian approximation

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The Lagrangian governing the dynamics of a compact binary system is explicitly computed in the post-Minkowskian approximation, and at any post-Newtonian order, by means of the effective field theory approach. This result is then specialized to the fifth post-Newtonian order and allows us to determine one formerly unknown coefficient of the energy expression for binary point masses on the circular orbit as a function of the orbital angular frequency.

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I. INTRODUCTION

The post-Newtonian (PN) approach to the 2-body problem in general relativity (see [1] and [2] for two excellent reviews) has been the object of revived attention in recent years.

On one side, this is certainly related to the start of the era of gravitational wave interferometers like LIGO and Virgo [3], whose output data analysis requires a very accurate knowledge of the binary dynamics, hence the necessity to carry on calculations at very high order in the PN expansion parameter $v^2 \sim GM/r$ [4].

Besides this, the advent of the effective field theory (EFT) method developed in [5], [6] (see also [7] for a recent review) opened the way to new computational possibilities at high PN orders, while also triggering a healthy competition with more traditional approaches, thus leading to the production of several new results both in the conservative [8], [9], [10], [11] and in the radiative [12] sector.

Another useful approach to the 2-body problem is the post-Minkowskian one, namely, an expansion in powers of G only, discarding the link with v expressed by the virial theorem. Initially introduced to study the gravitational field generated by the binary system in the radiation zone [13], it can provide useful information also to the conservative part of the binary dynamics. A closed formula for the first post-Minkowskian (henceforth, simply denoted as PM) Hamiltonian, and *exact* in terms of the stars' momenta has been written in [14] within the Arnowitt-Deser-Misner formalism. In the present work, an analogous expression within the Lagrangian formalism is first derived via use of EFT methods and then the fifth post-Newtonian (5PN) part, containing terms of order Gv^{10}, is explicitly extracted. This allows us to determine one formerly unknown coefficient of the energy-frequency relation for circular orbits at the fifth PN order.

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II. MAIN

A. Setup

The derivation is done in the EFT framework along the lines of [15], which is here briefly reviewed in its basic aspects; the conventions of this paper for what concerns the metric signature, the definition of curvature tensors and other definitions are thus the same as [15]. The starting point is the action $S = S_{pp} + S_{\rm bulk}$, the first term being the worldline point particle action

$$S_{pp} = -\sum_{a=1,2} m_a \int {\rm d}\tau_a = -\sum_{a=1,2} m_a \int \sqrt{-g_{\mu\nu}(x_a^\mu){\rm d}x_a^\mu {\rm d}x_a^\nu}, \tag{1}$$

while the second is the usual Einstein-Hilbert action [16] plus a harmonic gauge fixing term

$$S_{\rm bulk} = \frac{1}{16\pi G}\int {\rm d}^4x\sqrt{-g}\left[R - \frac{1}{2}\Gamma_\mu\Gamma^\mu\right], \tag{2}$$

with $\Gamma^\mu \equiv \Gamma^\mu_{\alpha\beta} g^{\alpha\beta}$. Notice that in the PM limit there are no divergent contributions in dimensional regularization, so one can work in four space-time dimensions right from the beginning.

Following [17], the standard Kaluza-Klein parametrization of the metric [18], [19] (a somehow similar parametrization was first applied within the framework of a PN calculation in [20]) is adopted here:

$$g_{\mu\nu} = e^{2\phi/m_p}\begin{pmatrix} -1 & A_j/m_p \\ A_i/\Lambda & e^{-4\phi/m_p}\gamma_{ij} - A_iA_j/m_p^2 \end{pmatrix}, \tag{3}$$

with $\gamma_{ij} = \delta_{ij} + \sigma_{ij}/m_p$, $m_p^{-2} = 32\pi G$, and the indices i, j running over the 3 spatial dimensions. In terms of the metric parametrization (3), the worldline coupling to the gravitational degrees of freedom ϕ, A_i, σ_{ij} reads

