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Static cylindrically symmetric dyonic wormholes in six-dimensional Kaluza-Klein theory: Exact solutions

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We study cylindrically symmetric Abelian wormholes in $(4+n)$-dimensional Kaluza-Klein theory. It is shown that static, four-dimensional, cylindrically symmetric solutions in $(4+n)$-dimensional Kaluza-Klein theory with maximal Abelian isometry group $U(1)^n$ of the internal space with diagonal internal metric can be obtained, as in the case of a supersymmetric static black hole [M. Cvetič and D. Youm, Phys. Rev. D **52**, 2144 (1995)], only if the isometry group of the internal space is broken down to the $U(1)_e \times U(1)_m$ gauge group; they correspond to dyonic configurations with one electric (Q_e) and one magnetic (Q_m) charge that are related either to the same $U(1)_e$ or $U(1)_m$ gauge field or to different factors of the $U(1)_e \times U(1)_m$ gauge group of the effective six-dimensional Kaluza-Klein theory. We find new exact solutions of the six-dimensional Kaluza-Klein theory with two Abelian gauge fields, a dilaton field and a scalar field, associated with the internal metric. We obtain new types of cylindrically symmetric wormholes supported by the radial and longitudinal electric and magnetic fields.

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I. INTRODUCTION

As a starting point we recall the comment in a paper [1,2] where a class of static spherically symmetric solutions in $(4+n)$-dimensional Kaluza-Klein theory with Abelian isometry was studied: ‘‘We assumed that the internal isometry group G is Abelian. In this case, different supersymmetric static spherical solutions spontaneously break G down to different $U(1)_E \times U(1)_M$ factors as the vacuum configurations. We suspect that the same thing will happen for axially symmetric stationary configurations, but it remains to be proven.’’ This article is motivated by the desire to find out to what extent the numerous results (e.g. Refs. [3–6], and others) obtained for (spherically symmetric) black holes in higher-dimensional unified theories, related to supergravity and string theory, can be transferred to axially symmetric wormholes. We started with a cylindrically symmetric wormhole; the first task was the derivation from the higher-dimensional theory of an effective four-dimensional theory of pure gravity, including ‘‘external gravitons’’—the space-time metric—and ‘‘internal gravitons’’—the scalar and gauge fields associated with the extra dimensions—as well as finding the exact solutions of the resulting theory. This problem is solved in this paper. The study of the geodesic structure, singularity structure and thermal properties of the wormhole solutions we find, as well as the inclusion of fermions in the theory and finding connections between Kaluza-Klein wormhole solutions and string theory, would be the next step in the research of cylindrically symmetric wormholes within supersymmetric unified field theories.

By wormholes one usually means topological features in the form of ‘‘handles’’ (throats) connecting different regions of the space-time continuum.

The typically discussed wormhole models are endowed with spherical symmetry (see, for example, the survey [7,8]). It has been shown [9] that spherically symmetric wormholes can exist only in the presence of so-called ‘‘exotic matter,’’ which refers to a variety of field configurations that have, for example, negative energy density and negative pressure.

A cylindrically symmetric space-time has a preferred direction—the axis of (axial) symmetry. Examples of axially symmetric systems include cosmic strings, among others. We consider a static cylindrically symmetric space-time metric [10]

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\omega(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2, \tag{1}$$

where u is an arbitrary cylindrical radial coordinate, $z \in (-\infty, +\infty)$ is the longitudinal coordinate, and $\phi \in [0, 2\pi]$ is the angular coordinate. The ‘‘circle radius’’ $R(u) := e^{\beta(u)}$ is non-negative and tends to $+\infty$ when $u \to \pm\infty$. The space-time (1) has one timelike Killing vector $\xi_1 = \partial_t$ and two spacelike Killing vectors $\xi_2 = \partial_\phi$, $\xi_3 = \partial_z$, which define the axial symmetry. By a suitable choice of the coordinate u the equality

$$\omega(u) = \beta(u) + \gamma(u) + \xi(u)$$

can be satisfied (in what follows we assume that this equality holds), and from Eq. (1) we have

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2[\beta(u)+\gamma(u)+\xi(u)]}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2. \tag{2}$$

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Following Bronnikov and Lemos [10] we make the following definitions.

Definition 1.—We say that the metric (1) describes a cylindrically symmetric wormhole if the circle radius $R(u)$ has an absolute minimum $R(u_0) > 0$ at some point $u = u_0$ and for all possible values of u the metric functions $\omega(u)$, $\beta(u)$, $\gamma(u)$, and $\xi(u)$ in Eq. (1) are smooth and finite.

Definition 2.—The throat of a cylindrically symmetric wormhole with the metric (1) is a cylindrical hypersurface defined by the equation

$$u = u_0.$$

It has been shown in Ref. [10] that the existence of the static, cylindrically symmetric wormholes does not require violation of the weak or null energy conditions near the throat, and the cylindrically symmetric geometry of wormhole configurations can be generated by less exotic sources compared to the case of spherical symmetry. The exact solutions of Einstein's theory of gravity with scalar, spinor, and electromagnetic fields describing static cylindrically symmetric wormholes with the metric (2) have been obtained in Ref. [10], and all the solutions are not asymptotically flat. It has been proven that in the absence of material fields that violate the weak or null energy conditions, i.e. in the case of an everywhere non-negative energy density of matter, flat asymptotic behavior on both sides of a cylindrically symmetric wormhole is impossible [10].

In this article we discuss static, cylindrically symmetric space-times (2) within $(4 + n)$-dimensional Kaluza-Klein theory with Abelian isometry, and our results confirm the validity of the above "no-go" statement.

The paper is organized as follows. In Sec. II we discuss the dimensional reduction of $(4 + n)$-dimensional gravity with a four-dimensional space-time metric (2) and gauge and scalar fields that are compatible with cylindrical symmetry. We show that the isometry group $U(1)^n$ of the internal space with diagonal metric is broken down to the $U(1)_e \times U(1)_m$ gauge group and obtain the constraints on charges in the case of compactification on a 2-torus. In Sec. III we derive and integrate equations of motion for cylindrically symmetric configurations with two Abelian gauge fields, a dilaton field and a scalar field. We find the exact solutions describing cylindrically symmetric wormholes in the six-dimensional effective Kaluza-Klein theory with radial electric and magnetic fields. In Sec. IV the exact solutions describing cylindrically symmetric six-dimensional Kaluza-Klein wormholes with the longitudinal electric and magnetic fields are found. Conclusions are given in Sec. V.

In this paper we use units in which the speed of light in vacuum c and gravitational constant G are taken to be one: $c = G = 1$.

II. DIMENSIONAL REDUCTION OF (4 + N)-DIMENSIONAL GRAVITY WITH DIAGONAL INTERNAL METRIC

The effective four-dimensional Kaluza-Klein theory is obtained from $(4 + n)$-dimensional pure gravity with the Einstein-Hilbert action [11],

$$S_{4+n} = \frac{1}{16\pi G_{4+n}} \int \sqrt{-g^{(4+n)}} R^{(4+n)} d^{4+n}x, \qquad (3)$$

by compactifying the n extra spatial coordinates on a compact manifold. Here G_{4+n} is the gravitational constant in $4 + n$ dimensions, $R^{(4+n)}$ is the Ricci scalar and $g^{(4+n)}$ is the determinant of a metric g_{AB}[1] of the unified space M^{4+n}.

