

**Modeling scalar fields consistent with positive mass**Masato Nozawa<sup>1</sup> and Tetsuya Shiromizu<sup>2</sup><sup>1</sup>*Theory Center, KEK, Tsukuba 305-0801, Japan*<sup>2</sup>*Department of Physics, Kyoto University, Kyoto 606-8502, Japan*

(Received 14 October 2013; published 27 January 2014)

This paper explores the conditions under which modified gravitational theories admit positive mass. Following Witten's spinor argument, it is argued that a single condition should be imposed upon a gauge connection in the supercovariant derivative. Under this condition, we present a simple formula for the divergence of the Nester tensor in Einstein's gravity with a general source. Applying this prescription to the Einstein-scalar system, we find that for a certain class of the gauge connection, a special kind of noncanonical scalar-field theory admits the positivity property in addition to the ordinary canonical scalar-field system. In both cases, the scalar potential can be written in terms of a superpotential. In the noncanonical case, we obtain the most general Bogomol'nyi-Prasad-Sommerfield solutions which preserve at least half of the supersymmetry.

DOI: [10.1103/PhysRevD.89.023011](https://doi.org/10.1103/PhysRevD.89.023011)

PACS numbers: 95.30.Sf, 04.20.-q

**I. INTRODUCTION**

The observational evidence of the present-day acceleration of the Universe [1] brought us a tantalizing and profound puzzle in modern cosmology. A possible resolution to this problem may be ascribed to the presence of an unknown component of the energy-momentum tensor mimicking a positive cosmological constant dubbed “dark energy.” A simple instance is provided by the scalar fields, which are ubiquitous in string theory. A plethora of scalar-field models of dark energy are able to alleviate coincidence problems and fine-tuning. Another approach is to modify general relativity at the cosmological distance. However, most of the modified theories of gravity that were invented to account for the late-time acceleration of the Universe are phenomenological and lack a fundamental theoretical description. Sometimes modified gravities require nonlinear effects to recover general relativity, resulting in poor theoretical predictability. Moreover, many of these theories do not respect suitable energy conditions. These properties cast doubt on the reliability of gravitational theories. In general relativity, on the other hand, the positive mass theorem guarantees the classical nonperturbative stability of spacetimes [2,3], and the zero-mass ground state is the flat Minkowski spacetime. At the classical level, the positivity of mass undoubtedly places a strong underpinning for viable gravitational theories, and at the quantum level, the instability of classical theory could cause unitary violation. A nontrivial example is Starobinsky's  $R + \alpha R^2$  gravity [4]. If  $1 + 2\alpha R > 0$  (this condition also assures that the Cauchy problem is well posed), one can perform the conformal transformation of the metric to bring this theory into the scalar-tensor theory in which the energy-momentum tensor obeys the dominant energy condition. Hence, this theory admits the mass positivity [5].

The present paper aims to extend Witten's argument [3] to gravitational theories incorporating a noncanonical scalar field. Specifically, our discussion will primarily concentrate on a theory described by the Lagrangian  $R + 2K(\phi, X)$ , where  $X := -(1/2)(\nabla\phi)^2$ . This class of Lagrangian has been intensively studied in a cosmological setting since it can drive inflation without a finely constructed scalar potential [6]. This is the special class of Horndeski's scalar-tensor theory [7], which is the most general scalar-tensor theory maintaining the second-order equations of motion. It was recently realized that Horndeski's theory could be obtained in the decoupling limit of nonlinear massive gravity [8]. We would like to address the viability of these models in the context of classical stability. In this paper, we study the minimal condition under which the mass positivity is guaranteed in the  $R + 2K(\phi, X)$  theory. This result generalizes Boucher's work [9] in which the conditions for canonical scalar fields to possess positive mass were worked out (see, also, Ref. [10]). We obtain a useful formula for the mass positivity that can be used in a broad context. Utilizing this formula, we find that in a simple class of the gauge connection, the nontrivial form of  $K$  consistent with the positive mass falls into two families. One is the canonical scalar field with its potential given in terms of a superpotential recovering the result in the literature [9,10]. Another class of Lagrangian is the noncanonical theory, which we shall focus on in the body of the text. We obtain the general Bogomol'nyi-Prasad-Sommerfield (BPS) metric obeying the first-order differential equations in the noncanonical case.

The rest of this paper is organized as follows. In Sec. II, we provide the outline of proof of the positive mass theorem in general gravitational theories and present a useful formula with wide applicability. Section III

examines the Einstein gravity with a noncanonical scalar field and explores which types of theories are compatible with the positive mass theorem. Section IV addresses some issues when the BPS inequality is saturated. We discuss the multiple scalar generalization in Sec. V. The final section summarizes our work and illustrates the recipe of the extension to more general cases.

We adopt the convention with a mostly plus metric signature. The Clifford algebra is  $\{\gamma_a, \gamma_b\} = 2g_{ab}$ , and the Riemann curvature is given by  $2\nabla_{[\rho}\nabla_{\sigma]}V^\mu = R^\mu{}_{\nu\rho\sigma}V^\nu$ . Greek indices  $(\mu, \nu, \dots)$  refer to the spacetime component, whereas the latin indices  $(a, b, \dots)$  refer to the frame indices. We work in units of  $c = 8\pi G = 1$  throughout the paper.