$$S_{pp} = -\sum_{a=1,2} m_a \int \mathrm{d}\tau_a = -m_a \int \mathrm{d}t_a e^{\phi/m_p} \sqrt{\left(1 - \frac{A_i}{m_p} v_a^i\right)^2 - e^{-4\phi/m_p}\left(v_a^2 + \frac{\sigma_{ij}}{m_p} v_a^i v_a^j\right)}, \tag{4}$$

and its Taylor expansion provides the following particle-gravity vertices:

$$V_\phi(v_a) = -\frac{m_a}{m_p}\frac{1+v_a^2}{\sqrt{1-v_a^2}}, \qquad V_A^i(v_a) = \frac{m_a}{m_p}\frac{v_a^i}{\sqrt{1-v_a^2}},$$
$$V_\sigma^{ij}(v_a) = \frac{m_a}{m_p}\frac{v_a^i v_a^j}{2\sqrt{1-v_a^2}}. \tag{5}$$

The propagators for the gravitational degrees of freedom are needed as well. They can be derived by inverting the quadratic part of the pure gravity action S_{bulk} written in terms of the Kaluza-Klein variables:

$$S_{\mathrm{bulk,free}} \supset \frac{1}{32\pi G}\int \mathrm{d}^4x \left\{ \frac{1}{4}[(\vec{\nabla}\sigma)^2 - 2(\vec{\nabla}\sigma_{ij})^2 - (\dot{\sigma}^2 - 2(\dot{\sigma}_{ij})^2)] - 4[(\vec{\nabla}\phi)^2 - \dot{\phi}^2] + \left[\frac{F_{ij}^2}{2} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}^2\right]\right\}. \tag{6}$$

The propagators are fully relativistic and their spatial Fourier transforms are expanded in the quasistatic approximation as a sum of instantaneous contributions along the following pattern:

$$\mathcal{P}(\mathbf{p}^2, t_a, t_b) = \frac{i}{\mathbf{p}^2 - \partial_{t_a}\partial_{t_b}} \simeq \frac{i}{\mathbf{p}^2}\left(1 + \frac{\partial_{t_a}\partial_{t_b}}{\mathbf{p}^2} + \frac{\partial_{t_a}^2\partial_{t_b}^2}{\mathbf{p}^4}\ldots\right)$$
$$\simeq \frac{i}{\mathbf{p}^2}\left(1 - \frac{\partial_{t_a}^2}{\mathbf{p}^2} + \frac{\partial_{t_a}^4}{\mathbf{p}^4}\ldots\right) \tag{7}$$

(the two expressions providing Lagragians terms that differ only by a total derivative) up to the desired post-Newtonian order.

B. The post-Minkowskian Lagrangian

The relevant diagrams for the PM approximation are the three ones shown in Fig. 1. By means of the Feynman rules derived above, one finds amplitudes of this kind:

$$A \sim G m_1 m_2 \int_{\mathbf{p},t_1,t_2} \sum_{n\geq 0} \frac{(\partial_{t_1}\partial_{t_2})^n}{\mathbf{p}^{2(n+1)}} \times \delta(t_1 - t_2) \mathrm{e}^{i\mathbf{p}\cdot\mathbf{r}_{12}} V_k(v_1) V_k(v_2), \tag{8}$$

where the measure of momentum integration is $\mathrm{d}^3p/(2\pi)^3$, $\mathbf{r}_{12} \equiv \vec{x}_1(t_1) - \vec{x}_2(t_2)$, $v_{1,2} \equiv v_{1,2}(t_{1,2})$, and $k = \{\phi, A, \sigma\}$.

