Consistent probabilities in perfect fluid quantum universes

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(Received 26 August 2013; published 21 January 2014)

Recently it has been claimed that the Wheeler–DeWitt quantization of gravity is unable to avoid cosmological singularities. However, in order to make this assertion, one must specify the underlying interpretation of quantum mechanics which has been adopted. For instance, several nonsingular models were obtained in Wheeler–DeWitt quantum cosmology in the framework of the de Broglie–Bohm quantum theory. Conversely, there are specific situations where the singularity cannot be avoided in the framework of the consistent histories approach to quantum mechanics. In these specific situations, the matter content is described by a scalar field, and the Wheeler–DeWitt equation looks like a Klein–Gordon equation. The aim of this work is to study a possible singular behavior of quantum cosmological models in which the matter content is described by a hydrodynamical perfect fluid, where the resulting wave equation is a genuine Schrödinger equation. In this case, it is shown that the conclusions of the consistent histories and the de Broglie–Bohm approaches coincide in the quantum cosmological models where the curvature of the spatial sections is not positive definite, namely, that the cosmological singularities are eliminated. In the case of positive spatial curvature, the family of histories is no longer consistent, and no conclusion can be given in this framework.

DOI: 10.1103/PhysRevD.89.023514 PACS numbers: 98.80.Cq, 04.60.Ds

I. INTRODUCTION

The standard cosmological model has been successfully tested until the nucleosynthesis era, if one assumes the existence of the so-called dark sector. However, the extrapolation of this scenario to higher energies leads to a singularity, where the volume of the Universe becomes null, and physical quantities such as energy densities and the space-time curvature diverge in a finite cosmic time. A natural alternative is to investigate possible quantum effects in this high-energy limit, which may avoid such cosmological singularities. Indeed, many investigations have shown the elimination of the singularities in quantum cosmological models using the Wheeler–DeWitt (WDW) [1] and loop quantum gravity quantization schemes [2]. However, some authors have argued that in the WDW approach the singularity is not avoided [2–6].

To deal with these questions, one must define precisely how one can obtain physical information from the quantum state of the Universe: as we know, the standard Copenhagen interpretation cannot be used in quantum cosmology because it needs a classical external domain where measurements are made outside the quantum system in order to give a physical meaning to the quantum state, and in cosmology one is dealing with a closed quantum system, the Universe, without any external classical observers. Hence, one has to use an alternative quantum theory, where external agents do not play any special role, and there are many viable proposals in the literature, for example, the ones proposed in Refs. [7,8].

Among the papers mentioned above questioning the WDW approach, only two (Refs. [2,3]) have specified the interpretation of quantum mechanics they are using, which was the consistent histories approach [9,10]. In this framework, probabilities are not assigned to events as in usual quantum mechanics, but to whole histories. A history of a closed physical system is a succession of properties of this system occurring at different times. An example of a property of a system is the sentence "the eigenvalue of the observable $\hat{B}$ is in the set D." To each property is associated a projector operator. In the above example, it would be the projector $\hat{P}$ onto the subspace of the Hilbert space containing all eigenvectors of $\hat{B}$ with eigenvalues in the set D. However, as we know, we cannot assign probabilities to every history in quantum mechanics. The interference figure obtained from the two slit experiment is an evidence of this fact. Hence, we must establish what are the conditions on families of histories in order to be possible to assign probabilities to all members of such families. Once we obtain these conditions, we will have the possibility of saying that a history of the Universe is more probable than another one, without mentioning observers or measurements.

The Wheeler–DeWitt quantization of a spatially flat Friedmann cosmological model with a massless free scalar field was implemented under this framework. The resulting Wheeler–DeWitt equation is a Klein–Gordon-like equation. In that case, as shown by Craig and Singh [3], the cosmological singularities are not avoided if one takes the square root of the Klein–Gordon equation and their associated Newton–Wigner states and histories concerning properties of the quantum system at only two specific moments of time: the infinity past and the infinity future. However, as shown in Ref. [11], if one considers the two frequency sectors of the resulting Klein–Gordon-like equation and/or three or more instants of time, the family of histories is not consistent anymore, and no conclusion about the existence of singularities drawn. Furthermore, if one analyzes the same physical system within the de Broglie–Bohm quantum theory, in which trajectories can be unambiguously defined through the so-called guidance relations, there are plenty of states where quantum bounces occur in both quantization schemes, and the cosmological singularity disappears. Also, in Ref. [11] the reasons for these discrepant results coming from the two different quantum approaches were discussed: it is due to the fact that the resulting Wheeler–DeWitt equation of the minisuperspace quantum cosmological model considered is a Klein–Gordon-like equation, not an usual Schrödinger-like equation. As we know, the predictions of these two approaches agree in usual physical systems investigated in the laboratory, which obey the Schrödinger equation.

The aim of this paper is to discuss carefully this question concerning the existence of singularities in Friedmann–Lemaître–Robertson–Walker (FLRW) quantum universes, but now considering the other widely studied global description of the matter content of the very hot universe, namely, the perfect hydrodynamical fluid representation.[1] Hydrodynamical fluids in a homogeneous and isotropic universe can be described by a scalar field without potential but with a nontrivial power law kinetic term. This scalar field is connected to the velocity potential of the fluid and defines a preferred time parameter, as we will see. Quantization of hydrodynamical fluids in this context has been justified in physical terms in Ref. [27]. In this case it is well known that the quantization procedure yields a Schrödinger-like equation. In this situation, we expect that the consistent histories approach should give the same results as the ones already obtained within the de Broglie–Bohm quantum theory, namely, that the cosmological singularities are eliminated [12–15]. And indeed, our analysis have shown that in the cases where it is possible to define consistent probabilities for histories (the models where the spatial curvature is not positive definite) evaluated in two specific moments ($t_1 \to -\infty$ and $t_2 \to +\infty$), the histories without singularities have probability 1 to occur, as in the de Broglie–Bohm case. For positive spatial curvature, we have shown that no consistent probability can be associated to histories concerning those two moments.