The dimensional reduction of the $(4 + n)$-dimensional gravity is achieved by requiring right invariance of the metric g_{AB} under the action of an isometry group G_n with Killing vector fields $X_a (a = 1, 2, \ldots, n)$ and structure constants f^c_{ab}: $L_{X_a} g_{AB} = 0$, $[X_a, X_b] = f^c_{ab} X_c$. M^{4+n} is considered as a principal fiber bundle $P(M^4, G_n)$ with the four-dimensional space-time M^4 as the base manifold and G_n as the structure group. In a local direct-product coordinate basis $(\partial_\mu, \partial_a)$ the metric g_{AB} is written as

$$\begin{pmatrix} e^{-\sigma/\alpha} g_{\mu\nu} + e^{2\sigma/(n\alpha)} \rho_{bc} A^b_\mu A^c_\nu & e^{2\sigma/(n\alpha)} \rho_{cb} A^c_\mu \\ e^{2\sigma/(n\alpha)} \rho_{ac} A^c_\nu & e^{2\sigma/(n\alpha)} \rho_{ab} \end{pmatrix},$$

where σ is the dilaton field and $A^1_\mu, \ldots, A^n_\mu$ are gauge potentials, $\alpha = \sqrt{1 + 2/n}$ is the coupling constant of the dilaton to the gauge fields, and ρ_{ab} is the unimodular part of the internal metric: $\det(\rho_{ab}) = 1$. The dependence of g_{AB} on internal space coordinates is determined by its right invariance under the action of G_n [2,11],

$$\partial_a g_{\mu\nu} = 0, \qquad \partial_a A^c_\mu = -f^c_{ab} A^b_\mu,$$
$$\partial_a \rho_{bc} = f^d_{ab} \rho_{dc} + f^d_{ac} \rho_{bd}.$$

When the isometry group G_n is unimodular, the $(4 + n)$-d Lagrangian density of the Einstein-Hilbert action (3) becomes explicitly independent of internal coordinates. The dimensional reduction is obtained by integrating over the internal space and, up to a total divergence, the Lagrangian of the effective four-dimensional Kaluza-Klein theory is given by [11]

[1] Uppercase letters $A, B, \ldots$ are used for the indices of the coordinates $x^0, \ldots, x^{3+n}$ in a $(4 + n)$-dimensional space. The greek letters $\mu, \nu, \ldots$ denote the indices of coordinates x^0, x^1, x^2, x^3 of the four-dimensional space-time and the lowercase letters $a, b, \ldots$ are used for the coordinates $x^4, \ldots, x^{3+n}$ of an internal space.

$$L = \frac{1}{16\pi}\sqrt{-g}\Big[R + e^{-\alpha\sigma}\tilde{R} - \frac{1}{4}e^{\alpha\sigma}\rho_{ab}F^a_{\mu\nu}F^{b\mu\nu} - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4}\rho^{ab}\rho^{cd}D_\mu\rho_{ac}D^\mu\rho_{bd} + \lambda(\det(\rho_{ab}) - 1)\Big], \tag{4}$$

where $g = \det(g_{\mu\nu})$, $\tilde{R}$ is the Ricci scalar of the unimodular part ρ_{ab} of the internal metric, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \hat{g}f^a_{bc}A^b_\mu A^c_\nu$ is the field strength of A^a_μ, $\hat{g}$ is the gauge coupling constant, $D_\mu\rho_{ab} = \partial_\mu\rho_{ab} - f^d_{cb}A^c_\mu\rho_{ad} - f^d_{ca}A^c_\mu\rho_{db}$ is the corresponding gauge covariant derivative, $\rho^{ac}\rho_{cb} = \delta^a_b$, and λ is a Lagrange multiplier; the 4-dimensional gravitational constant has been set equal to 1. When G_n is an Abelian isometry group (compactification on an n-torus T^n), all structure constants f^c_{ab} vanish, the gauge covariant derivatives in Eq. (4) become partial derivatives, and the Ricci scalar $\tilde{R}$ vanishes.

We use the diagonal internal metric *Ansatz* [2,12],

$$(\rho_{ab}) = \mathrm{diag}\Big(\rho_1, \ldots, \rho_{n-1}, \prod_{h=1}^{n-1}\rho_h^{-1}\Big), \tag{5}$$

and the static cylindrically symmetric *Ansätze* for the 4-dimensional space-time metric and for the gauge and scalar fields associated with the internal metric. We choose the space-time metric $g_{\mu\nu}$ of the form (2). The *Ansätze* for the electric and magnetic fields, compatible with cylindrical symmetry, are obtained by using the Yang-Mills equations $\nabla_\mu(e^{\alpha\sigma}\rho_a F^{a\,\mu\nu}) = 0$ derived from the Lagrangian (4) and are of the following three forms:

$$F^a_{tu} = \frac{Q^a_e e^{2\gamma(u)}}{e^{\alpha\sigma(u)}\rho_a(u)} \equiv \tilde{E}^a(u)e^{2\gamma(u)},$$
$$F^a_{z\phi} = Q^a_m \quad \text{for electric and magnetic radial fields,}$$

$$F^a_{tz} = Q^a_e,$$
$$F^a_{u\phi} = \frac{Q^a_m e^{2\beta(u)}}{e^{\alpha\sigma(u)}\rho_a(u)} \equiv \tilde{H}^a(u)e^{2\beta(u)}$$
$$\text{for electric and magnetic longitudinal fields,}$$

$$F^a_{t\phi} = Q^a_e,$$
$$F^a_{uz} = \frac{Q^a_m e^{2\xi(u)}}{e^{\alpha\sigma(u)}\rho_a(u)} \equiv \tilde{B}^a(u)e^{2\xi(u)}$$
$$\text{for electric and magnetic azimuthal fields,}$$

where $a = 4, \ldots, n+3$, the constant Q^a_m is the magnetic charge and the constant Q^a_e is the electric charge of the configuration; the other components of the strength fields $F^a_{\mu\nu}$ are equal to zero [see arguments after Eqs. (13) and (44)].

The allowed charge configurations are restricted by the special choice of the internal metric (5). Really, the Euler-Lagrange equations for $\rho_{ab}(u)$, derived from the Lagrangian (4), read

$$e^{\alpha\sigma(u)+2\gamma(u)}[Q^a_m Q^b_m - \tilde{E}^a(u)\tilde{E}^b(u)] + \lambda\rho^{ab}(u) = \frac{d^2}{du^2}\rho^{ab}(u) \quad \text{for the radial fields,}$$

$$e^{\alpha\sigma(u)+2\beta(u)}[\tilde{H}^a(u)\tilde{H}^b(u) - Q^a_e Q^b_e] + \lambda\rho^{ab}(u) = \frac{d^2}{du^2}\rho^{ab}(u) \quad \text{for the longitudinal fields,}$$

$$e^{\alpha\sigma(u)+2\xi(u)}[\tilde{B}^a(u)\tilde{B}^b(u) - Q^a_e Q^b_e] + \lambda\rho^{ab}(u) = \frac{d^2}{du^2}\rho^{ab}(u) \quad \text{for the azimuthal fields.}$$

It follows from this that for the diagonal metric (5) the following constraints have to be satisfied:

$$Q^a_e Q^b_e - e^{2\alpha\sigma}\rho_a\rho_b Q^a_m Q^b_m = 0$$
$$\text{for the radial fields,} \quad \text{when } a \neq b,$$

$$Q^a_m Q^b_m - e^{2\alpha\sigma}\rho_a\rho_b Q^a_e Q^b_e = 0$$
$$\text{for the longitudinal and azimuthal fields,} \quad \text{when } a \neq b,$$

whence for radial fields (i) $Q^a_e Q^b_e = Q^a_m Q^b_m = 0$ if $a \neq b$, or, generally, (ii) $Q^a_e Q^b_e - \kappa_a\kappa_b Q^a_m Q^b_m = 0$ if $a \neq b$, with $\kappa_a \equiv e^{\alpha\sigma}\rho_a = \text{const}$. The latter case would imply the equation of motion for $e^{\alpha\sigma}\rho_a$ with $Q^a_e = Q^a_m = 0$. Thus, the constraint (ii) reduces to the subset of constraints (i) which imply that the same (electric or magnetic) type of charge can appear in at most one gauge field. The same result is valid for longitudinal and azimuthal electric and magnetic fields. Consequently, the internal isometry group $\mathrm{U}(1)^n$ is broken down to at most $\mathrm{U}(1) \times \mathrm{U}(1)$, with only one electric and one magnetic charge. Without loss of generality, we associate two U(1) factors of the internal isometry group $\mathrm{U}(1) \times \mathrm{U}(1)$ with the $(n-1)$th and the nth internal dimensions, i.e., with gauge fields A^{n-1}_μ and A^n_μ. When the first $(n-2)$ gauge fields are turned off the first $(n-2)$ components of the diagonal internal metric become constant: $e^{2\sigma/(n\alpha)}\rho_a = \text{const}$, $a = 1, \ldots n-2$. As a result the solutions of $(4+n)$-dimensional Kaluza-Klein theory are those of effective six-dimensional Kaluza-Klein theory with action

$$S_4 = \frac{1}{16\pi}\int\sqrt{-g}[R - 2(\nabla\psi)^2 - 4(\nabla\chi)^2 - e^{2\sqrt{2}(\psi+\chi)}K^{\mu\nu}K_{\mu\nu} - e^{2\sqrt{2}(\psi-\chi)}F^{\mu\nu}F_{\mu\nu}]d^4x, \tag{6}$$

which is obtained from Eq. (4) by using the following field redefinition: $\chi_a \equiv (1/\sqrt{2})[\ln\rho_a + 2\sigma/(n\alpha)] = \text{const}$, $a = 1, \ldots, n-2$, $\chi_{n-1} \equiv (1/\sqrt{2})[\ln\rho_{n-1} + (2-n)\sigma/(n\alpha)] \equiv \chi$,