## II. POSITIVE MASS THEOREM

Although we will focus on a simple case later, we shall not specify the gravitational theory in this section and give a versatile formula in the proof of the mass positivity. We attempt to keep the discussion as general as possible. Nonetheless, the final upshot is quite simple and universal and has a potentially wide applicability. We follow Witten's spinor argument [3], which provides a covariant integral expression for gravitational energy-momentum.

Let us consider the  $d$ -dimensional Lorentzian spacetime  $(M, g_{\mu\nu})$  admitting a spin structure. In terms of a (commuting) Dirac spinor  $\epsilon$ , we define the Nester tensor as [11]

$$N^{\mu\nu} := -i\left(\bar{\epsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon - \overline{\hat{\nabla}_\rho\epsilon}\gamma^{\mu\nu\rho}\epsilon\right), \quad (1)$$

where  $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$ , and  $\hat{\nabla}_\mu$  is the supercovariant derivative operator defined by

$$\hat{\nabla}_\mu\epsilon = (\nabla_\mu + \mathcal{A}_\mu)\epsilon. \quad (2)$$

Here,  $\nabla_\mu$  is an ordinary Levi-Civita covariant derivative, and  $\mathcal{A}_\mu$  is the connection of the spinor bundle taking values in  $\text{GL}(2^{[d/2]}, \mathbb{C})$ . The Dirac conjugate of a spinor is given by  $\bar{\epsilon} = i\epsilon^\dagger\gamma^0$ ; hence, we have

$$\overline{\hat{\nabla}_\mu\epsilon} = i(\hat{\nabla}_\mu\epsilon)^\dagger\gamma^0 = \overline{\nabla_\mu\epsilon} - \bar{\epsilon}\gamma^0\mathcal{A}_\mu^\dagger\gamma^0 = \overline{\nabla_\mu\epsilon} - \bar{\epsilon}\bar{\mathcal{A}}_\mu, \quad (3)$$

where  $\bar{\mathcal{A}}_\mu := \gamma^0\mathcal{A}_\mu^\dagger\gamma^0$ . The index 0 stands for the time component in the local vielbein.

Let  $\Sigma$  denote the  $(d-1)$ -dimensional spacelike partial Cauchy surface with boundary  $\partial\Sigma$  at infinity. From Stokes's theorem, we obtain

$$-\int_\Sigma \nabla_\nu N^{\mu\nu}u_\mu d\Sigma = \frac{1}{2}\int_{\partial\Sigma} N_{\mu\nu}dS^{\mu\nu}, \quad (4)$$

where  $u_\mu$  is the future-pointing unit normal to  $\Sigma$ . For the asymptotically flat/anti-de Sitter spacetimes, the right-hand side of the above equation gives rise to the globally

conserved energy-momentum (contracted by a generator of asymptotic symmetry) with the appropriate form of  $\mathcal{A}_\mu$  and with a required falloff [12]. Here we suppose that the right-hand side of Eq. (4) yields the finite energy-momentum for a given  $\mathcal{A}_\mu$ . If we are able to show the positivity of the left-hand side, we can then prove that the spacetime energy-momentum is timelike, i.e., the positive mass follows.

We turn to evaluate the left-hand side of Eq. (4). First of all, we notice the following identity:

$$\hat{\nabla}_{[\mu}\hat{\nabla}_{\nu]}\epsilon = \left(\frac{1}{8}R_{\mu\nu\alpha\beta}\gamma^{\alpha\beta} + \frac{1}{2}\mathcal{F}_{\mu\nu}\right)\epsilon, \quad (5)$$

where  $\mathcal{F}_{\mu\nu}$  is the curvature of the spinor bundle,

$$\mathcal{F}_{\mu\nu} = 2(\nabla_{[\mu}\mathcal{A}_{\nu]} + \mathcal{A}_{[\mu}\mathcal{A}_{\nu]}). \quad (6)$$

Using the identity (5), a straightforward computation brings the divergence of the Nester tensor into the following form:

$$\begin{aligned} \nabla_\nu N^{\mu\nu} &= 2i\overline{\hat{\nabla}_\rho\epsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\nu\epsilon - G^\mu{}_\nu V^\nu \\ &\quad - \frac{i}{2}\bar{\epsilon}[\bar{\mathcal{F}}_{\nu\rho}\gamma^{\mu\nu\rho} + \gamma^{\mu\nu\rho}\mathcal{F}_{\nu\rho}]\epsilon \\ &\quad - i\bar{\epsilon}(\bar{\mathcal{A}}_\nu\gamma^{\mu\nu\rho} - \gamma^{\mu\nu\rho}\mathcal{A}_\nu)\hat{\nabla}_\rho\epsilon \\ &\quad + i\overline{\hat{\nabla}_\rho\epsilon}(\bar{\mathcal{A}}_\nu\gamma^{\mu\nu\rho} - \gamma^{\mu\nu\rho}\mathcal{A}_\nu)\epsilon, \end{aligned} \quad (7)$$

where  $V^\mu = i\bar{\epsilon}\gamma^\mu\epsilon$  is the future-directed timelike vector and  $\bar{\mathcal{F}}_{\mu\nu} := \gamma^0\mathcal{F}_{\mu\nu}^\dagger\gamma^0$ . Here we assume that  $\epsilon$  satisfies the Dirac-Witten condition [3]