The paper will be divided as follows. In Sec. II we present the consistent histories framework. In Sec. IIIA we present the classical cosmological model, and in Sec. IIIB we discuss the quantization of the latter. In Sec. IV we discuss the FLRW model in the consistent histories interpretation, and then in Sec. IVA, we present the criterion for a nonsingular universe in this context. In Sec. IVB we evaluate the probabilities of the FLRW quantum universes to reach a singularity in $t \to \pm\infty$ in flat and open universes, and in Sec. IVC we discuss the closed case. Finally in Sec. V we present our final remarks. Numerical calculations supporting our results are shown in the Appendix.

[1]We will choose this fluid to be radiation, because at high energies the rest energy of any particle becomes negligible with respect to its total kinetic energy. Furthermore, other possibilities will lead essentially to the same results.

II. CONSISTENT HISTORIES

The consistent histories approach, also known as generalized quantum mechanics, was proposed initially by Griffiths [9] and Omnes [10] and later developed in the gravity context mainly by Hartle and Gell-Mann [16,17] and Halliwell [18–20]. It is a framework in which probabilistic predictions can be made about quantum systems avoiding the need of external classical observers and preserving most of the conceptual structure of the standard Copenhagen interpretation. In this section we will present some general aspects of this interpretation that are relevant to the issue of the presence of singularities in quantum cosmology. In this approach, instead of associating probabilities to a single eigenvalue, we assign them to a history, that is, a sequence of time ordered eigenvalue intervals. For a given set of observables A^α with eigenvalue intervals Δa_k^α, a history h may be represented by a class operator C_h defined as

$$C_h = P^{\alpha_1}_{\Delta a_{k_1}}(t_1)\ldots P^{\alpha_{n-1}}_{\Delta a_{k_{n-1}}}(t_{n-1})P^{\alpha_n}_{\Delta a_{k_n}}(t_n), \tag{1}$$

where $P^\alpha_{\Delta a_k}(t)$ is the projector onto the subspace for which the kth eigenvalue of the observable A^α in time t is in the interval Δa_k. The time dependent projectors are defined using the Heisenberg operators,

$$P^\alpha_{\Delta a_k}(t) = U^\dagger(t)P^\alpha_{\Delta a_k}U(t), \tag{2}$$

where $U(t)$ is the evolution operator of the quantum system with Hamiltonian H,

$$U(t_i - t_j) = \exp\,[-\imath H(t_i - t_j)], \tag{3}$$

where we considered $\hbar = 1$ and $\imath$ is the imaginary unity.

We are interested in asking questions about the eigenvalue intervals of the scale factor operator, $\hat{R}$, at some moments of time. The time independent projector operator associated with this is

$$P_{\Delta R} = \int_{\Delta R} dR|R\rangle\langle R|. \tag{4}$$

At this point, we may distinguish two kinds of histories. The fine-grained histories are the most precise descriptions of a physical system and deal with the smallest intervals of eigenvalues. On the other hand, the coarse-grained histories are partitions of the fine-grained ones, considering larger intervals of eigenvalues. Those histories are defined to answer physically relevant questions, which, in our case, is whether the Universe is singular or not.

The wave function that represents the amplitude for a given initial state to follow the history h, the branch wave function, is defined as

$$|\Psi_h\rangle = C_h^\dagger|\Psi\rangle, \tag{5}$$

which is not normalized. The branch wave function in Eq. (5) represents a physical system evolving through a set of instants of time t_i, generally discontinuous. Since it is not a function of time t, the branch wave function does not satisfy the Schrödinger equation.

To have a meaningful notion of probability, we must require some reasonable conditions upon the probability p we want to assign to each history, (6)

$$0 \leq p(C_h) \leq 1, \tag{6a}$$

$$p(\mathbb{1}) = 1, \tag{6b}$$

$$p(C_h + C_{h'}) = p(C_h) + p(C_{h'}), \tag{6c}$$

where $\mathbb{1}$ is the identity operator and h, h' are disjoint histories. Following the Hartle–Gell-Mann [16] approach, we define the decoherence functional for a set of histories $\{h_i\}$, considering pure states only, as

$$D(h,h') = \langle\Psi_h|\Psi_{h'}\rangle. \tag{7}$$

It has been shown that in order to satisfy Eq. (6), the sufficient condition is

$$D(h,h') \approx 0, \quad h \neq h'. \tag{8}$$

If the family of histories satisfies Eq. (8), we say that this family is consistent. Since $\sum_k P^{\alpha}_{\Delta a_k} = 1$, we have

$$\sum_h C_h = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} P^{\alpha_1}_{\Delta a_{k_1}}(t_1)P^{\alpha_2}_{\Delta a_{k_2}}(t_2)\ldots P^{\alpha_n}_{\Delta a_{k_n}}(t_n) = 1, \tag{9}$$

and the decoherence functional is normalized because

$$\sum_h |\Psi_h\rangle = \sum_h C_h^\dagger|\Psi\rangle = |\Psi\rangle. \tag{10}$$

The probability of a history can then be written as

$$p = |P^{\alpha_1}_{\Delta a_{k_1}}(t_1)\ldots P^{\alpha_{n-1}}_{\Delta a_{k_{n-1}}}(t_{n-1})P^{\alpha_n}_{\Delta a_{k_n}}(t_n)|\Psi\rangle|^2 = |C_h^\dagger|\Psi\rangle|^2 = \langle\Psi_h|\Psi_h\rangle. \tag{11}$$

Using Eqs. (8), (10), and (11) and considering a consistent family of histories, we can write the decoherence functional as

$$D(h,h') = \langle\Psi_h|\Psi_{h'}\rangle \approx p(h)\delta_{h'h}, \tag{12}$$

where $p(h) \equiv p(C_h)$ represents the probability that the system follows a history h. One can interpret condition (8) as demanding that no interference can occur between histories.

In what follows, a singular history will be defined as the one which contains at least one projector associated with operator eigenvalues characterizing a space-time singularity.

III. RADIATION DOMINATED FLRW

With the quantum interpretation specified in the last section, we will calculate the probabilistic predictions concerning the existence of singularities in specific cosmological models. In this section we will introduce FLRW models where the matter content is described by a radiation fluid with $p = \rho/3$ [21,22]. The result can easily be generalized to any equation of state $p = w\rho$ [23,24].