$\chi_n = -\chi$, $\psi \equiv \sqrt{2}\alpha\sigma$, $F^{n-1}_{\mu\nu} \equiv 2K_{\mu\nu}$, $F^n_{\mu\nu} \equiv 2F_{\mu\nu}$ (cf. Ref. [2]; see also Refs. [6,12]).

Note that the above result can also be obtained by solving the Killing spinor equations for supersymmetric Kaluza-Klein configurations with cylindrical symmetry (the proof will be given in a separate paper).

III. EXACT SOLUTIONS OF EINSTEIN–YANG-MILLS–DILATON EQUATIONS WITH RADIAL ELECTRIC AND MAGNETIC FIELDS

In this section we will obtain the Einstein equations, Yang-Mills equations for the gauge fields A_μ, B_ν, and the equations for the dilaton ψ and scalar field χ. We will find exact solutions of the Einstein–Yang-Mills–dilaton (EYMD) equations in the case of the radial electric and magnetic fields.

We consider the four-dimensional action (6) of six-dimensional Kaluza-Klein theory with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad K_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad (7)$$

where A_μ, B_ν are the two Abelian gauge fields, and ψ (dilaton) and χ are two scalar fields. The nonzero components of the Ricci tensor R^μ_ν and the Ricci scalar $R \equiv R^\mu_\mu$ of the metric (2) are

$$\begin{aligned}
&R^t_t = e^{-2(\beta+\gamma+\xi)}\gamma'',\\
&R^u_u = e^{-2(\beta+\gamma+\xi)}[\gamma'' + \xi'' + \beta'' - 2(\beta'\xi' + \xi'\gamma' + \gamma'\beta')],\\
&R^z_z = e^{-2(\beta+\gamma+\xi)}\xi'', \qquad R^\phi_\phi = e^{-2(\beta+\gamma+\xi)}\beta'',\\
&R = 2e^{-2(\beta+\gamma+\xi)}(\gamma'' + \xi'' + \beta'' - \beta'\xi' - \xi'\gamma' - \gamma'\beta'),
\end{aligned}$$

where the prime denotes differentiation with respect to u. The Euler-Lagrange equations for the scalar fields ψ and χ, derived from the the action (6), are given by

$$\begin{aligned}
&\Box\psi = (1/\sqrt{2})(e^{2\sqrt{2}(\psi+\chi)}K_{\mu\nu}K^{\mu\nu} + e^{2\sqrt{2}(\psi-\chi)}F_{\mu\nu}F^{\mu\nu}),\\
&\Box\chi = (1/(2\sqrt{2}))(e^{2\sqrt{2}(\psi+\chi)}K_{\mu\nu}K^{\mu\nu} - e^{2\sqrt{2}(\psi-\chi)}F_{\mu\nu}F^{\mu\nu})
\end{aligned} \quad (8)$$

($\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$). Variation in the fields A_ν, B_ν gives the Yang-Mills equations,

$$\begin{aligned}
&\nabla_\nu F^{\mu\nu} = 2\sqrt{2}F^{\mu\nu}\nabla_\nu(\chi - \psi),\\
&\nabla_\nu K^{\mu\nu} = -2\sqrt{2}K^{\mu\nu}\nabla_\nu(\psi + \chi),
\end{aligned} \quad (9)$$

where ∇_ν is the covariant derivative with respect to x^ν.

Finally, varying the metric $g_{\mu\nu}$, we obtain the Einstein equations

$$R^\mu_\nu = \tilde{T}^\mu_\nu, \qquad \tilde{T}^\mu_\nu := T^\mu_\nu - (1/2)\delta^\mu_\nu T \qquad (T \equiv T^\lambda_\lambda), \quad (10)$$

with the energy-momentum tensor

$$\begin{aligned}
T_{\mu\nu} = {}& 2\nabla_\mu\psi\nabla_\nu\psi - g_{\mu\nu}(\nabla\psi)^2 + 4\nabla_\mu\chi\nabla_\nu\chi\\
&- 2g_{\mu\nu}(\nabla\chi)^2 + e^{2\sqrt{2}(\psi+\chi)}(2K_{\mu\tau}K_\nu{}^\tau\\
&- (1/2)g_{\mu\nu}K_{\sigma\tau}K^{\sigma\tau}) + e^{2\sqrt{2}(\psi-\chi)}(2F_{\mu\tau}F_\nu{}^\tau\\
&- (1/2)g_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}).
\end{aligned} \quad (11)$$

We expand the gauge fields A_μ and B_μ into temporal and spatial components, $A_\mu = (A_t, \vec{A})$, $B_\mu = (B_t, \vec{B})$, where $\vec{A} = (A_u, A_z, A_\phi)$, $\vec{B} = (B_u, B_z, B_\phi)$, and consider the *Ansatz* for the form of the next static Abelian gauge fields, a dilaton field and a scalar field,

$$\begin{aligned}
&A_\mu = (A_t(u), 0, 0, 0), \qquad B_\mu = (0, 0, 0, \mathrm{const}\cdot z),\\
&\psi = \psi(u), \qquad \chi = \chi(u).
\end{aligned} \quad (12)$$

We introduce the three-dimensional vector fields

$$\begin{aligned}
&\vec{E}_A := -\vec{\partial}A_t + \frac{\partial\vec{A}}{\partial t}, \qquad \vec{H}_A := [\vec{\partial}, \vec{A}],\\
&\vec{E}_B := -\vec{\partial}B_t + \frac{\partial\vec{B}}{\partial t}, \qquad \vec{H}_B := [\vec{\partial}, \vec{B}],
\end{aligned} \quad (13)$$

where $\vec{\partial} = (\partial_u, \partial_z, \partial_\phi)$, $([\vec{\partial}, \vec{A}])^h := e^{-[\gamma+2(\xi+\beta)]}\varepsilon^{hkl}\partial_k A_l$ with totally antisymmetric ε^{hkl}: $\varepsilon^{uz\phi} = 1$, and h, k, l running over u, z, ϕ; components are defined similarly for $[\vec{\partial}, \vec{B}]$. By an analogy with the electromagnetic field, $\vec{E}_A$, $\vec{E}_B$ are called electric fields and $\vec{H}_A$, $\vec{H}_B$ are called magnetic fields. Since from Eqs. (12) and (13) only the components E^u_A, H^u_B are nonzero, we call the fields A_μ, B_μ radial.