$$\gamma^i\hat{\nabla}_i\epsilon = 0, \quad (8)$$

where the latin index  $i$  stands for the spatial component. We suppose that this differential equation has no  $L^2$ -normalizable zero mode subjected to the boundary condition such that the spacetime energy-momentum is finite. This implies that the solution of Eq. (8) exists (this is indeed the case for asymptotically Minkowski/anti-de Sitter spacetime in general relativity [13]). It follows that the zeroth component of the first term on the right-hand side of Eq. (7) is non-negative  $2i\overline{\hat{\nabla}_\rho\epsilon}\gamma^{0\nu\rho}\hat{\nabla}_\nu\epsilon = 2g^{ij}(\hat{\nabla}_i\epsilon)^\dagger\hat{\nabla}_j\epsilon \geq 0$ . Our remaining task is to work out the condition under which the zeroth component of the residual terms is non-negative. In the simplest  $\mathcal{A}_\mu = 0$  case, the Einstein equations  $G_{\mu\nu} = T_{\mu\nu}^{(\text{matter})}$  enable us to establish this claim, provided  $T_{\mu\nu}^{(\text{matter})}$  satisfies the dominant energy condition. This proves that the Arnowitt-Deser-Misner mass is positive in general relativity [3].

Next, we are interested in the gravitational theory for which  $\mathcal{A}_\mu$  is nonvanishing and the positivity of the mass

holds. Here we require that the fourth and fifth terms in the right-hand side of Eq. (7) should vanish, since they have no sign control. This is realized if

$$\bar{A}_\nu \gamma^{\mu\nu\rho} = \gamma^{\mu\nu\rho} A_\nu. \quad (9)$$

This condition is indeed satisfied for the Einstein-Maxwell system with or without a negative cosmological constant [14,15]. It deserves to notice that the Einstein- $\Lambda(>0)$  system fails to fulfill Eq. (9) [16,17]. This accords with our intuition because the de Sitter universe does not have a globally timelike Killing field.

When Eq. (9) is satisfied, Eq. (7) is considerably simplified to

$$\nabla_\nu N^{\mu\nu} = 2i\bar{\nabla}_\rho \epsilon \gamma^{\mu\nu\rho} \hat{\nabla}_\nu \epsilon - G^\mu{}_\nu V^\nu + S^\mu, \quad (10)$$

where

$$S^\mu := -i\bar{\epsilon} \gamma^{\mu\nu\rho} \mathcal{F}_{\nu\rho} \epsilon. \quad (11)$$

Although the positivity of the zeroth component of  $-G^\mu{}_\nu V^\nu + S^\mu$  depends sensitively on the theories under consideration, the above formula (10), nevertheless, shows the broad utility in a variety of gravitational theories due to its simplicity.

### III. EINSTEIN-SCALAR SYSTEM

In this section, we apply the general formula obtained in the preceding section to specific gravitational theory. As a first stage of progress, we focus on a quite simple but still nontrivial system described by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [R + 2K(\phi, X) + 2\mathcal{L}_{\text{matter}}], \quad (12)$$

where  $\phi$  is a real scalar field,  $X := -(1/2)(\nabla\phi)^2$ , and  $\mathcal{L}_{\text{matter}}$  is the Lagrangian for ordinary matters which decouples with the scalar field. This is the simplest case of the Horndeski theory (for example, see Refs. [18,19]). The gravitational field equation derived from the action is

$$G_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{matter})}, \quad (13)$$

with

$$T_{\mu\nu}^{(\phi)} = K_X \nabla_\mu \phi \nabla_\nu \phi + K g_{\mu\nu}, \quad (14)$$

where  $T_{\mu\nu}^{(\text{matter})}$  is the energy-momentum tensor for matters satisfying the dominant energy condition. Here and in what follows, the subscript denotes the partial derivative with respect to the corresponding argument, e.g.,  $K_X = \partial_X K(\phi, X)$ . The evolution equation for the scalar field reads

$$\nabla^\mu (K_X \nabla_\mu \phi) + K_\phi = 0. \quad (15)$$

It is obvious that the general form of  $K$  does not ensure the mass positivity. For example,  $K = -X$  gives rise to a ghost. In order to prevent this, we demand that the scalar field satisfies the null energy condition. This is tantamount to imposing a single condition

$$K_X \geq 0. \quad (16)$$

We will use this equation later.

We now wish to examine the acceptable form of  $\mathcal{A}_\mu$ . Since the energy-momentum tensor (14) does not involve the second derivative of  $\phi$ ,  $\mathcal{A}_\mu$  cannot have the derivative of  $\phi$ . Otherwise, the curvature  $\mathcal{F}_{\mu\nu}$  in Eq. (11) gives rise to a second derivative of  $\phi$ , which cannot be canceled. Hence, the simplest form of  $\mathcal{A}_\mu$  is

$$\mathcal{A}_\mu = W(\phi) \gamma_\mu, \quad (17)$$

for which  $\mathcal{F}_{\mu\nu} = 2W_\phi \nabla_{[\mu} \phi \gamma_{\nu]} + 2W^2 \gamma_{\mu\nu}$ . This  $\mathcal{A}_\mu$  satisfies the condition of Eq. (9).