A. Classical model

The line element of the homogeneous and isotropic FLRW geometries can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2(t)dt + R^2(t)\sigma_{ij}dx^i dx^j, \tag{13}$$

where $N(t)$ is the lapse function, $R(t)$ is the scale factor, and $R^2\sigma_{ij}$ is the metric of the homogeneous and isotropic 3-space of constant curvature $k = -1$, 0, 1, corresponding to the hyperbolic, flat, and spherical geometries, respectively. We also set $c = 1$. The Latin indices vary from 1 to 3, and the Greek indexes vary from 0 to 3. The matter content is a perfect fluid, where we use the Schutz [25] formalism for relativistic fluids. The total action reads

$$\begin{aligned} S &= \int d^4x\mathcal{L} \\ &= \int d^4x(\mathcal{L}_g + \mathcal{L}_f) \\ &= \int_M d^4x\sqrt{-g}\,{}^4R + 2\int_{\partial M^4} d^3x\sqrt{-h}h_{ij}K^{ij} \\ &\quad + \int_M d^4x\sqrt{-g}p, \end{aligned} \tag{14}$$

where the last term corresponds to the fluid action, p is the pressure, $g = \mathrm{Det}(g_{\mu\nu})$, K_{ij} is the extrinsic curvature, h_{ij} is the 3-metric, $h = \mathrm{Det}(h_{ij})$, ∂M^4 is the boundary of the manifold M^4, and ${}^4R$ is the Ricci curvature escalar constructed with the metric $g_{\mu\nu}$.

For the geometry represented by Eq. (13) and fluid described by radiation, one can show that the above action (14) reduces to [21,22,26,27]

$$S = \int dt(p_R\dot{R} + p_T\dot{T} - N\mathcal{H}), \tag{15}$$

where T is the degree of freedom associated to the radiation fluid, p_R and p_T are the momenta canonically conjugate to R and T, respectively, and it can be seen that the lapse function $N(t)$ is the Lagrange multiplier of the super-Hamiltonian

$$\mathcal{H} = -\frac{p_R^2}{24R} - 6kR + \frac{p_T}{R}. \tag{16}$$

The momentum p_T is linear in the super-Hamiltonian, so we choose the time gauge $T = t$, inducing $N = R$ (conformal time), which reduces the action to its canonical form

$$S = \int dt\left(p_R\dot{R} + \frac{p_R^2}{24} + 6kR^2\right), \tag{17}$$

where one can read the canonical Hamiltonian

$$H \equiv -\frac{p_R^2}{24} - 6kR^2$$

and t is the conformal time.

The equations of motion are

$$\frac{dR}{dt} = -\frac{p_R}{12}, \qquad \frac{dp_R}{dt} = 12kR. \tag{18}$$

The differential equation for R can be written as

$$\ddot{R} + kR = 0, \tag{19}$$

which has the following solutions:

$$R(t) = R_0\begin{cases} \sinh t & k = -1, \\ t & k = 0, \\ \sin t & k = +1. \end{cases} \tag{20}$$

In all cases the Universe reaches an unavoidable classical singularity in $t = 0$. In the next subsection, we present the quantum model for the radiation dominated FLRW universe in order to discuss whether the singularity persists in the quantum regime.

B. Canonical quantization and boundary conditions

As we have reduced the action to its canonical form, the canonical quantization will lead to a Schrödinger-like equation:[2]

$$i\frac{\partial\Psi}{\partial t} = \hat{H}\Psi. \tag{21}$$

The above Schrödinger equation is simple to solve, but one must take care of the fact that R is defined only for $R \geq 0$. As a result, the description of the quantum system requires not only the canonical quantization procedure but also the imposition of boundary conditions to ensure that the wave functions Ψ are square integrable and the Hamiltonian operator is self-adjoint. The self-adjointness of the Hamiltonian is a necessary condition to have unitary evolution. We may construct the Hamiltonian operator by performing the substitution $p_R \to -\imath\frac{d}{dR}$ yielding

$$\hat{H} = \frac{1}{24}\frac{d^2}{dR^2} - 6kR^2, \tag{22}$$

where $\hbar = 1$. The self-adjointeness condition may be expressed as

$$\langle\Psi_1|\hat{H}\Psi_2\rangle = \langle\hat{H}\Psi_1|\Psi_2\rangle. \tag{23}$$

From this point, in order to simplify the notation, we will omit the (̂). The condition (23) is equivalent to

$$\int_0^\infty dR\Psi^\star(R)\frac{d^2\Psi(R)}{dR^2} = \int_0^\infty dR\frac{d^2\Psi^\star(R)}{dR^2}\Psi(R). \tag{24}$$

After integration by parts twice, we get

$$\begin{aligned}&\left(\Psi_1^\star(R)\frac{d\Psi_2(R)}{dR} - \frac{d\Psi_1^\star(R)}{dR}\Psi_2(R)\right)(+\infty)\\ &\quad= \left(\Psi_1^\star(R)\frac{d\Psi_2(R)}{dR} - \frac{d\Psi_1^\star(R)}{dR}\Psi_2(R)\right)(0).\end{aligned} \tag{25}$$

As we want to find only square integrable solutions, the left side of Eq. (25) must vanish, so

$$\left(\Psi_1^\star(R)\frac{d\Psi_2(R)}{dR} - \frac{d\Psi_1^\star(R)}{dR}\Psi_2(R)\right)(0) = 0. \tag{26}$$

The necessary and sufficient condition for Eq. (26) to be satisfied is

[2]One could also employ a Dirac quantization procedure through the imposition of the condition $\mathcal{H}\Psi = 0$, with $\mathcal{H}$ given by Eq. (16). As the momentum p_T appears linearly in $\mathcal{H}$, the resulting Wheeler–DeWitt equation reduces to the Schrödinger equation written below. Hence, both quantization schemes yield the same quantum equation.

$$\Psi'(0) = \eta\Psi(0), \tag{27}$$

where η is in the interval $(-\infty, +\infty)$ and $'$ denotes a derivative with respect to R. For simplicity we restrict ourselves to the case $\eta = 0$, using the explicit results presented in Ref. [21].