Using Eqs. (7) and (12) we find the solutions of the Yang-Mills equations (9), describing the radial electric and magnetic fields,

$$\begin{aligned}
&K^{z\phi} = Q_m e^{-2(\beta+\xi)},\\
&K^{t\phi} = K^{tz} = K^{tu} = K^{uz} = K^{u\phi} = 0, \qquad Q_m = \mathrm{const},\\
&F^{ut} = Q_e e^{-2[\beta+\gamma+\xi+\sqrt{2}(\psi-\chi)]},\\
&F^{t\phi} = F^{tz} = F^{uz} = F^{u\phi} = F^{z\phi} = 0, \qquad Q_e = \mathrm{const}.
\end{aligned}$$

The nonzero components of the tensor $\tilde{T}^\nu_\mu$ are

$$\begin{aligned}
\tilde{T}^\phi_\phi &= \tilde{T}^z_z = -\tilde{T}^t_t\\
&= Q_e^2 e^{-2[\beta+\xi+\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{-2[\beta+\xi-\sqrt{2}(\psi+\chi)]},\\
\tilde{T}^u_u &= -2e^{-2(\gamma+\xi+\beta)}(\psi'^2 + 2\chi'^2) - Q_e^2 e^{-2[\beta+\xi+\sqrt{2}(\psi-\chi)]}\\
&\quad - Q_m^2 e^{-2[\beta+\xi-\sqrt{2}(\psi+\chi)]}.
\end{aligned}$$

Equations (8) give

$$\chi'' = -(1/\sqrt{2})(Q_e^2 e^{2[\gamma-\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{2[\gamma+\sqrt{2}(\psi+\chi)]}), \quad (14)$$

$$\psi'' = \sqrt{2}(Q_e^2 e^{2[\gamma-\sqrt{2}(\psi-\chi)]} - Q_m^2 e^{2[\gamma+\sqrt{2}(\psi+\chi)]}). \quad (15)$$

The Einstein equations $R_\nu^\mu = \tilde{T}_\nu^\mu$ (10) reduce to the next four equations,

$$\gamma'' = -Q_e^2 e^{2[\gamma-\sqrt{2}(\psi-\chi)]} - Q_m^2 e^{2[\gamma+\sqrt{2}(\psi+\chi)]} \quad (R_t^t = \tilde{T}_t^t), \quad (16)$$

$$\xi'' = Q_e^2 e^{2[\gamma-\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{2[\gamma+\sqrt{2}(\psi+\chi)]} \quad (R_z^z = \tilde{T}_z^z), \quad (17)$$

$$\beta'' = Q_e^2 e^{2[\gamma-\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{2[\gamma+\sqrt{2}(\psi+\chi)]} \quad (R_\phi^\phi = \tilde{T}_\phi^\phi), \quad (18)$$

$$\begin{aligned}\gamma'' + \xi'' + \beta'' - 2(\beta'\xi' + \xi'\gamma' + \gamma'\beta') &= -2\psi'^2 - 4\chi'^2 - Q_e^2 e^{2[\gamma-\sqrt{2}(\psi-\chi)]} \\ &\quad - Q_m^2 e^{2[\gamma+\sqrt{2}(\psi+\chi)]} \quad (R_u^u = \tilde{T}_u^u). \quad (19)\end{aligned}$$

The other six Einstein equations are satisfied identically. Substituting Eqs. (16)–(18) into Eq. (19), we obtain

$$\begin{aligned}\gamma'\beta' + \xi'\gamma' + \beta'\xi' = \psi'^2 + 2\chi'^2 + Q_e^2 e^{2[\gamma-\sqrt{2}(\psi-\chi)]} \\ + Q_m^2 e^{2[\gamma+\sqrt{2}(\psi+\chi)]}. \quad (20)\end{aligned}$$

We notice that the equation obtained by differentiating Eq. (20) is satisfied identically as a consequence of Eqs. (14)–(18), which imposes restrictions on the initial data for this system. By integrating Eqs. (14) and (16) we see that $\gamma'' = \sqrt{2}\chi''$. Setting the constants of integration equal to zero, we obtain

$$\gamma = \sqrt{2}\chi. \quad (21)$$

In addition, from Eqs. (16)–(18) we get $\beta = -\gamma + au + \beta_0$, $\xi = -\gamma + bu + \xi_0$, where a, b, β_0, ξ_0 are the constants of integration. By scale transformations of coordinates z and ϕ the additive constants β_0, ξ_0 can be made to vanish, so we have

$$\beta = -\gamma + au, \qquad \xi = -\gamma + bu. \quad (22)$$

From Eqs. (15), (16), and (21) it follows that

$$\begin{aligned}\gamma'' &= -Q_e^2 e^{2(2\gamma-\sqrt{2}\psi)} - Q_m^2 e^{2(2\gamma+\sqrt{2}\psi)}, \\ \psi'' &= \sqrt{2}(Q_e^2 e^{2(2\gamma-\sqrt{2}\psi)} - Q_m^2 e^{2(2\gamma+\sqrt{2}\psi)}),\end{aligned} \quad (23)$$

supplemented by the condition (20), which in view of Eqs. (21) and (22) takes the form

$$-2\gamma'^2 + ab = \psi'^2 + Q_m^2 e^{2(2\gamma+\sqrt{2}\psi)} + Q_e^2 e^{2(2\gamma-\sqrt{2}\psi)}. \quad (24)$$

The system (23) is equivalent to the following system:

$$\eta'' = -2Q_e^2 e^{4\eta}, \qquad \zeta'' = -2Q_m^2 e^{4\zeta}, \quad (25)$$

where

$$\eta = \gamma - (1/\sqrt{2})\psi, \qquad \zeta = \gamma + (1/\sqrt{2})\psi. \quad (26)$$

Consider all possible cases: (1) $Q_eQ_m \neq 0$, (2) $Q_e \neq 0$, $Q_m = 0$, (3) $Q_e = 0$, $Q_m \neq 0$, and (4) $Q_e = Q_m = 0$.

Solving the system (25) in the case $Q_eQ_m \neq 0$ and taking into account that, by Eq. (26), $\gamma = (\zeta + \eta)/2$, $\psi = (\zeta - \eta)/\sqrt{2}$, we find

$$\begin{aligned}\gamma(u) = -(1/4)\ln(4|q_eq_m|\cosh[h_e(u-u_e)] \\ \times\cosh[h_m(u-u_m)]), \quad (27)\end{aligned}$$

$$\begin{aligned}\psi(u) = -(1/(2\sqrt{2}))\ln(|q_m/q_e| \\ \times\cosh[h_m(u-u_m)]/\cosh[h_e(u-u_e)]), \quad (28)\end{aligned}$$

where $h_e \neq 0$, $h_m \neq 0$, u_e, u_m are constants of integration, $Q_e/h_e \equiv q_e$ and $Q_m/h_m \equiv q_m$. Without loss of generality we can assume that $h_e > 0$, $h_m > 0$. From Eq. (22) we obtain

$$\begin{aligned}\beta(u) = (1/4)\ln(4|q_eq_m|\cosh[h_e(u-u_e)] \\ \times\cosh[h_m(u-u_m)]) + au, \quad (29)\end{aligned}$$

$$\begin{aligned}\xi(u) = (1/4)\ln(4|q_eq_m|\cosh[h_e(u-u_e)] \\ \times\cosh[h_m(u-u_m)]) + bu. \quad (30)\end{aligned}$$

Substituting Eqs. (27)–(30) into Eq. (20), we have

$$4ab = h_e^2 + h_m^2.$$

From Eq. (29) we derive

$$\begin{aligned}\beta'(u) = (1/4)(h_e\tanh[h_e(u-u_e)] \\ + h_m\tanh[h_m(u-u_m)]) + a. \quad (31)\end{aligned}$$

Due to the monotonicity of tanh the derivative $\beta'(u)$ can vanish at no more than one point. If

$$|a| < (h_e + h_m)/4 \quad (32)$$

then there exists a (unique) point $u_0 \in \mathbb{R}$ for which $\beta'(u_0) = 0$. If the condition (32) is not satisfied, then there is no point at which the derivative (31) vanishes.

Note that because of the transcendence of the equation $\beta'(u) = 0$ the value u_0 can be found analytically only for some particular values of the parameters h_e, h_m.

Since, by Eq. (18), the second derivative of $\beta(u)$ is positive, taking into account the condition (32), the function $\beta(u)$ has an absolute minimum at $u = u_0$. In accordance with Definition 1, the metric (2) with the functions (27), (29), and (30) together with condition (32) describes a family of cylindrically symmetric wormholes characterized by an electric charge Q_e, a magnetic charge Q_m and parameters h_e, h_m, a and b. Note that this solution is not asymptotically flat.