As momentum integration gives

$$\int_{\mathbf{p}} \frac{1}{\mathbf{p}^{2(n+1)}} \mathrm{e}^{i\mathbf{p}\cdot\mathbf{r}_{12}} = \frac{\Gamma(3/2-n-1)}{\sqrt{4\pi}^3\Gamma(n+1)}\left(\frac{r_{12}}{2}\right)^{2n-1}, \tag{9}$$

the complete PM potential (at all post-Newtonian orders) can be written in the following way:

$$V_{\mathrm{PM}} = -G m_1 m_2 \sum_{n\geq 0} \frac{\Gamma(3/2-n-1)}{4^n\sqrt{\pi}\Gamma(n+1)} \times \{(\partial_{t_1}\partial_{t_2})^n [r_{12}^{2n-1} f(v_1, v_2)]\}_{t_1=t_2}, \tag{10}$$

$$f(v_1, v_2) \equiv \frac{1 + v_1^2 + v_2^2 - 4\vec{v_1}\cdot\vec{v_2} - v_1^2 v_2^2 + 2(\vec{v_1}\cdot\vec{v_2})^2}{\sqrt{(1-v_1^2)(1-v_2^2)}}. \tag{11}$$

The above equation, which contains higher-than-second derivatives, can be manipulated by adding suitable total derivative terms and/or using the double zero trick [21], that transforms terms quadratic in the accelerations into $\mathcal{O}(G^2)$ ones. The result is a Lagrangian which is classically equivalent to the above expression, but which is just linear in the accelerations and does not contain higher derivatives. Such a possibility will be exploited in the next subsection to determine the binary dynamics at 5PN in the post-Minkowskian approximation in the center-of-mass frame.

Before doing so, it is worth noticing that one can eliminate also the residual accelerations [22] in the Lagrangian by means of a series of contact transformations

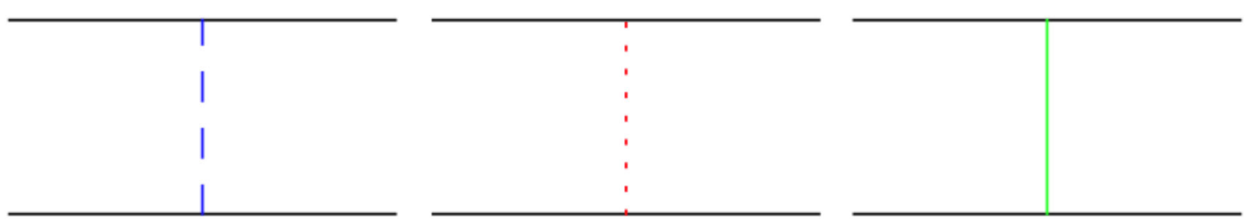

FIG. 1 (color online). The three diagrams giving the post-Minkowskian dynamics. The ϕ, A and σ propagators are represented, respectively, by blue dashed, red dotted and green solid lines.

[23] bringing from $(\vec{r}, \vec{v}_{1,2})$ to some new variables $(\vec{R}, \vec{V}_{1,2})$, such that one is left with an acceleration-free generalized potential $V^*_{\rm PM}(\vec{R}, \vec{V}_{1,2})$ which can assume this remarkably simple form:

$$V^*_{\rm PM} = -\frac{Gm_1m_2}{R}\frac{f(V_1, V_2)}{\sqrt{1 - V_1^2 + (\frac{\vec{R}\cdot\vec{V}_1}{R})^2}}. \tag{12}$$

It has to be said that the above expression is not unique (that is, there are infinite contact transformations that eliminate accelerations), and that a price has to be paid: the contact transformations generally do not respect the harmonic gauge conditions; more than that, the new gauge condition has been proven to explicitly break Lorentz invariance [24].

Along with the kinetic term $m_a\sqrt{1 - v_a^2}$, Eqs. (10) or (12) make the Lagrangian counterpart of the expression derived in [14] within the Hamiltonian formalism. To prove that the two formalisms are indeed equivalent, one should show explicitly that they are related by a Legendre transformation, eventually after a further contact transformation (or canonical transformation, on the Hamiltonian side). An alternative way (actually more viable at high PNs) is to show directly that the two formalisms provide the same predictions for gauge invariant quantities, like the energy of circular orbits as a function of the orbital frequency (measured at spatial infinity); this method has indeed been used to prove that [9] and [10] are two equivalent descriptions of the binary dynamics at the fourth post-Newtonian (4PN) in the post-post-Minkoskian approximation. The next subsection will be thus devoted to work out the prediction of the PM Lagrangian formalism for the energy of circular orbits at 5PN.