Now the vector field  $S^\mu$  given by Eq. (11) becomes

$$\begin{aligned} S^\mu &= -4i\bar{\epsilon} \gamma^{\mu\nu} \epsilon \nabla_\nu \phi W_\phi + 12V^\mu W^2 \\ &= i\bar{\delta\lambda} \gamma^\mu \delta\lambda + V^\nu \left[ f^2 \nabla^\mu \phi \nabla_\nu \phi \right. \\ &\quad \left. + \left( -\frac{1}{2} f^2 (\nabla\phi)^2 - 8f^{-2} W_\phi^2 + 12W^2 \right) \delta^\mu{}_\nu \right], \end{aligned} \quad (18)$$

where we have defined  $\delta\lambda$  as

$$\delta\lambda := \frac{1}{\sqrt{2}} [f(\phi, X) \gamma^\mu \nabla_\mu \phi - 4f^{-1}(\phi, X) W_\phi] \epsilon. \quad (19)$$

The second term in Eq. (19) has been chosen to cancel the  $i\bar{\epsilon} \gamma^{\mu\nu} \epsilon$  term at the first equality of Eq. (18). Comparing Eq. (18) with Eq. (14),  $S^\mu$  can be written into

$$S^\mu = i\bar{\delta\lambda} \gamma^\mu \delta\lambda + T^{(\phi)\mu}{}_\nu V^\nu, \quad (20)$$

provided  $K$  satisfies

$$f^2 = K_X(\phi, X), \quad (21)$$

and

$$-\frac{1}{2} f^2 (\nabla\phi)^2 - 8f^{-2} W_\phi^2 + 12W^2 = K. \quad (22)$$

If  $K$  satisfies the above two relations, the use of Einstein's equations then allows us to find

$$\nabla_\nu N^{\mu\nu} = 2i\bar{\nabla}_\rho \epsilon \gamma^{\mu\nu\rho} \hat{\nabla}_\nu \epsilon - T_\nu^{(\text{matter})} V^\nu + i\bar{\delta\lambda} \gamma^\mu \delta\lambda. \quad (23)$$

Since the zeroth component of the last term is non-negative and  $T_{\mu\nu}^{(\text{matter})}$  is supposed to satisfy the dominant energy condition, we can show the positivity of the mass.

We now look at the two conditions (21) and (22) required above in more detail. Obviously, Eq. (21) requires that  $K_X$  is non-negative, which is assured if we impose the null energy condition for the scalar field (16). Equations (21) and (22) lead to the following equation that  $K$  should satisfy:

$$XK_X - K - \frac{8W_\phi^2}{K_X} = -12W^2(\phi). \quad (24)$$

From the above equation, we can obtain the integrability condition

$$0 = \partial_X \left( XK_X - K - \frac{8W_\phi^2}{K_X} \right) = K_{XX} \left( X + \frac{8W_\phi^2}{K_X^2} \right). \quad (25)$$

The general solutions to this equation fall into two families:

$$(i) K_{XX} = 0, \quad (ii) XK_X^2 + 8W_\phi^2 = 0. \quad (26)$$

The case (i) reduces to  $K = X - U(\phi)$  (the coefficient of  $X$  can be set to unity by absorbing into the rescaling of  $\phi$ ); i.e., this corresponds to the canonical scalar field. In this case, Eq. (24) implies

$$U(\phi) = 8W_\phi^2 - 12W^2. \quad (27)$$

This recovers the result obtained in Refs. [9,10]. The potential  $U(\phi)$  is now expressed in terms of a single superpotential  $W(\phi)$ .

The solution to case (ii) depends on the sign of  $X$ . For the  $X > 0$  case, one can see immediately that  $K_X = W_\phi = 0$ , thus,  $K = K(\phi)$  and  $W = \text{constant}$ . The field equation for  $\phi$  shows  $K = \text{constant}$ . This is the case of the Einstein theory with a negative cosmological constant. Thus, there is no nontrivial scalar field in this case.

For  $X < 0$ , on the other hand, the differential equation in (26) can be integrated to give  $K = 4\sqrt{2}W_\phi(-X)^{1/2} + K_1(\phi)$ . Plugging this back into Eq. (24), we obtain  $K_1 = 12W^2(\phi)$ , i.e.,

$$K = 4\sqrt{2}W_\phi(-X)^{1/2} + 12W^2(\phi). \quad (28)$$

Note that this excludes the homogeneous-isotropic solution with  $\phi = \phi(t)$  because of  $X = \dot{\phi}^2/2 > 0$ . This means that the case (ii) cannot be applied to a cosmological argument.

Since the Lagrangian for  $X < 0$  takes a complicated form, it is far from obvious whether this theory admits asymptotically Minkowski/anti-de Sitter solutions. To see this more precisely, let us focus on the spherically symmetric static spacetimes, for which

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega_2^2, \quad \phi = \phi(r). \quad (29)$$

In this case, we obtain

$$\begin{aligned} \sqrt{-g}K &= 4r^2 \sin \theta e^{\nu+\lambda} [-e^{-\lambda} W_\phi \phi' + 3W^2] \\ &\sim r^2 [-W'(r)(r/\ell)^n + W^2], \end{aligned} \quad (30)$$

where we have denoted  $n = 0$  for Minkowski and  $n = 1$  for anti-de Sitter with the curvature radius  $\ell$ . We also assumed  $\phi'(r) < 0$ , but this is not essential. When  $n = 0$  and  $W_\phi \neq 0$ , we have  $W \sim 1/r$ , while the  $n = 1$  case yields  $W \sim 1/\ell + 1/r$  to maintain the asymptotic conditions. Hence, the noncanonical case also may allow asymptotically Minkowski/anti-de Sitter solutions, as far as the superpotential  $W(r)$  displays the above falloff rate.