We put $r_e = \exp(u_e)$, $r_m = \exp(u_m)$ and introduce a new "radial" coordinate $r := \exp(u) \in [0, +\infty)$. With these new coordinates the metric (2) with the functions (27), (29), and (30) takes the form

$$ds^2 = \kappa\Omega\left(\frac{dt^2}{\kappa^2\Omega^2} - r^{2(a+b-1)}dr^2 - r^{2b}dz^2 - r^{2a}d\phi^2\right), \tag{33}$$

where

$$\kappa = \sqrt{|q_e q_m|},$$
$$\Omega = \sqrt{[(r/r_e)^{h_e} + (r/r_e)^{-h_e}][(r/r_m)^{h_m} + (r/r_m)^{-h_m}]},$$
$$4ab = h_e^2 + h_m^2,$$

and $h_e h_m Q_e Q_m \neq 0$. If $4|a| < h_e + h_m$ then Eq. (33) is the metric of the wormhole with the throat radius r_0 defined by the equation

$$(4a + h_e + h_m)(r_0/r_e)^{2h_e}(r_0/r_m)^{2h_m} + (4a - h_e + h_m)(r_0/r_m)^{2h_m} + (4a + h_e - h_m)(r_0/r_e)^{2h_e} + 4a - h_e - h_m = 0.$$

The wormhole (33) with $4|a| < h_e + h_m$ which we denote by $\mathrm{WhCR}^{e;m}$ is generated by the following Abelian gauge fields and scalar fields:

$$A_\mu = (-(1/(4q_e))[(r/r_e)^{h_e} - (r/r_e)^{-h_e}]/|[(r/r_e)^{h_e} + (r/r_e)^{-h_e}], 0, 0, 0),$$
$$B_\mu = (0, 0, 0, h_m q_m z),$$
$$\psi = (1/(2\sqrt{2}))\ln(|q_e/q_m|[(r/r_e)^{h_e} + (r/r_e)^{-h_e}]/[(r/r_m)^{h_m} + (r/r_m)^{-h_m}]),$$
$$\chi = -(1/(4\sqrt{2}))\ln(|q_e q_m|[(r/r_e)^{h_e} + (r/r_e)^{-h_e}][(r/r_m)^{h_m} + (r/r_m)^{-h_m}]). \tag{34}$$

In the second case when $Q_e \neq 0$ and $Q_m = 0$ we obtain

$$\gamma = \sqrt{2}\chi = -(1/4)\ln D(u) + cu, \qquad \beta = -\gamma + au,$$
$$\xi = -\gamma + bu, \qquad \psi = (1/(2\sqrt{2}))\ln D(u) + \sqrt{2}cu,$$

where $D(u) \equiv 2|q_e|\cosh[h_e(u - u_e)]$ and, by Eq. (20), $4(ab - 4c^2) = h_e^2$. We have

$$\beta'(u) = (h_e/4)\tanh[h_e(u - u_e)] + a - c.$$

The single point at which the derivative of $\beta(u)$ vanishes exists only if $4|a - c| < h_e$, and it is given by

$$u_0 = u_e - (1/h_e)\mathrm{artanh}[4(a - c)/h_e].$$

The hypersurface $u = u_0$ defines a throat of the cylindrically symmetric wormhole, which we denote by WhCR^e. Its metric in coordinates t, r, z, ϕ is

$$ds^2 = k_e\Lambda_e\left(\frac{r^{2c}dt^2}{k_e^2\Lambda_e^2} - r^{2(a+b-c-1)}dr^2 - r^{2(b-c)}dz^2 - r^{2(a-c)}d\phi^2\right) \qquad (a, b, c = \mathrm{const}), \tag{35}$$

where

$$k_e = \sqrt{|q_e|}, \qquad \Lambda_e = \sqrt{(r/r_e)^{h_e} + (r/r_e)^{-h_e}},$$
$$4ab = h_e^2 + 16c^2 \qquad (h_e, q_e = \mathrm{const}, h_e q_e \neq 0). \tag{36}$$

The throat radius of the WhCR^e is

$$r_0 = r_e\left(\frac{h_e - 4(a - c)}{h_e + 4(a - c)}\right)^{1/(2h_e)}, \qquad h_e > 4|a - c|, \tag{37}$$

and the wormhole is generated by the following single Abelian gauge field A_μ, a dilaton field ψ and a scalar field χ:

$$A_\mu = (-(1/(4q_e))[(r/r_e)^{h_e} - (r/r_e)^{-h_e}][(r/r_e)^{h_e} + (r/r_e)^{-h_e}]^{-1}, 0, 0, 0),$$
$$\psi = (1/(2\sqrt{2}))\ln(|q_e|r^{4c}[(r/r_e)^{h_e} + (r/r_e)^{-h_e}]),$$
$$\chi = -(1/(4\sqrt{2}))\ln(|q_e|r^{-4c}[(r/r_e)^{h_e} + (r/r_e)^{-h_e}]). \tag{38}$$

The wormhole WhCR^m corresponding to the third case ($Q_e = 0$, $Q_m \neq 0$) is defined by the metric

$$ds^2 = k_m\Lambda_m\left(\frac{r^{2c}dt^2}{k_m^2\Lambda_m^2} - r^{2(a+b-c-1)}dr^2 - r^{2(b-c)}dz^2 - r^{2(a-c)}d\phi^2\right) \qquad (a, b, c = \mathrm{const}), \tag{39}$$

where

$$k_m = \sqrt{|q_m|}, \qquad \Lambda_m = \sqrt{(r/r_m)^{h_m} + (r/r_m)^{-h_m}},$$
$$4ab = h_m^2 + 16c^2 \qquad (h_m, q_m = \mathrm{const}, h_m q_m \neq 0), \tag{40}$$

and by the formulas

$$B_\mu = (0, 0, 0, h_m q_m z),$$
$$\psi = -(1/(2\sqrt{2}))\ln(|q_m|r^{4c}[(r/r_m)^{h_m} + (r/r_m)^{-h_m}]),$$
$$\chi = -(1/(4\sqrt{2}))\ln(|q_m|r^{-4c}[(r/r_m)^{h_m} + (r/r_m)^{-h_m}]),$$
$$r_0 = r_m\left(\frac{h_m - 4(a - c)}{h_m + 4(a - c)}\right)^{1/(2h_m)}, \quad h_m > 4|a - c|. \tag{41}$$

Finally, in the case $Q_e = Q_m = 0$ we have nonwormhole exact solutions of the Einstein-dilaton-scalar field equations,

$$\begin{aligned} ds^2 &= r^{2c}dt^2 - r^{2(a+b+c-1)}dr^2 - r^{2b}dz^2 - r^{2a}d\phi^2 \\ &\quad (a, b, c = \text{const}, r \in (0, +\infty)), \qquad \psi = k_1 \ln r + k_0, \\ \chi &= s_1 \ln r + s_0, \qquad ab + ac + bc = k_1^2 + 2s_1^2 \\ &\quad (k_0, k_1, s_0, s_1 = \text{const}). \end{aligned} \tag{42}$$

Let us consider the question of the stability of the solutions. Introducing the variables $\gamma_1 := \gamma'$, $\psi_1 := \psi'$, we rewrite the system (23) in the form

$$\begin{cases} \gamma' = \gamma_1, \\ \gamma_1' = Q_e^2 e^{2(2\gamma - \sqrt{2}\psi)} + Q_m^2 e^{2(2\gamma + \sqrt{2}\psi)}, \\ \psi' = \psi_1, \\ \psi_1' = \sqrt{2}(Q_e^2 e^{2(2\gamma - \sqrt{2}\psi)} - Q_m^2 e^{2(2\gamma + \sqrt{2}\psi)}). \end{cases} \tag{43}$$

Given that the equilibrium point of a system, $d\vec{x}/d\tau = \vec{f}(\vec{x}(\tau))$, where $\vec{x}$, $\vec{f}$ are vector strings, is the point at which the right-hand side vanishes, $\vec{f}(\vec{x}) = 0$ [13], we find that the equilibrium points of the system (43) are defined by the equations $\gamma_1 = \psi_1 = Q_e^2 e^{2(2\gamma - \sqrt{2}\psi)} = Q_m^2 e^{2(2\gamma + \sqrt{2}\psi)} = 0$. We can see from here that, if at least one of the charges Q_e or Q_m is nonzero, then the equilibrium points are absent, indicating the stability of our wormhole solutions.