C. Dynamics at 5PN

As previously mentioned, Eq. (10) can be transformed, within the harmonic gauge, into in expression which is linear in the accelerations. Such a Lagrangian (whose expression is rather long and not worth reporting) can be used, along the lines described in [25], to determine the center-of-mass position at the needed order, 5PN in this case [26], as well as the energy in a generic frame. Then, the knowledge of the center-of-mass position is exploited (along with the equations of motion) to write the energy in the center-of-mass frame, as a function of the relative position r and velocity v, as well as of the reduced mass μ and of the symmetric mass ratio of the system $\nu \equiv \mu/M$:

$$\begin{aligned}
E^{\rm PM}_{\rm 5PN} = {} & \frac{\mu v^{12}}{1024}(231 - 5401\nu + 49632\nu^2 - 224675\nu^3 + 502425\nu^4 - 445005\nu^5) \\
& + \frac{GM\mu}{256r}[v^{10}(1197 - 14505\nu + 64818\nu^2 - 103695\nu^3 - 41377\nu^4 + 195615\nu^5) \\
& - v^8v_r^2\nu(2541 - 17233\nu + 12815\nu^2 + 125626\nu^3 - 232875\nu^4) \\
& + 2v^6v_r^4\nu(495 - 1410\nu - 12471\nu^2 + 58959\nu^3 - 66825\nu^4) \\
& + 10v^4v_r^6\nu(11 - 410\nu + 2701\nu^2 - 6062\nu^3 + 4131\nu^4) \\
& - 35v^2v_r^8\nu(7 - 80\nu + 313\nu^2 - 473\nu^3 + 207\nu^4) \\
& + 9v_r^{10}\nu(7 - 63\nu + 189\nu^2 - 210\nu^3 + 63\nu^4)] + \mathcal{O}(G^2),
\end{aligned} \tag{13}$$

with $v_r \equiv \vec{v}\cdot\vec{r}/r$. Neglecting gravitational wave radiation, the PM energy is conserved under the action of the PM equation of motion, which has the following form in the center-of-mass frame:

$$\begin{aligned}
\vec{a} = {} & \vec{a}_{0\text{–}3\rm PN} + \vec{a}^{\rm PM}_{\rm 4PN} + \vec{a}^{\rm PM}_{\rm 5PN} + \mathcal{O}(6{\rm PN}), \\
\vec{a}^{\rm PM}_{\rm 4PN} = {} & \frac{G\mu}{256r^3}\{2[16v^8(-21 + 175\nu - 488\nu^2 + 432\nu^3) + 24v^6v_r^2(56 - 397\nu + 872\nu^2 - 528\nu^3) \\
& - 120v^4v_r^4(18 - 125\nu + 269\nu^2 - 178\nu^3) + 280v^2v_r^6(5 - 34\nu + 69\nu^2 - 42\nu^3) \\
& - 315v_r^8(1 - 7\nu + 14\nu^2 - 7\nu^3)]\vec{r} \\
& + [16v^6(157 - 864\nu + 1104\nu^2 + 384\nu^3) - 48v^4v_r^2(120 - 623\nu + 688\nu^2 + 288\nu^3) \\
& + 240v^2v_r^4(22 - 111\nu + 115\nu^2 + 28\nu^3) - 560v_r^6(3 - 14\nu + 11\nu^2 + 2\nu^3)]v^r\vec{v}\},
\end{aligned}$$