It should be also emphasized, however, that under these settings, the  $\phi(r)$  dependence disappears from Eq. (30), which now gives a governing equation for  $W = W(r)$  [of course, this gives an implicit equation for  $\phi$  once we fix  $W = W(\phi)$ ]. This is quite strange because the solution itself is insensitive to the explicit form of  $W = W(\phi)$  and  $\phi = \phi(r)$  is undetermined. This is due to the property that the theory (28) is invariant under the field redefinition

$$\phi \rightarrow \Phi(\phi). \quad (31)$$

We will see in the next section that the supersymmetric solution illustrates this feature.

Finally, let us consider the case in which the mass vanishes. Equation (23) indicates that  $\hat{\nabla}_i \epsilon = \delta\lambda = T_{\mu\nu}^{(\text{matter})} = 0$  holds on  $\Sigma$ . A slight deformation of the time slices implies that  $\hat{\nabla}_\mu \epsilon = \delta\lambda = T_{\mu\nu}^{(\text{matter})} = 0$  holds. As shown in Ref. [10], they imply that the spacetime is anti-de Sitter/Minkowski, and  $\phi$  is constant if the spacetime is asymptotically globally anti-de Sitter/Minkowski.<sup>1</sup> This statement means that the ground state of the spacetime is maximally symmetric. This is a quite convincing and important consequence.

#### IV. SUPERSYMMETRIC BACKGROUNDS

Once an inequality has been proved for the system of Eq. (12), one is next interested in the cases where the inequality is saturated. In the previous section, we demonstrated that this is the case only for anti-de Sitter/Minkowski, provided the spacetime globally approaches asymptotically the anti-de Sitter/Minkowski spacetime. In this section, we relax the asymptotic boundary conditions and argue the interrelationships between the positive mass theorem and supersymmetry.

In supergravity, the gravitational background is said to preserve supersymmetry or be in a BPS state if the variation

<sup>1</sup>This is not the case for asymptotically locally anti-de Sitter [20].

of the fermionic configurations vanishes. This forces the spacetime to obey the first-order differential equations,

$$\hat{\nabla}_\mu \epsilon = 0, \quad \delta\lambda = 0. \quad (32)$$

In the present case, we have

$$(\nabla_\mu + W(\phi)\gamma_\mu)\epsilon = 0, \quad (33a)$$

$$\left(\gamma^\mu \nabla_\mu \phi - \frac{4W_\phi}{K_X}\right)\epsilon = 0, \quad (33b)$$

where  $K$  is given by Eqs. (27) or (28). The spinor  $\epsilon$  obeying these equations is conventionally called a Killing spinor. This is because  $V^\mu = i\bar{\epsilon}\gamma^\mu\epsilon$  is always a causal Killing vector when the theory can be embedded into the genuine supergravity theories (see Ref. [21]). In the generic gravitational theories,  $V^\mu = i\bar{\epsilon}\gamma^\mu\epsilon$  becomes a Killing vector of BPS spacetime if the following condition holds:

$$\bar{\mathcal{A}}_{(\mu}\gamma_{\nu)} = \gamma_{(\mu}\mathcal{A}_{\nu)}. \quad (34)$$

This condition may be also used to constrain the possible form of  $\mathcal{A}_\mu$  and is fulfilled for the present model (17). We shall refer to Eq. (33a) as a gravitino Killing spinor equation and Eq. (33b) as a dilatino Killing spinor equation, although they might not have a supergravity origin.

It deserves to notice that the existence of the nontrivial BPS geometries is not always assured [22]. To check the consistency, we compute the integrability condition for the dilatino Killing spinor. Acting  $\nabla^\nu\phi\nabla_\nu$  to Eq. (33b), we have

$$\begin{aligned} & \gamma^\mu \nabla_\mu X \epsilon \\ &= \left( \frac{8XW_{\phi\phi}}{K_X} + \frac{4W_\phi}{K_X^2} \nabla^\mu \phi \nabla_\mu K_X + \frac{16WW_\phi^2}{K_X^2} + 2XW \right) \epsilon, \end{aligned} \quad (35)$$

whereas acting  $\gamma^\nu \nabla_\nu$  to  $\delta\lambda = 0$  yields

$$\begin{aligned} 0 &= \gamma^\nu \nabla_\nu \left[ \left( \sqrt{K_X} \gamma^\mu \nabla_\mu \phi - \frac{4W_\phi}{\sqrt{K_X}} \right) \epsilon \right] \\ &= \left( \sqrt{K_X} \nabla^2 \phi - \frac{16W_\phi W_{\phi\phi}}{K_X^{3/2}} + \frac{24WW_\phi}{\sqrt{K_X}} + \frac{4W_\phi}{K_X^{3/2}} \gamma^\mu \nabla_\mu K_X \right) \epsilon. \end{aligned} \quad (36)$$

Here we have used  $\hat{\nabla}_\mu \epsilon = 0$  and  $\delta\lambda = 0$ . Inserting Eq. (35) into the last term of Eq. (36), we obtain

$$\frac{1}{\sqrt{K_X}} (K_X \nabla^2 \phi + \nabla^\mu \phi \nabla_\mu K_X + K_\phi) \epsilon = 0. \quad (37)$$

It turns out that the scalar-field equation (15) is automatically satisfied if there exists a spinor  $\epsilon$  satisfying Eqs. (33a) and (33b).