Let $G\,(R, R', t)$ be the propagator of the Hamiltonian (22) in the Hilbert space $L^2(-\infty, +\infty)$. The propagator $G^{(a)}$ (R, R', t) which satisfies $\Psi'(0) = 0$ in the half-line Hilbert space $L^2(0, +\infty)$ is

$$G^{(a)}(R, R', t) = G(R, R', t) + G(R, -R', t), \tag{28}$$

where $G(R, R', t)$ is the usual quantum harmonic oscillator propagator for which the mass is $m = 12$ and the frequency is $\omega = \sqrt{k}$ [28]. It can be written as

$$G(R,R',t) = \left(\frac{6\sqrt{k}}{\pi\imath \sin\sqrt{k}t}\right)^{1/2} \times \exp\left\{\frac{6\imath\sqrt{k}}{\sin\sqrt{k}t}((R^2+R'^2)\cos\sqrt{k}t - 2RR')\right\}. \tag{29}$$

In the next section, we present the consistent histories approach to deal with the singularity issue in this context.

IV. CONSISTENT PROBABILITIES IN RADIATION DOMINATED FLRW QUANTUM MODELS

In this section we present a precise criterion to determine if a given quantum model can avoid the singularity. This criterion is developed in Ref. [3]. In the Sec. IVA, we specify this criterion and perform some calculations in order to simplify the problem and make it more tractable. After that, we evaluate the quantum model presented in Sec. IIIB.

A. Singularity in consistent histories

To address the question whether the Universe is singular or not, one must construct a suitable family of histories. To pursue that we focus on the behavior of the scale factor. We say that a history is singular if the scale factor is within an interval $\Delta R_\star = [0, R_\star]$, where $R_\star$ is a fiducial value, which can be arbitrarily small. This interval represents the Universe arbitrarily near the singularity. We also define the complementary interval $\overline{\Delta R_\star} = (R_\star, +\infty)$, where the Universe is out of the singularity. The class operator that represents the history without singularity is

$$C_b(t_1, \ldots, t_n) = P_{\overline{\Delta R_\star}}(t_n) P_{\overline{\Delta R_\star}}(t_{n-1}) \ldots P_{\overline{\Delta R_\star}}(t_1), \tag{30}$$

for n moments in time, where the $P_{\overline{\Delta R_\star}}(t)$ are the projectors in the interval $\overline{\Delta R_\star}$. Any history for which at least one projector defined in the singularity interval gives a non-null value when applied to the state of the system is a singular history. From now on, for simplicity, we consider histories specifying two moments t_1 and t_2 evaluated in the limits $t_1 \to -\infty$ and $t_2 \to +\infty$. The class operator for the singularity is

$$C_s(t_1, t_2) = P^1_{\Delta R_\star} P^2_{\overline{\Delta R_\star}} + P^1_{\overline{\Delta R_\star}} P^2_{\Delta R_\star} + P^1_{\Delta R_\star} P^2_{\Delta R_\star}, \tag{31}$$

with $P^1_{\Delta R} := P_{\Delta R}(t_1)$ and $P^2_{\Delta R} := P_{\Delta R}(t_2)$. The class operator in Eq. (31) can be more conveniently written as [3]

$$C_s(t_1, t_2) = P^1_{\Delta R_\star} + P^2_{\Delta R_\star} - P^1_{\Delta R_\star} P^2_{\Delta R_\star}, \tag{32}$$

where we considered $P^1_{\Delta R_\star} P^2_{\Delta R_\star} + P^1_{\Delta R_\star} P^2_{\Delta \bar{R}_\star} = P^1_{\Delta R_\star}$. The criterion for consistency of the family is

$$\begin{aligned}\langle\Psi|C_b C_s^\dagger|\Psi\rangle &= \langle\Psi|P^1_{\Delta\bar{R}_\star} P^2_{\Delta\bar{R}_\star} P^1_{\Delta R_\star}|\Psi\rangle \\ &\quad + \langle\Psi|P^1_{\overline{\Delta R_\star}} P^2_{\overline{\Delta R_\star}} P^2_{\Delta R_\star}|\Psi\rangle \\ &\quad - \langle\Psi|P^1_{\overline{\Delta R_\star}} P^2_{\overline{\Delta R_\star}} P^2_{\Delta R_\star} P^1_{\Delta R_\star}|\Psi\rangle = 0.\end{aligned} \tag{33}$$

However, the last two terms are null since they contain products of projectors in orthogonal subspaces. The relation for the consistency of the family of histories may also be written in terms of projectors in the singularity interval only,

$$\begin{aligned}\langle\Psi|(1 - C_s)C_s^\dagger|\Psi\rangle &= \langle\Psi|C_s^\dagger|\Psi\rangle - \langle\Psi|C_s C_s^\dagger|\Psi\rangle \\ &= \langle\Psi|\Psi_s\rangle - \langle\Psi_s|\Psi_s\rangle \\ &= \langle\Psi|P^2_{\Delta R_\star} P^1_{\Delta R_\star}|\Psi\rangle \\ &\quad - \langle\Psi|P^1_{\Delta R_\star} P^2_{\Delta R_\star} P^1_{\Delta R_\star}|\Psi\rangle,\end{aligned} \tag{34}$$

where we used that $C_s + C_b = 1$.

If the histories decohere, the probability for having the singularity is

$$\begin{aligned}p_s &\equiv \langle\Psi|C_s C_s^\dagger|\Psi\rangle \\ &= -\langle\Psi|P^1_{\Delta R_\star} P^2_{\Delta R_\star} P^1_{\Delta R_\star}|\Psi\rangle \\ &\quad + \langle\Psi|P^2_{\Delta R_\star}|\Psi\rangle + \langle\Psi|P^1_{\Delta R_\star}|\Psi\rangle.\end{aligned} \tag{35}$$

The probability that the Universe is not singular is given by

$$\begin{aligned}p_b &\equiv \langle\Psi|(1 - C_s)(1 - C_s^\dagger)|\Psi\rangle \\ &= 1 - 2\,\mathrm{Re}(\langle\Psi|\Psi_s\rangle) + \langle\Psi_s|\Psi_s\rangle,\end{aligned} \tag{36}$$

where we considered again that $C_s + C_b = 1$. In the next section, we evaluate these probabilities and check whether the histories decohere in the FLRW quantum model filled with radiation.