In this section we have found three new types of static cylindrically symmetric dilatonic wormholes: dyonic $\mathrm{WhCR}^{e;m}$ defined by Eqs. (33) and (34), WhCR^{e} (35)–(38) with nonzero electric charge and WhCR^{m} (39)–(41) with nonzero magnetic charge. All found solutions are Jacobi stable.

The purely dilatonic exact solutions determined by Eq. (42) turn out to be nonwormholes. So the wormhole solutions are not available when both electric and magnetic charges are equal to zero. Hence, one can conclude that the wormhole properties of our solutions are generated by electric and/or magnetic charges of static Abelian gauge fields.

IV. EXACT SOLUTIONS OF EINSTEIN–YANG-MILLS–DILATON EQUATIONS WITH LONGITUDINAL ELECTRIC AND MAGNETIC FIELDS

In this section we write and integrate the Einstein–Yang-Mills equations and dilaton and scalar field equations for the following *Ansatz*:

$$\begin{aligned} A_\mu &= (\text{const} \cdot z, 0, 0, 0), \qquad B_\mu = (0, 0, 0, B_\phi(u)), \\ \psi &= \psi(u), \qquad \chi = \chi(u). \end{aligned} \tag{44}$$

Of the components (13) only E_A^z and H_B^z are nonzero, so one can speak of longitudinal electric and magnetic fields. Using Eq. (7) and solving the Yang-Mills equations (9) we have

$$\begin{aligned} K^{u\phi} &= Q_m e^{-2[\beta+\gamma+\xi+\sqrt{2}(\psi+\chi)]}, \\ K^{u\phi} &= K^{tz} = K^{tu} = K^{uz} = K^{u\phi} = 0, \qquad Q_m = \text{const}, \\ F^{zt} &= Q_e e^{-2(\gamma+\xi)}, \\ F^{t\phi} &= F^{uz} = F^{ut} = F^{u\phi} = F^{z\phi} = 0, \qquad Q_e = \text{const}. \end{aligned}$$

The Eqs. (8) give

$$\chi'' = -(1/\sqrt{2})(Q_e^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}), \tag{45}$$

$$\psi'' = \sqrt{2}(Q_e^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} - Q_m^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}). \tag{46}$$

The nontrivial Einstein equations (10) are

$$-\gamma'' = Q_e^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}, \tag{47}$$

$$-\xi'' = Q_e^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}, \tag{48}$$

$$\beta'' = Q_e^2 e^{2[\beta+\sqrt{2}(\psi-\chi)]} + Q_m^2 e^{2[\beta-\sqrt{2}(\psi+\chi)]}, \tag{49}$$

$$\begin{aligned} &\gamma'' + \xi'' + \beta'' - 2(\beta'\xi' + \xi'\gamma' + \gamma'\beta') \\ &\quad = -2\psi'^2 - 4\chi'^2 + Q_m^2 e^{2\beta} e^{-2\sqrt{2}(\psi+\chi)} \\ &\qquad + Q_e^2 e^{2\beta} e^{2\sqrt{2}(\psi-\chi)}. \end{aligned} \tag{50}$$

From Eqs. (47)–(50) we get

$$\begin{aligned} \gamma'\beta' + \xi'\gamma' + \beta'\xi' &= \psi'^2 + 2\chi'^2 - Q_m^2 e^{2\beta} e^{-2\sqrt{2}(\psi+\chi)} \\ &\quad - Q_e^2 e^{2\beta} e^{2\sqrt{2}(\psi-\chi)}. \end{aligned} \tag{51}$$

By virtue of Eqs. (45)–(49) the differential consequence of the last equation is satisfied identically, which imposes restrictions on the initial data for the system. From Eqs. (45) and (49) it follows that $\beta'' = -\sqrt{2}\chi''$. We put

$$\beta = -\sqrt{2}\chi \tag{52}$$

and, from Eqs. (47)–(49),

$$\gamma = -\beta + au, \qquad \xi = -\beta + bu \quad (a, b = \text{const}), \tag{53}$$

where the additive integration constants are eliminated by the rescaling of t and z. The system (45)–(49) reduces to the form

$$\begin{aligned} \beta'' &= Q_e^2 e^{2(2\beta - \sqrt{2}\psi)} + Q_m^2 e^{2(2\beta - \sqrt{2}\psi)}, \\ (1/\sqrt{2})\psi'' &= Q_e^2 e^{2(2\beta - \sqrt{2}\psi)} - Q_m^2 e^{2(2\beta - \sqrt{2}\psi)}. \end{aligned} \tag{54}$$

From Eq. (51) we have

$$Q_e^2 e^{2(2\beta + \sqrt{2}\psi)} + Q_m^2 e^{2(2\beta - \sqrt{2}\psi)} + ab = \psi'^2 + 2\beta'^2. \tag{55}$$

In variables $\bar{\eta} = \beta + (1/\sqrt{2})\psi$, $\bar{\zeta} = \beta - (1/\sqrt{2})\psi$, Eq. (54) takes the form

$$\bar{\eta}'' = 2Q_e^2 e^{4\bar{\eta}}, \qquad \bar{\zeta}'' = 2Q_m^2 e^{4\bar{\zeta}}. \tag{56}$$

We assume $Q_e Q_m \neq 0$. From Eq. (56) it follows that

$$\bar{\eta}'^2 - Q_e^2 e^{4\bar{\eta}} = (\varepsilon/4)h_e^2, \qquad \bar{\zeta}'^2 - Q_m^2 e^{4\bar{\zeta}} = (\bar{\varepsilon}/4)h_m^2,$$

where h_e, h_m are positive integration constants and ε, $\bar{\varepsilon}$ take values 0, ±1. Considering all possible cases we obtain the solutions $\bar{\eta}_k = -\ln \Lambda_{k|e}$, $\bar{\zeta}_k = -\ln \Lambda_{k|m}$, where $k = 1, 2, 3$ and

$$\begin{aligned}
\Lambda_{1|e} &= \sqrt{2}\sqrt{|Q_e(u-u_e)|},\\
\Lambda_{2|e} &= \sqrt{2}\sqrt{|q_e \sinh[h_e(u-u_e)]|},\\
\Lambda_{3|e} &= \sqrt{2}\sqrt{|q_e \sin[h_e(u-u_e)]|},\\
\Lambda_{1|m} &= \sqrt{2}\sqrt{|Q_m(u-u_m)|},\\
\Lambda_{2|m} &= \sqrt{2}\sqrt{|q_m \sinh[h_m(u-u_m)]|},\\
\Lambda_{3|m} &= \sqrt{2}\sqrt{|q_m \sin[h_m(u-u_m)]|}
\end{aligned} \tag{57}$$

($q_e \equiv Q_e/h_e, q_m \equiv Q_m/h_m, h_e > 0, h_m > 0$). Substituting these solutions into the formulas $\beta = (\bar{\eta} + \bar{\zeta})/2$, $\psi = (\bar{\eta} - \bar{\zeta})/\sqrt{2}$ and using Eqs. (52) and (53), we obtain in the case $Q_e Q_m \neq 0$ nine types $(k|e; j|m)$ of exact solutions of the EYMD equations marked with two indices k, j running over 1, 2, 3,

$$\begin{aligned}
ds^2_{k|e;j|m} &= \Lambda_{k|e}\Lambda_{j|m}(e^{2au}dt^2 - e^{2(a+b)u}du^2 - e^{2bu}dz^2)\\
&\quad - \frac{d\phi^2}{\Lambda_{k|e}\Lambda_{j|m}}, \qquad 4ab = \varepsilon_k h_e^2 + \bar{\varepsilon}_j h_m^2,\\
\psi_{k|e;j|m} &= (1/\sqrt{2})\ln(\Lambda_{j|m}/\Lambda_{k|e}),\\
\chi_{k|e;j|m} &= (1/(2\sqrt{2}))\ln(\Lambda_{k|e}\Lambda_{j|m}),\\
A_\mu &= (-h_e q_e z, 0, 0, 0), \qquad B_\mu^{j|m} = (0, 0, 0, B_\phi^{j|m}),
\end{aligned} \tag{58}$$

where $\varepsilon_1 = \bar{\varepsilon}_1 = 0$, $\varepsilon_2 = \bar{\varepsilon}_2 = 1$, $\varepsilon_3 = \bar{\varepsilon}_3 = -1$ and

$$\begin{aligned}
B_\phi^{1|m} &= -1/(4Q_m(u-u_m)),\\
B_\phi^{2|m} &= -(1/(4q_m))\coth[h_m(u-u_m)],\\
B_\phi^{3|m} &= -(1/(4q_m))\cot[h_m(u-u_m)].
\end{aligned} \tag{59}$$