Similarly, the integrability condition of the gravitino Killing spinor equation (5) implies

$$0 = \gamma^\nu \hat{\nabla}_{[\mu} \hat{\nabla}_{\nu]} \epsilon = -\frac{1}{4} (R_{\mu\nu} \gamma^\nu - 2\gamma^\nu \mathcal{F}_{\mu\nu}) \epsilon. \quad (38)$$

Substitution of the current model gives

$$E_{\mu\nu} \gamma^\nu \epsilon = 0, \quad (39)$$

where

$$E_{\mu\nu} := R_{\mu\nu} - [K_X \nabla_\mu \phi \nabla_\nu \phi + (XK_X - K)g_{\mu\nu}]. \quad (40)$$

The vanishing  $E_{\mu\nu}$  amounts to requiring Einstein's equations. When  $V^\mu = i\bar{\epsilon}\gamma^\mu\epsilon$  is timelike,  $E_{\mu\nu} = 0$  follows automatically, whereas when  $V^\mu$  is null, all but the single component of Einstein's equations are satisfied. This is a common feature in supergravity and provides a consistency check [21].

We now classify all the BPS geometries satisfying Eq. 33. To this end, we introduce the tensorial bilinears constructed from a Killing spinor [23],

$$\begin{aligned} E &= \bar{\epsilon}\epsilon, & B &= i\bar{\epsilon}\gamma_5\epsilon, & V_\mu &= i\bar{\epsilon}\gamma_\mu\epsilon, \\ U_\mu &= i\bar{\epsilon}\gamma_5\gamma_\mu\epsilon, & \Phi_{\mu\nu} &= i\bar{\epsilon}\gamma_{\mu\nu}\epsilon, \end{aligned} \quad (41)$$

where  $\gamma_5 = i\gamma_{0123}$  is a chiral matrix ( $\gamma_5^2 = 1$ ). In our convention, all the above tensorial fields are real. Our strategy to classify the BPS solutions is to constrain the possible form of the metric by deriving algebraic and differential conditions for the bilinears. The algebraic relations of the bilinears are obtained from Fierz identities and read

$$\begin{aligned} V \cdot V &= -U \cdot U = -(E^2 + B^2), & V \cdot U &= 0, \\ (E^2 + B^2)\Phi_{\mu\nu} &= 2BV_{[\mu}U_{\nu]} - E\epsilon_{\mu\nu\rho\sigma}V^\rho U^\sigma. \end{aligned} \quad (42)$$

The gravitino Killing spinor equation now puts differential constraints upon the bilinears,

$$\nabla_\mu E = 0, \quad (43a)$$

$$\nabla_\mu B = -2WU_\mu, \quad (43b)$$

$$\nabla_\mu V_\nu = 2W\Phi_{\mu\nu}, \quad (43c)$$

$$\nabla_\mu U_\nu = -2WBg_{\mu\nu}, \quad (43d)$$

$$\nabla_\mu \Phi_{\nu\rho} = 4Wg_{\mu[\nu}V_{\rho]}. \quad (43e)$$

It follows that  $V^\mu$  is a Killing vector, and  $U^\mu$  is a conformal Killing vector. Similarly, the dilatino equation gives

$$V^\mu \nabla_\mu \phi = 0, \quad (44a)$$

$$EW_\phi = 0, \quad E\nabla_\mu \phi = 0, \quad (44b)$$

$$U^\mu \nabla_\mu \phi - \frac{4W_\phi B}{K_X} = 0, \quad (44c)$$

$$\Phi_{\mu\nu} \nabla^\nu \phi - \frac{4W_\phi}{K_X} V_\mu = 0, \quad (44d)$$

$$B\nabla_\mu \phi - \frac{4W_\phi}{K_X} U_\mu = 0. \quad (44e)$$

In this paper, we are interested in supersymmetric solutions in the noncanonical case. As we will see below, we are able to find the general BPS metric in an explicit form utilizing the algebraic and differential relations obtained above. Due to Eq. (43a), we can set  $E = 0$  or  $E = 1$ . In the latter case, Eq. (44b) implies  $\phi = W = \text{constant}$ , which is not our main concern here. Hence, we shall consider the  $E = 0$  case in what follows.