$$\begin{aligned}\vec{a}^{\rm PM}_{\rm 5PN} = \frac{G\mu}{256r^3}\{&[4v^{10}(-163+2093\nu-10124\nu^2+21360\nu^3-16192\nu^4)\\
&+6v^8v_r^2(576-6163\nu+24084\nu^2-39184\nu^3+20416\nu^4)\\
&-120v^6v_r^4(64-635\nu+2287\nu^2-3509\nu^3+1896\nu^4)\\
&+140v^4v_r^6(56-521\nu+1713\nu^2-2345\nu^3+1120\nu^4)\\
&-630v^2v_r^8(6-54\nu+167\nu^2-205\nu^3+79\nu^4)+693v_r^{10}(1-9\nu+27\nu^2-30\nu^3+9\nu^4)]\vec{r}\\
&+2[v^8(1493-13616\nu+41376\nu^2-37120\nu^3-14080\nu^4)\\
&-12v^6v_r^2(384-3209\nu+8660\nu^2-6536\nu^3-2816\nu^4)\\
&+30v^4v_r^4(208-1621\nu+3953\nu^2-2712\nu^3-672\nu^4)\\
&-140v^2v_r^6(28-201\nu+421\nu^2-213\nu^3-40\nu^4)+315v_r^8(3-20\nu+36\nu^2-12\nu^3-2\nu^4)]v^r\vec{v}\},\end{aligned}\tag{14}$$

where $\vec{a}_{0-3\rm PN}$ can be read off [1] (Sec. 9.3), $\vec{a}^{\rm PM}_{\rm 4PN}$ can be deduced from the results of [9] and [7], and finally $\vec{a}^{\rm PM}_{\rm 5PN}$ has been determined from the same Lagrangian that led to Eq. (13).

Equations (13) and (14) (together with the known results at lower PNs) are sufficient to determine the ν^5 coefficient of the energy-frequency relation for circular orbits, where the frequency has been traded off of the adimensional quantity $x \equiv (GM\omega_{\rm circ})^{2/3}$. Indeed, $\mathcal{O}(G^2)$ energy terms, even if generally dependent on ν^5 in this gauge, do not give a contribution to the energy-frequency relation at this PN order, as it has been explicitly checked for any possible $\mathcal{O}(G^2v^8)$ Lagrangian structure compatible with Lorentz invariance: this fact is more transparent in the Hamiltonian approach, where it is clear that at the nth PN order the ν^n terms of the center-of-mass Hamiltonian can have at most one power of G; see for instance [11].

A tedious but straightforward calculation finally leads to the determination of the ν^5 coefficient of the energy-frequency relation for circular orbits at 5PN. This has to be added to other previously known terms to give

$$\begin{aligned}E(x)^{\rm circ}_{\rm 5PN} = -\frac{\mu x^6}{2}\left[-\frac{45927}{512}-\left(55.13(3)+\frac{4988}{35}\log x\right)\nu\right.\\
\left.+\left(c_2-\frac{656}{5}\log x\right)\nu^2+c_3\nu^3+c_4\nu^4+\frac{3121}{32}\nu^5\right],\end{aligned}\tag{15}$$

where the first coefficient, corresponding to the limit $\nu \to 0$, is easily derivable from the test particle dynamics in the Schwarzschild background, while the other formerly known terms have been computed in [27], except for the logarithmic term in the ν coefficient which has been derived in [28].

III. CONCLUSIONS

The 5PN dynamics is not of immediate interest for the upcoming gravitational wave detection and phenomenology, but it may become relevant for the next generation of detectors. It is also worth reminding that the effacement principle ceases its effect precisely at this order for spinless stars, thus making the 5PN dynamics very relevant for possibly getting information about the internal structure of compact objects (see [29,30] for two very recent works in this direction). As can be seen from Eq. (15), the test-particle limit and the post-Minkowskian approximation attack the energy-frequency relation from opposite sides; as already happened in the 4PN case, a good road map to determine the remaining unknown coefficients (which depend on the 2nd Newtonian Love number of the compact star) is to work using self-force methods on one side (low ν), and Arnowitt-Deser-Misner or EFT approaches on the other one (terms with high powers of ν).

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APPENDIX: DERIVATION OF $V^*_{\rm PM}$

Since all the Lagrangian terms containing accelerations can be removed by means of the double zero trick and by a suitable contact transformation, one can obtain $V^*_{\rm PM}$ simply by setting to zero all such terms that are generated by doing the derivatives in Eq. (10), thus implying also that one can completely avoid to apply the derivatives to the function $f(v_1, v_2)$.