In the same way we obtain three types $(k|e)$, $k = 1, 2, 3$, of exact solutions of the EYMD equations in the case $Q_e \neq 0$, $Q_m = 0$,

$$\begin{aligned}
ds^2_{k|e} &= e^{cu}\Lambda_{k|e}(e^{2au}dt^2 - e^{2(a+b)u}du^2 - e^{2bu}dz^2)\\
&\quad - \frac{e^{-cu}d\phi^2}{\Lambda_{k|e}}, \qquad 4(ab - c^2) = \varepsilon_k h_e^2,\\
\psi_{k|e} &= (1/\sqrt{2})(cu - \ln \Lambda_{k|e}),\\
\chi_{k|e} &= (1/(2\sqrt{2})(\ln \Lambda_{k|e} + cu),\\
A_\mu &= (-h_e q_e z, 0, 0, 0), \qquad B_\mu = 0,\\
\varepsilon_1 &= 0, \varepsilon_2 = 1, \varepsilon_3 = -1,
\end{aligned} \tag{60}$$

and three types $(k|m)$, $k = 1, 2, 3$, of exact solutions of the EYMD equations in the case $Q_e = 0$, $Q_m \neq 0$,

$$\begin{aligned}
ds^2_{k|m} &= e^{cu}\Lambda_{k|m}(e^{2au}dt^2 - e^{2(a+b)u}du^2 - e^{2bu}dz^2)\\
&\quad - \frac{e^{-cu}d\phi^2}{\Lambda_{k|m}}, \qquad 4(ab - c^2) = \bar{\varepsilon}_k h_m^2,\\
\psi_{k|m} &= (1/\sqrt{2})(\ln \Lambda_{k|m} - cu),\\
\chi_{k|m} &= (1/(2\sqrt{2})(\ln \Lambda_{k|m} + cu), \qquad A_\mu = 0,\\
B_\mu &= (0, 0, 0, B_\phi^{k|m}), \qquad \bar{\varepsilon}_1 = 0, \bar{\varepsilon}_2 = 1, \bar{\varepsilon}_3 = -1,
\end{aligned} \tag{61}$$

where $\Lambda_{k|e}$, $\Lambda_{k|m}$, $B_\phi^{k|m}$ are defined by Eqs. (57) and (59), $c = \text{const}$, and $k = 1, 2, 3$.

In the case $Q_e = Q_m = 0$ we again obtain nonwormhole exact solutions (42) of the Einstein-dilaton-scalar field equations.

Arguing similarly to the previous case, it is easy to see that all the above solutions are Jacobi stable.

We now turn to a discussion of the wormhole properties of the exact solutions found in this section. We require that the radicand of $\Lambda_{k|e}$, $\Lambda_{k|m}$ [Eq. (57)], $k = 1, 2, 3$, be nonzero. The sets $U_{k|e}$ of the roots of the equations $\Lambda_{k|e} = 0$ are $U_{1|e} = U_{2|e} = \{u_e\}$, $U_{3|e} = \{u_e + \pi N/h_e | N \in \mathbb{Z}\}$, and the sets $U_{k|m}$ of the roots of the equations $\Lambda_{k|m} = 0$ are $U_{1|m} = U_{2|m} = \{u_m\}$, $U_{3|m} = \{u_m + \pi N/h_m | N \in \mathbb{Z}\}$. For $Q_e Q_m \neq 0$ the domains of the functions $\ln[(\Lambda_{k|e}\Lambda_{j|m})^{-1/2}] = \beta(u)$ and $\psi_{k|e;j|m}$ [Eq. (58)] are $D_{k|e;j|m} \equiv \mathbb{R}\backslash(U_{k|e} \cup U_{j|m})$. Using Eq. (57) we find for the types $(j|e; 3|m)$, $j = 1, 2, 3$,

$$\beta'(u) = \begin{cases} -(1/4)((u-u_e)^{-1} + h_m \cot[h_m(u-u_m)]) & \text{for type } (1|e;3|m),\\ -(1/4)(h_e \coth[h_e(u-u_e)] + h_m \cot[h_m(u-u_m)]) & \text{for type } (2|e;3|m),\\ -(1/4)(h_e \cot[h_e(u-u_e)] + h_m \cot[h_m(u-u_m)]) & \text{for type } (3|e;3|m) \qquad (h_e > 0, h_m > 0). \end{cases} \tag{62}$$

We put $V_N^m \equiv (u_m + \pi N/h_m, u_m + \pi(N+1)/h_m)$ for $N \in \mathbb{Z}$. Suppose for $j = 1, 2$ we have $u_e \in V_{N_0}^m$ for some $N_0 \in \mathbb{Z}$. Then the domain $D_{j|e;3|m}$ of $\beta(u)$ is the set of the intervals $V_-^m \equiv (u_m + \pi N_0/h_m, u_e)$, $V_+^m \equiv (u_e, u_m + \pi(N_0 + 1)/h_m)$ and V_N^m for $N \neq N_0$: $D_{j|e;3|m} = V_-^m \cup V_+^m \cup_{N\neq N_0} V_N^m$. At the boundaries of each of the above intervals the function $\beta(u)$ and the circle radius $R(u) = e^{\beta(u)}$ tend to $+\infty$. It follows from Eq. (62) that $\beta''(u) > 0$ in the domain of $\beta(u)$, and therefore $\beta'(u)$ grows monotonically in each interval of $D_{j|e;3|m}$. Since the monotone continuous function $\beta'(u)$ tends to $-\infty$ at the left boundary of each interval of $D_{j|e;3|m}$ and tends to $+\infty$ at the right boundary of the interval, there exists a single point for each interval, $u_{0-} \in V^m_-$, $u_{0+} \in V^m_+$, $u_{0N} \in V^m_N$, $N \neq N_0$, at which $\beta'(u)$ vanishes and $\beta(u)$ has a minimum. According to Definitions 1 and 2 there is a throat in each interval of $D_{j|e;3|m}$, and the space-time "consists" of an infinite countable number of "universes" (58) with $u \in V^m_{N\in\mathbb{Z}}$, $N \neq N_0$ or $u \in V^m_-$ or $u \in V^m_+$; each universe has a throat. Similarly, when $u_e = u_m + \pi\tilde{N}/h_m$ for some $\tilde{N} \in \mathbb{Z}$ we obtain an infinite countable number of universes (58) with $u \in V^m_N$, $N \in \mathbb{Z}$. We denote these wormhole solutions by $\mathrm{WhCL}^{j|e;3|m}$, $j = 1, 2$. The solutions $\mathrm{WhCL}^{j|m;3|e}$, $j = 1, 2$, are similarly defined.