Let us focus on the  $B \neq 0$  case, for which  $V^\mu$  is timelike and  $U^\mu$  is spacelike.<sup>2</sup> Equations (42) and (43c) imply that  $V^\mu$  is hypersurface orthogonal; hence, the spacetime is static. It is, therefore, convenient to introduce a local coordinate system for which  $V^\mu = (\partial_t)^\mu$  with  $V_\mu = -B^2 \nabla_\mu t$ , implying that  $g_{\mu\nu}$  and  $\phi$  are  $t$  independent [see Eqs. (43c) and (44a)]. Equation (43d) implies that there exists a local function  $z$  such that  $U = dz$ .<sup>3</sup> Now we can write the metric as

$$ds^2 = -B^2 dt^2 + B^{-2} [dz^2 + B^4 e^{2\psi} (dx^2 + dy^2)], \quad (45)$$

where we have exploited the freedom of  $(x, y)$  to eliminate the cross term  $dx dz$ , etc., and fix to the conformally flat gauge. Equations (43b) and (44e) imply that  $B$  and  $\phi$  are functions of  $z$  only, whereas Eq. (43d) implies that  $\psi$  is independent of  $z$ . As pointed out in Ref. [24], the trace of Einstein's equation assures the integrability condition of the Killing spinor. This yields  $(\partial_x^2 + \partial_y^2)\psi = 0$ ; hence, we have  $\psi = F(\zeta) + \bar{F}(\bar{\zeta})$ , where  $F$  is an analytic function of  $\zeta := x + iy$ . Using the remaining freedom  $\zeta = x + iy \rightarrow f(\zeta)$ , which keeps the conformally flat form, we can set  $\psi = 0$  without loss of generality. Thus, the most general timelike class of the BPS solution reads

<sup>2</sup>The null family  $V^\mu V_\mu = E = B = 0$  gives the plane-fronted wave. The explicit metric can be obtained in a similar fashion, but we shall not attempt to do this here.

<sup>3</sup>It is worthwhile to notice that the fact that  $U_\mu$  is closed  $\nabla_{[\mu} U_{\nu]} = i\bar{\epsilon}\gamma_5\gamma_{[\mu} A_{\nu]}\epsilon - i\bar{\epsilon}\tilde{A}_{[\mu}\gamma_{\nu]}\gamma_5\epsilon = 0$  is a direct consequence of the condition (9).

$$ds^2 = B^{-2}(z)dz^2 + B^2(z)(-dt^2 + dx^2 + dy^2), \quad (46)$$

with

$$W = -\frac{1}{2}B'(z), \quad \phi = \phi(z). \quad (47)$$

Here,  $B(z)$  and  $\phi(z)$  are arbitrary functions. In this case, one can integrate the Killing spinor equation directly and find the solution,

$$\epsilon = B^{1/2}(z)(1 + \gamma_3)\epsilon_0, \quad (48)$$

where  $\epsilon_0$  is a constant spinor. Hence, the metric preserves 1/2 supersymmetry. This is in contrast with the  $\mathcal{N} = 2$  gauged supergravity, for which the general BPS solutions preserve only one quarter of the supersymmetry [23]. One can also verify that the maximally supersymmetric solution is exhausted by the anti-de Sitter spacetime.

As noticed in the previous section,  $W = W(\phi)$  remains unfixed. In this sense, the noncanonical theory obtained here is not qualified as a well-defined gravitational theory.

## V. MULTIPLE SCALAR FIELDS

We discuss in this section the generalization of the argument in Sec. III to the multiple scalar system. We shall consider the following action:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [R + 2K(\phi^I, X^{JK}) + 2L_{\text{matter}}], \quad (49)$$

where  $\phi^I$  ( $I = 1, \dots, N$ ) are real scalar fields and  $X^{IJ} := -(1/2)\nabla^\mu \phi^I \nabla_\mu \phi^J$ . The stress-energy tensor derived from the action reads

$$T_{\mu\nu}^{(\phi)} = K_{IJ} \nabla_\mu \phi^I \nabla_\nu \phi^J + K g_{\mu\nu}, \quad (50)$$

where we have denoted  $K_{IJ} := \partial K / \partial X^{IJ}$ . We assume that the metric  $K_{IJ}$  is invertible and positive definite.

Following the parallel argument given in Sec. III, let us discuss the condition under which the gravitational theory (49) admits the positive mass. We define the supercovariant derivative as

$$\hat{\nabla}_\mu \epsilon = (\nabla_\mu + W(\phi^I)\gamma_\mu)\epsilon, \quad (51)$$

and the variation of dilatino as

$$\delta\lambda^I = \frac{1}{\sqrt{2}} (\gamma^\mu \nabla_\mu \phi^I - 4K^{IJ} \partial_J W) \epsilon, \quad (52)$$

where  $K^{IJ}$  is the inverse metric of  $K_{IJ}$  and  $\partial_I W := \partial W / \partial \phi^I$ . It follows that the vector field  $S^\mu$  can be written into the desired form,

$$S^\mu = T^{(\phi)\mu}{}_\nu V^\nu + iK_{IJ}\overline{\delta\lambda^I}\gamma^\mu\delta\lambda^J, \quad (53)$$

provided that  $K(\phi^I, X^{JK})$  satisfies the following equation:

$$X^{IJ}K_{IJ} - K - 8K^{IJ}\partial_I W\partial_J W = -12W^2. \quad (54)$$

The same line of argument given in Sec. III enables us to find two classes of solutions to this differential equation. One is the canonical scalar fields, for which  $K$  is given by

$$K = G_{IJ}(\phi)X^{IJ} - U(\phi), \quad (55)$$

where  $G_{IJ}$  describes the positive-definite moduli space metric and

$$U(\phi) = 8G^{IJ}\partial_I W\partial_J W - 12W^2. \quad (56)$$

Here,  $G^{IJ}$  is the inverse of  $G_{IJ}$ . This recovers the result in Refs. [9,10].