B. Singularity in radiation dominated FLRW models

Consider the wave function

$$\langle R|\Psi(t)\rangle = \Psi(R,t) = \int_0^{+\infty} dR\cdot G^{(a)}(R,R',t)\Psi_0^{(a)}(R), \quad (37)$$

which is square integrable for a given initial condition $\Psi_0^{(a)}(R) = \langle R|\Psi\rangle$. If we apply the projector in the singularity interval to a state $|\Psi\rangle$, we get

$$|\Phi(t)\rangle \equiv P_{\Delta R_\star}(t)|\Psi\rangle = U^\dagger(t)\int_0^{R_\star} dR|R\rangle U(t)\langle R|\Psi\rangle. \quad (38)$$

Inserting the resolution of the identity $\int_0^{+\infty} dR'|R'\rangle\langle R'|$ in the equation above yields

$$\begin{aligned}|\Phi(t)\rangle &= \int_0^{+\infty} dR'|R'\rangle\langle R'|U^\dagger\int_0^{R_\star} dR|R\rangle\langle R|\Psi(t) \\ &= \int_0^{+\infty} dR'|R'\rangle\int_0^{R_\star} G^{(a)}(R,R',-t)\Psi(R,t)dR. \quad (39)\end{aligned}$$

Let us evaluate the integral

$$\int_0^{R_\star} G^{(a)}(R,R',-t)\Psi(R,t)dR \quad (40)$$

in the limits $t\to\pm\infty$ by considering its square modulus. It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}&\int_0^{R_\star} |G^{(a)}(R,R',-t)\Psi(R,t)|^2 dR \\ &\le \left(\int_0^{R_\star} |G^{(a)}(R,R',-t)|^2 dR\right)\times\left(\int_0^{R_\star}|\Psi(R,t)|^2 dR\right). \quad (41)\end{aligned}$$

The last integral in the inequality above is a real and positive number c^2 with $0\le c^2\le 1$ for a given time t. Let us analyze the square modulus of $G^{(a)}$ $(R, R', -t)$, which yields[3]

$$\begin{aligned}|G^{(a)}(R,R',t)|^2 &= |G(R,R',t)+G(R,-R',t)|^2 \\ &= \frac{12\sqrt{|k|}}{\pi|\sin(\sqrt{k}t)|}\left(1+\cos\left(\frac{24\sqrt{k}RR'}{\sin(\sqrt{k}t)}\right)\right). \quad (42)\end{aligned}$$

Note that the first term in Eq. (42) does not depend on R. We may rewrite the integral of Eq. (42) as

$$\begin{aligned}&\frac{12\sqrt{|k|}}{\pi|\sin(\sqrt{k}t)|}\int_0^{R_\star} dR\left(1+\cos\left(\frac{24\sqrt{k}RR'}{\sin(\sqrt{k}t)}\right)\right) \\ &= \frac{12}{\pi|\sin(\sqrt{k}t)|}\Big|_0^{R^\star} + \frac{\sin(\sqrt{k}t)}{2R'|\sin\sqrt{k}t|}\int_0^{u^\star}\cos(u)du \\ &= \frac{12R^\star}{\pi|\sin(\sqrt{k}t)|} + \frac{1}{2R'}\sin\left(\frac{24R'R^\star}{\sin(\sqrt{k}t)}\right), \quad (43)\end{aligned}$$

where we defined $u = 24RR'/\sin(\sqrt{k}t)$ and $u^\star = 24R^\star R'/\sin(\sqrt{k}t)$. In the spatially flat and hyperbolic universes in the limits $t\to\pm\infty$, we have $u^\star\to 0$.

Using Eq. (43), we see that the limits considered for the integral of $|G^{(a)}(R,R',t)|^2$ in the spatially flat and hyperbolic universes are

$$\lim_{t\to\pm\infty}\int_0^{R^\star}|G^{(a)}(R,R',-t)|^2 dR = \begin{cases}0, & k=-1,\\ 0, & k=0.\end{cases} \quad (44)$$

As $\int_0^{R_\star}|\Psi(R,t)|^2 dR$ is limited in the interval [0, 1], the right side of Eq. (41) is null; hence, we have

$$\lim_{t\to\pm\infty}\left|\int_0^{R_\star} G^{(a)}(R,R',-t)\Psi(R,t)dR\right| = 0, \quad (45)$$

for $k=-1$ and $k=0$. Therefore, we have shown that $P_{\Delta R_\star}(t)|\Psi\rangle$ vanishes in the infinite past and in the infinite future whatever the initial state. According to Eqs. (34), (35), and (36), we then have 46

$$\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\Psi_b|\Psi_s\rangle = 0, \quad (46a)$$

$$\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\Psi_s|\Psi_s\rangle = 0, \quad (46b)$$

$$\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\Psi_b|\Psi_b\rangle = 1. \quad (46c)$$

That is, the set of histories is consistent, and the probability of having a singularity is null for the flat and open Friedmann universes, for any initial state.To check this result explicitly, let us consider the particular case of the reasonable wave function obtained from the initial Gaussian,

$$\Psi_0^{(a)}(R) = \sqrt{\frac{8\sigma}{\pi}}\exp(-\beta R^2), \quad (47)$$

where $\beta = \sigma + \imath p$ and $\sigma > 0$. Using the propagator defined in Eq. (28), we evaluate $\Psi(R, t)$,

$$\begin{aligned}\Psi(R,t) &= \int_0^{\infty} dR\cdot G^{(a)}(R,R',t)\Psi_0^{(a)}(R) \\ &= \int_{-\infty}^{+\infty} dR\cdot G(R,R',t)\Psi_0^{(a)}(R), \quad (48)\end{aligned}$$

[3]Since we are interested in the limits $t\to\pm\infty$, we will ignore the minus sign in t.