We put $V^e_N \equiv (u_e + \pi N/h_e, u_e + \pi(N+1)/h_e)$, $N \in \mathbb{Z}$. The domain $D_{3|e;3|m} = \cup_{N_1,N_2\in\mathbb{Z}}(V^m_{N_1} \cap V^e_{N_2})$ of the function $\beta(u)$ for the solution of type $(3|e; 3|m)$ is the infinite set of intervals (u^K_i, u^K_f), $K \in \mathbb{Z}$, where $u^K_i, u^K_f \in U_{3|e} \cup U_{3|m}$, $u^K_i < u^K_f$ and the interval (u^K_i, u^K_f) does not contain any element of $U_{3|e} \cup U_{3|m}$. It follows from Eq. (62) that $\beta(u)$ and the circle radius $R(u) = e^{\beta(u)}$ tend to $+\infty$ at the boundaries of each interval (u^K_i, u^K_f). As the monotonically increasing function $\beta'(u)$ tends to $-\infty$ when $u \to u^K_i + 0$ and tends to $+\infty$ when $u \to u^K_f - 0$, in each interval (u^K_i, u^K_f) there exists a single point u^K_0 at which $\beta'(u)$ vanishes and $\beta(u)$ has a minimum. As in the previous case the space-time splits into multiple universes (58) with $u \in (u^K_i, u^K_f)$, $K \in \mathbb{Z}$; each universe has a throat defined by the equation $u = u^K_0$. This wormhole solution is denoted by $\mathrm{WhCL}^{3|e;3|m}$.

Consider the solutions of the remaining types $(f|e; h|m)$, $f, h = 1, 2$. We have

$$\beta'(u) = \begin{cases} -(1/4)((u-u_e)^{-1} + (u-u_m)^{-1}) & \text{for type } (1|e;1|m), \\ -(1/4)((u-u_e)^{-1} + h_m \coth[h_m(u-u_m)]) & \text{for type } (1|e;2|m), \\ -(1/4)(h_e \coth[h_e(u-u_e)] + (u-u_m)^{-1}) & \text{for type } (2|e;1|m), \\ -(1/4)(h_e \coth[h_e(u-u_e)] + h_m \coth[h_m(u-u_m)]) & \text{for type } (2|e;2|m). \end{cases}$$

If $u_e = u_m$ then the domain of $\beta(u)$ is $(-\infty, u_e) \cup (u_e, +\infty)$. As the first derivative of the smooth function $\beta(u)$ does not vanish in the domain, it has no minimum and the solutions of the types $(f|e; h|m)$, $f, h = 1, 2$ with $u_e = u_m$ are nonwormholes. In the case $u_e \neq u_m$ we have $D_{f|e;h|m} = (-\infty, u_i) \cup (u_i, u_f) \cup (u_f, +\infty)$ where $u_i = \min\{u_e, u_m\}$ and $u_f = \max\{u_e, u_m\}$. The first derivative of the monotonically increasing function $\beta'(u) \in (-\infty, +\infty)$ vanishes at a single point $u_0 \in (u_i, u_f)$, and the hypersurface $u = u_0$ defines a throat of the wormhole denoted by $\mathrm{WhCL}^{f|e;h|m}$, $u_e \neq u_m$, $f, h = 1, 2$. The equation $\beta'(u) = 0$ can be solved analytically only for the wormhole $\mathrm{WhCL}^{1|e;1|m}$ with the throat $u_0 = (u_e + u_m)/2$.

In the case $Q_m \neq 0$, $Q_e = 0$ the domains $D_{j|m}$ of the function $\beta(u)$ for the solutions of the types $(j|m)$, $j = 1, 2, 3$ are $D_{1|m} = D_{2|m} = (-\infty, u_m) \cup (u_m, +\infty)$ and $D_{3|m} = \cup_{N\in\mathbb{Z}} V^m_N$. The first derivative of $\beta(u)$ is

$$\beta'(u) = \begin{cases} -(1/4)((u-u_m)^{-1} + 2c), & \text{for type } (1|m), \\ -(1/4)(h_m \coth[h_m(u-u_m)] + 2c), & \text{for type } (2|m), \\ -(1/4)(h_m \cot[h_m(u-u_m)] + 2c), & \text{for type } (3|m). \end{cases} \tag{63}$$

For type $(1|m)$ the throat $u = u_0 \equiv u_m - (2c)^{-1}$ exists when $c > 0$, $-\infty < u_0 < u_m$ or $c < 0$, $u_m < u_0 < +\infty$; the solution with $c = 0$ is a nonwormhole. Solutions of the type $(2|m)$ are wormholes with the throat $u = u_0 \equiv u_m - \mathrm{arcoth}(2c/h_m)$ only if $2c/h_m > 1$, $-\infty < u < u_m$ or $2c/h_m < -1$, $u_m < u < +\infty$; we have no wormholes when $|2c/h_m| \leq 1$. In the case $(3|m)$ we obtain an infinite countable number of the "universes" (61) with $u \in V^m_{N\in\mathbb{Z}}$, each containing a throat $u = u_0 \equiv u_m + h_m^{-1}(\pi N - \mathrm{arccot}(2c/h_m))$. We denote the above wormhole solutions by $\mathrm{WhCL}^{k|m}$, $k = 1, 2, 3$. The wormhole solutions $\mathrm{WhCL}^{k|e}$ (60) are similarly defined.

The purely dilatonic exact solutions (42) are nonwormholes. So the wormhole solutions do not exist if both electric and magnetic charges are equal to zero.

The formulas (58) in the case $Q_e Q_m \neq 0$, Eq. (60) in the case $Q_e \neq 0$, $Q_m = 0$ and Eq. (61) in the case $Q_m \neq 0$, $Q_e = 0$ define new cylindrically symmetric exact solutions of the EYMD equations (8)–(10) with longitudinal electric and magnetic fields as sources; these solutions determine wormholes of the types $\mathrm{WhCL}^{j|e;k|m}$, $\mathrm{WhCL}^{k|e}$ and $\mathrm{WhCL}^{k|m}$, $j, k = 1, 2, 3$. All of them are not asymptotically flat.

V. CONCLUSION

We studied static axially (cylindrically) symmetric solutions in $4+n$-dimensional Kaluza-Klein theory with n gauge fields in the case that the internal symmetry group is the maximal Abelian isometry group $\mathrm{U}(1)^n$. It was shown that, as in the case of spherical symmetry [6], the consistency of the Euler-Lagrange equations for pure Einstein-Poincare gravity in $4+n$ dimensions with a diagonal internal metric and cylindrically symmetric *Ansätze* for gauge, dilaton and internal scalar fields imposes constraints on the possible charge configurations of the solutions. Such solutions can exist only for configurations with no more than one nonzero electric (Q_e) and one nonzero magnetic (Q_m) charge. In fact, $4+n$-dimensional Kaluza-Klein theory is reduced to an effective six-dimensional Kaluza-Klein theory (6) with two Abelian gauge fields A_μ, B_μ, a dilaton field ψ and a scalar field χ. We solved the Einstein–Yang-Mills–dilaton equations (8)–(10) derived from the action (6) for the cylindrically symmetric space-time (2) and found the exact solutions of these equations in the cases of (cylindrically symmetric) radial and longitudinal magnetic and electric fields.

Following Definitions 1 and 2 [10] we investigated the wormole properties of the obtained solutions. In the case of radial fields we found three types of static cylindrically symmetric dilatonic wormholes: dyonic $\mathrm{WhCR}^{e;m}$ defined by Eqs. (33) and (34), WhCR^{e} [Eqs. (35)–(38)] with nonzero electric charge and WhCR^{m} [Eqs. (39)–(41)] with nonzero magnetic charge. Nine types of dyonic wormholes $\mathrm{WhCL}^{k|e;j|m}$, k, $j=1, 2, 3$, determined by Eq. (58) are found in the case of longitudinal gauge fields as well as the wormhole $\mathrm{WhCL}^{3|e}$ [Eq. (60)] with nonzero electric charge and the wormhole $\mathrm{WhCL}^{3|m}$ [Eq. (61)] with nonzero magnetic charge. All the solutions we found are Jacobi stable.

The purely dilatonic exact solutions (42) are nonwormholes. So wormhole solutions do not exist if both electric and magnetic charges are equal to zero. Hence, the wormhole properties of these solutions are generated only by electric and/or magnetic charges of static Abelian gauge fields.

All obtained wormhole solutions are asymptotically nonflat; this confirms the “no-go” statement [10] about the nonflat asymptotic behavior of a cylindrically symmetric wormhole in the case of an everywhere nonnegative energy density of matter, i.e. in the absence of ghost fields.

The detailed study of the structure of the new wormhole solutions obtained in this paper will be a task for the next paper.

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