Actually it is more convenient to take a step backwards, and apply the operator $(-1)^n(\partial_{t_1})^{2n}$ [which is equivalent to

$(\partial_{t_1}\partial_{t_2})^n$ modulo a derivation by parts in the final result [31]] directly in Eq. (9); again, neglecting accelerations one has

$$(-1)^n\partial_{t_1}^{2n}\int_{\mathbf{p}}\frac{1}{\mathbf{p}^{2(n+1)}}e^{i\mathbf{p}.R_{12}} \simeq V_1^{i_1}\ldots V_1^{i_{2n}} I^{i_1\ldots i_{2n}}, \quad \text{(A1)}$$

$$I^{i_1\ldots i_{2n}} \equiv \int_{\mathbf{p}}\frac{p^{i_1}\ldots p^{i_{2n}}}{\mathbf{p}^{2(n+1)}}e^{i\mathbf{p}.\mathbf{R}_{12}}. \quad \text{(A2)}$$

The value of the integral can be deduced from (9) by taking gradients with respect to $\vec{R}_{12}$ and the result can be written as

$$I^{i_1\ldots i_{2n}} = \frac{1}{2^{2n-1}}\frac{(-1)^n\Gamma(\frac{3}{2}-n-1)}{(4\pi)^{3/2}\Gamma(n+1)} \times\sum_{j\le n}\{\eta^{ab}\}^{n-j}\{R^c\}^{2j}P^{n+j}[R^{2n-1}], \quad \text{(A3)}$$

where $\{\eta^{ab}\}^{n-j}$ and $\{R^c\}^{2j}$ indicate the product of $(n-j)$ metric tensors (symmetrized over the indices) and of $2j$ three-vectors $\vec{R}$, respectively, while the operator $P[f]$ is defined as

$$P[f] \equiv \frac{1}{R}\frac{\mathrm{d}f}{\mathrm{d}R} \Rightarrow P^{n+j}[R^{2n-1}] = (-1)^{j+1}(2n-1)!!(2j-1)!!\frac{1}{R^{2j+1}}. \quad \text{(A4)}$$

The next step of the derivation involves combinatorics as, for given n and j, one can contract $2j$ $\vec{V}_1$ vectors with $\{R^c\}^{2j}$ in $B(2n,2j)$ ways [$B(a,b)$ denoting the binomial coefficient], while the remaining $2(n-j)$ $\vec{V}_1$'s contract with the $\{\eta^{ab}\}^{n-j}$ tensor in $(2n-2j-1)!!$ ways. One is thus led to

$$V^*_{\mathrm{PM}} = \frac{Gm_1m_2}{R}\sum_{\{n,j\}}C_{nj}\left(\frac{\vec{R}\cdot\vec{V}_1}{R}\right)^{2j}V_1^{2(n-j)}f(V_1,V_2), \quad \text{(A5)}$$

$$C_{nj} = \frac{(-1)^{n+j+1}}{4^n\sqrt{\pi}}\frac{\Gamma(\frac{3}{2}-n-1)}{\Gamma(n+1)}\frac{(2n-1)!!(2j-1)!!(2n)!(2n-2j-1)!!}{(2j)!(2n-2j)!}. \quad \text{(A6)}$$

Finally the following resummation formula:

$$\sum_{\{nj\}}C_{nj}x^{2j}y^{2(n-j)} = -\frac{1}{\sqrt{1+x^2-y^2}} \quad \text{(A7)}$$

brings us to Eq. (12), which can be eventually symmetrized:

$$V^*_{\mathrm{PM}} = -\frac{Gm_1m_2}{2R}\left[\frac{1}{\sqrt{1-V_1^2+(\frac{\vec{R}\cdot\vec{V}_1}{R})^2}}+\frac{1}{\sqrt{1-V_2^2+(\frac{\vec{R}\cdot\vec{V}_2}{R})^2}}\right]f(V_1,V_2). \quad \text{(A8)}$$

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