which results

$$\Psi(R,t)=\left\{\frac{12\sqrt{2\sigma}\sqrt{k}}{\sqrt{\pi}\cos(\sqrt{k}t)(\beta\tan(\sqrt{k}t)-6\iota\sqrt{k})}\right\}^{1/2}\exp\left\{R^2\frac{6\iota\sqrt{k}}{\tan(\sqrt{k}t)}\left(1+\frac{6\iota\sqrt{k}}{\cos^2(\sqrt{k}t)(\beta\tan(\sqrt{k}t)-6\iota\sqrt{k})}\right)\right\}, \tag{49}$$

from which we can calculate the mean values

$$\langle R(t)\rangle=\kappa\{\sigma^2\sin^2\sqrt{k}t+(6-p\tanh\sqrt{k}t)^2\cosh^2\sqrt{k}t\}^{1/2}, \tag{50}$$

where $\kappa\coloneqq\frac{\sqrt{2}}{12(\pi\sigma)^{1/2}}$. In all cases $k=-1,\,0,\,+1$, the mean values of R never reach the singular value $R=0$. However, this is not a sufficient condition to state that the present model is never singular. Turning back to the consistent histories approach, the projected state in the singularity interval is

$$P_{\Delta R_\star}(t)|\Psi\rangle=U^\dagger(t)\int_0^{R_\star}dR|R\rangle U(t)\langle R|\Psi\rangle. \tag{51}$$

As before, inserting the identity $\int_0^{+\infty}dR'|R'\rangle\langle R'|$, the equation above takes the form

$$\begin{aligned}&\int_0^{+\infty}dR'|R'\rangle\langle R'|U^\dagger(t)\int_0^{R_\star}dR|R\rangle\langle R|\Psi(t)\rangle\\&\quad=\int_0^{+\infty}dR'|R'\rangle\int_0^{R_\star}G^{(a)}(R,R',-t)\Psi(R,t)dR.\end{aligned} \tag{52}$$

We evaluate the integral

$$I=\int_0^{R_\star}G^{(a)}(R,R',-t)\Psi(R,t)dR, \tag{53}$$

which results for the $k\to0$ case in

$$\begin{aligned}I&=\frac{(2\pi)^{1/4}}{\sigma^{1/4}\exp(-\sigma R'^2)}\\&\quad\times\left[\mathrm{erf}\left(\frac{6\iota(R_\star-R')+\sigma R't}{\sqrt{\iota 6t-\sigma t^2}}\right)\right.\\&\quad\left.+\mathrm{erf}\left(\frac{6\iota(R_\star+R')-\sigma R't}{\sqrt{\iota 6t-\sigma t^2}}\right)\right].\end{aligned} \tag{54}$$

In the considered limits, I vanishes; hence, we have

$$\lim_{t\to\pm\infty}P_{\Delta R_\star}(t)|\Psi\rangle=0, \tag{55}$$

and the family of histories is consistent since

$$\langle\Psi C_bC_s^\dagger|\Psi\rangle=\langle\Psi|P^1_{\overline{\Delta R_\star}}P^2_{\overline{\Delta R_\star}}P^1_{\Delta R_\star}|\Psi\rangle=0. \tag{56}$$

According to Eqs. (35) and (36), we have (57)

$$\lim_{\substack{t_1\to-\infty\\t_2\to-\infty}}\langle\Psi_s|\Psi_s\rangle=0, \tag{57a}$$

$$\lim_{\substack{t_1\to-\infty\\t_2\to-\infty}}\langle\Psi_b|\Psi_b\rangle=1. \tag{57b}$$

The same result is found for the $k=-1$ case. Therefore, these results confirm our previous analysis, the set of histories is consistent, and the probability of finding a singularity is null for the flat and open Friedmann universes filled with radiation.

C. Spherical universe ($k=+1$)

For the spherical case, we evaluate the consistency of the families of histories directly by solving the integral in Eq. (33), where we consider the wave function (49). The solution is

$$\langle\Psi_b|\Psi_s\rangle \propto \frac{\csc(t_1)\csc(t_1-t_2)}{|\sqrt{\beta+12\imath\tan(t_1/2)}|^2}\times\left\{\left[\sqrt{|d(t_1,t_2)|}\mathrm{C}\left(\lambda\frac{1}{\sqrt{|d(t_1,t_2)|}}\right)+\imath\mathrm{S}\left(\lambda\frac{1}{\sqrt{|d(t_1,t_2)|}}\right)\mathrm{sgn}(1/d(t_1,t_2))\right]\right.$$
$$+\left[\frac{1}{\sqrt{|d(t_1,t_2)|}}\mathrm{C}(\lambda\sqrt{|d(t_1,t_2)|})-\imath\mathrm{S}(\lambda\sqrt{|d(t_1,t_2)|})\mathrm{sgn}(d(t_1,t_2))\right]\Bigg\}$$
$$\times\left\{\left[\sqrt{|d(t_1,t_2)|}\left(1-2\mathrm{C}\left(\lambda\frac{1}{\sqrt{|d(t_1,t_2)|}}\right)+\imath\left(-1+2\mathrm{S}\left(\lambda\frac{1}{\sqrt{|d(t_1,t_2)|}}\right)\right)\mathrm{sgn}(d(t_1,t_2))\right)\right]\right.$$
$$\left.+\left[\frac{1}{\sqrt{|d(t_1,t_2)|}}(1-2\mathrm{C}(\lambda\sqrt{|d(t_1,t_2)|})-\imath(-1+2\mathrm{S}(\lambda\sqrt{|d(t_1,t_2)|}))\mathrm{sgn}(d(t_1,t_2)))\right]\right\}, \tag{58}$$

where $\mathrm{S}(y)=\int_0^y dt\sin\frac{\pi t^2}{2}$ and $\mathrm{C}(y)=\int_0^y dt\cos\frac{\pi t^2}{2}$ are the Fresnel sine and cosine functions, respectively, and we defined the function $d(t_1,t_2):=\tan(\frac{t_1-t_2}{2})$ and the constant $\lambda:=2R_\star\sqrt{\frac{6}{\pi}}$. Equation (58) oscillates in the limits $t_1\to-\infty$ and $t_2\to+\infty$. Hence, the family of histories does not decohere, and it is not possible to assign meaningful probabilities in this case.