Another class is noncanonical and is reduced to

$$K = 4(-2\partial_I W\partial_J W X^{IJ})^{1/2} + 12W^2. \quad (57)$$

This is a simple generalization of the noncanonical scalar field given in Sec. III. This requires that  $X^{IJ}$  is negative definite and excludes the cosmological solution of the form  $\phi^I = \phi^I(t)$ .

## VI. SUMMARY AND DISCUSSION

Since the classical stability of spacetime is a fundamental issue, the positive mass may be a creditable guide for the construction of the gravitational theory. Based on this belief, we explored in this paper whether the gravitational theories with a noncanonical scalar field allow the positive mass. In the case of the simplest Einstein-scalar system  $R + 2K(\phi, X)$  or its obvious multiple extension, we found that there are only two types of possible theories if the gauge connection does not involve the derivatives of scalar fields [see Eqs. (17) and (51)]. One is the canonical scalar system with the specific potential of Eq. (27) [9,10]. For the other case with  $X = -(1/2)(\nabla\phi)^2 > 0$ , it turns out that there is no nontrivial solution for  $\phi$ . This is a quite striking result because this prohibits cosmological solutions. In the  $X < 0$  case, we have a noncanonical theory compatible with positive mass. But, this case also excludes the cosmological solution. We checked the consistency of the Killing spinor equations and obtained the most general BPS metrics in the noncanonical theory. The BPS solutions illustrate that this theory exhibits a strange behavior in the sense that the function  $\phi$  remains unfixed if it depends on a single valuable. With this issue in mind, we should discard this theory as a viable gravitational theory.

We obtained a single condition upon the gauge connection required for the positive mass (9) *à la* Witten-Nester argument. This restricts the plausible connection

$\mathcal{A}_\mu$ , but we have not fully understood the underlying mathematical and physical implications. We remarked that in four dimensions, the condition (9) is equivalent to the fact that the 1-form  $U_\mu = i\bar{\epsilon}\gamma_5\gamma_\mu\epsilon$  is closed when the Killing spinor equation is satisfied. This bilinear property has an obvious dimensional dependence. For  $d = 5$ , the BPS metric can be written as a fiber over the (hyper-) Kähler manifold with a (hyper-) Kähler form  $\Phi_{\mu\nu} = i\bar{\epsilon}\gamma_{\mu\nu}\epsilon$  [21,25], and Eq. (9) directly implies that  $\Phi_{\mu\nu}$  is closed. We believe that we can assign an underlying mathematical reason to these relations. We would like to address this issue in a future publication.

There is a significant difference from the similar study of the nonlinear massive gravity, etc., which has not been addressed so far from the viewpoint of positive mass. However, the current study will give us a hint because the Horndeski theory may be regarded as a proxy theory for them.

There are many remaining issues. We employed the supercovariant derivative as Eq. (2), which yields a universal formula (10) with the extra condition (9). In this equation, the Einstein tensor appears explicitly; hence, this formula is useful for the Einstein frame. In other theories, the Jordan-frame formulation may be more advantageous, which motivates us to seek the suitable Jordan-frame supercovariant derivative. We leave this issue to a future study.

In addition, we employed the connection  $\mathcal{A}_\mu$  given by Eq. (17). This is the minimal ansatz to have the positive-definite form of the mass, and other forms of the gauge connection are conceivable. The Horndeski theory [18] may be tractable in an analogous fashion. The strategy is as follows:

- (i) Find the appropriate connection  $\mathcal{A}_\mu$  satisfying Eq. (9).
- (ii) Prove the positivity of  $S^0$  given by Eq. (11) modulo field equations and the dominant energy condition for additional matter fields.

In the process of (i), the condition (35) will be of help, since this should be satisfied for all supergravity theories. For example, Eq. (9) is satisfied for  $\mathcal{A}_\mu = f(\phi, X)(\gamma_{\mu\nu}\nabla^\nu\phi - \nabla_\mu\phi)$  but Eq. (34) is not. In (ii), a feasible blueprint is to write  $S^\mu$  into the form

$$S^\mu = S^\mu{}_\nu V^\nu + i\bar{\delta\lambda}\gamma^\mu\delta\lambda, \quad (58)$$

as we did in the main text. Here,  $S_{\mu\nu}$  should yield Einstein's equations  $G_{\mu\nu} = S_{\mu\nu} + T_{\mu\nu}^{(\text{matter})}$  and be derived from a covariant action. This will enable us to find the modified gravitational theories admitting the positive mass in a methodical manner. We hope to visit these issues in a future publication.

## ACKNOWLEDGMENTS

M. N. is grateful to Hideo Kodama for helpful conversations. T. S. thanks Tsutomu Kobayashi, Ryo Saito, and

Yuki Sakakihara for fruitful discussions in the early stage of this work. We thank the Yukawa Institute, Kyoto University, where this work was initiated during the two workshops on “Nonlinear massive gravity and its observational test”(YITP-T-12-04) and “String theory, black holes and holography”(YITP-W-13-02). The work of M.

S. is partly supported by the MEXT Grant-in-Aid for Scientific Research on Innovative Areas, Grant No. 21111006. T.S. is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan (Grants No. 21244033 and No. 25610055).

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