V. CONCLUSION

In the present work, we have shown that claims asserting that the Wheeler–DeWitt quantization does not eliminate the classical cosmological singularity depend on the quantum theory one is using and on the cosmological model that is being analyzed. In a recent paper [11], it was shown that for a universe filled with a scalar field, different quantum theories can give different results as long as the resulting Wheeler–DeWitt equation is not reduced to a trivial Schrödinger-like equation. However, if one considers a FLRW model where the matter content is described by a perfect fluid, where the resulting equation reduces to a trivial Schrödinger-like equation, then, for flat and hyperbolic spatial sections, the conclusions of the consistent histories approach calculated in this paper coincide, as it should, with the de Broglie–Bohm calculations made before, namely, that such models never reach a singularity in the two moments considered. As far as we know, this is the first nonsingular solution in the consistent histories approach concerning quantum cosmology using standard quantization procedures. For the $k\to 1$ case, the existence of singularities cannot be answered due to the fact that the family of histories is not consistent, and then this approach is silent about that. Since in this model we expect oscillations between the contracting and expanding phases, the lack of consistency may be related to the specific moments in time we chose.

Our results do not depend on the choice of the initial wave function; hence, they are general. Note that in the framework of the de Broglie–Bohm theory one can also prove that singularities are removed irrespective of the initial wave function that is used; see Ref. [29].

ACKNOWLEDGMENTS

C. R. B. was funded by a FAPERJ "Nota 10" fellowship. G. B. S. was funded by a PCI grant from MCTi/CNPq of Brazil, and N. P.-N. would like to thank CNPq for financial support. We also thank A. Scardua for useful discussions.

APPENDIX: NUMERICAL CALCULATIONS

According to Eqs. (34), (35), and (36), in order to check the consistency of the family of histories and calculate the probabilities p_s, p_b, one must evaluate the integrals

$$\langle\Psi|P_{\Delta R}(t)|\Psi\rangle=\langle\Psi U^\dagger\int_{\Delta R}dR'|R'\rangle\langle R'|U|\Psi\rangle = \int_{\Delta R}dR'|\Psi(R',t)|^2, \tag{A1a}$$

$$\langle\Psi|P^2_{\Delta R}P^1_{\Delta R}|\Psi\rangle=\int_{\Delta R}dR\int_{\Delta R}dR'G^{(a)}(R,R',t_2-t_1)\times\Psi(R',t_1)\Psi^\star(R,t_1), \tag{A1b}$$

$$\langle\Psi|P^1_{\Delta R}P^2_{\Delta R}P^1_{\Delta R}|\Psi\rangle = \int_{\Delta R}dR\int_{\Delta R}dR'\int_{\Delta R}dR''\times\Psi^\star(R,t_1)G^{(a)}(R,R',t_1-t_2)\times G^{(a)}(R',R'',t_2-t_1)\Psi(R'',t_1). \tag{A1c}$$

For the wave function (49) in the $k\to 0$ case, from Eq. (A1a) it follows that

$$\langle\Psi|P_{\Delta R^\star}(t)|\Psi\rangle=\int_0^{R^\star}dR|\Psi(R,t)|^2 = \int_0^{X'}dx\frac{2}{\sqrt{\pi}}\exp(-x^2), \tag{A2}$$

where $x=\sqrt{\frac{72}{\sigma^2t^2+(6-pt)^2}}R$. In the considered limits, we have

$$\lim_{t\to\pm\infty}\int_0^{X'}dx\frac{2}{\sqrt{\pi}}\exp(-x^2)=0. \tag{A3}$$

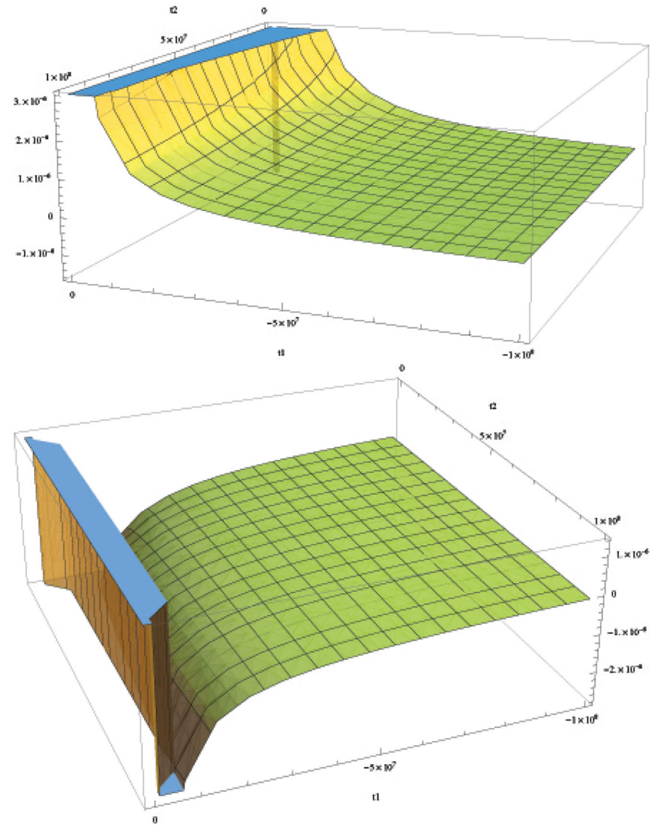

FIG. 1 (color online). Real and imaginary parts of A_0 (t_1, t_2), respectively, where $R_\star = 200$, $p = 0$ and $\sigma = 15$.

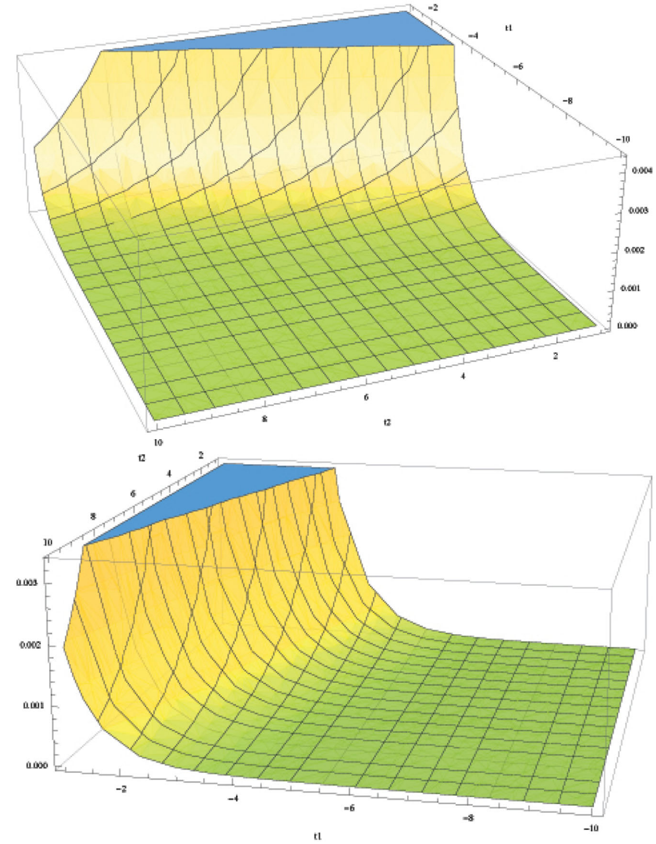

FIG. 2 (color online). Real and imaginary parts of A_{-1} (t_1, t_2), respectively, where $R_\star = 200$, $p = 0$ and $\sigma = 15$.

Likewise, for the hyperbolic case, Eq. (A1a) yields

$$\langle\Psi|P_{\Delta R^\star}(t)|\Psi\rangle = \int_0^{R^\star} dR|\Psi(R,t)|^2 = \int_0^{Y'} dy\frac{2}{\sqrt{\pi}}\exp(-y^2), \qquad (A4)$$

where

$$y = \frac{\sqrt{72\sigma}}{\cosh t(\sigma^2\tanh^2 t + (6 - p\tanh t)^2)^{1/2}}R.$$

Also in this case, we get

$$\lim_{t\to\pm\infty}\int_0^{Y'} dy\frac{2}{\sqrt{\pi}}\exp(-y^2) = 0. \qquad (A5)$$

To determine the consistency of the family of histories, we evaluate Eq. (A1b) numerically. We present the results for the $k \to 0$ case in Fig. 1, where $A_0(t_1, t_2) {:=} \langle\Psi|P^2_{\Delta R}P^1_{\Delta R}|\Psi\rangle$, and we have used $R_\star = 200$ (of order of hundreds of Planck lengths) and $\sigma = 15$. The constant p is set to 0 since it can be eliminated by a time translation. In Fig. 2 we present the results for the $k \to -1$ case, assuming the same set of constants. In both cases we have

$$\lim_{\substack{t_1\to-\infty \\ t_2\to+\infty}}\langle\Psi|P^2_{\Delta R}P^1_{\Delta R}|\Psi\rangle = 0. \qquad (A6)$$

To calculate the probability of a singular universe and to determine the consistency of family of histories, it also is necessary to solve Eq. (A1c). The solution for this equation for the flat universe is

$$\langle\Psi|P^1_{\Delta R_*}P^2_{\Delta R_*}P^1_{\Delta R_*}|\Psi\rangle = \frac{\sigma R_*}{\sqrt{2\pi^3}|\beta|^2 t_1(t_1 - t_2)} \times (12R_*\mathrm{sgn}(t_1 - t_2) + \sqrt{3\pi||t_1 - t_2||})^2\left|\left(\mathrm{C}\left(\frac{\sqrt{3}\cdot 4R_*}{\pi||t_1 - t_2||}\right)\mathrm{sgn}(t_1 - t_2) + \imath\mathrm{S}\left(\frac{\sqrt{3}\cdot 4R_*}{\pi||t_1 - t_2||}\right)\right)\right|^2. \qquad (A7)$$

In the hyperbolic case, the solution found is

$$\begin{aligned}\langle\Psi|P^1_{\Delta R_*}P^2_{\Delta R_*}P^1_{\Delta R_*}|\Psi\rangle &= \sqrt{3}/2\lambda\left|\sqrt{\frac{-\cosh(t_1)}{\beta - 12\imath\tanh(t_1/2)}}\right|^2\cosh(t_1-t_2)\\ &\times\bigg[\mathrm{C}^2(\lambda\sqrt{1/b(t_1,t_2)}) + \frac{2\mathrm{C}(\sqrt{1/b(t_1,t_2)})\mathrm{C}(\lambda\sqrt{1/b(t_1,t_2)})}{|b(t_1,t_2)|} + \frac{1}{b^2(t_1,t_2)}(\mathrm{C}^2(\lambda\sqrt{b(t_1,t_2)})\\ &+\mathrm{S}^2(\lambda\sqrt{b(t_1,t_2)} + \mathrm{S}^2(\lambda\sqrt{1/b(t_1,t_2)})\mathrm{sgn}^2(1/b(t_1,t_2))\\ &+\frac{2}{b^{3/2}(t_1,t_2)}\mathrm{S}^2(\lambda\sqrt{1/b(t_1,t_2)})\times\mathrm{S}^2(\lambda\sqrt{b(t_1,t_2)})\mathrm{sgn}(b^{-1}(t_1,t_2))\times\sqrt{b(t_1,t_2)}\bigg]b(t_1,t_2), \end{aligned} \tag{A8}$$

where we defined $b(t_1,t_2) \coloneqq \tanh(\frac{t_1-t_2}{2})$. Once more for both flat and hyperbolic cases, it follows that

$$\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\Psi|P^1_{\Delta R}P^2_{\Delta R}P^1_{\Delta R}|\Psi\rangle = \begin{cases}0 & t_1\to+\infty\\ 0 & t_1\to-\infty\\ 0 & t_2\to+\infty\\ 0 & t_2\to-\infty.\end{cases} \tag{A9}$$

With the help of Eqs. (34), (35), and (36), we find that (A10)

$$\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\Psi_b|\Psi_s\rangle = 0, \tag{A10a}$$

$$\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\Psi_s|\Psi_s\rangle = 0, \tag{A10b}$$

$$\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\Psi_b|\Psi_b\rangle = 1. \tag{A10c}$$

The family of histories is consistent, and the Universe never reaches a singularity in those two limits considered for flat and hyperbolic universes.

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