

Power counting and perturbative one pion exchange in heavy meson molecules

M. Pavón Valderrama*

*Instituto de Física Corpuscular (IFIC), Centro Mixto CSIC—Universidad de Valencia,
Institutos de Investigación de Paterna, Apartado 22085, E-46071 Valencia, Spain*

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We discuss the possible power counting schemes that can be applied in the effective field theory description of heavy meson molecules, such as the $X(3872)$ or the recently discovered $Z_b(10610)$ and $Z_b(10650)$ states. We argue that the effect of coupled channels is suppressed by at least two orders in the effective field theory expansion, meaning that they can be safely ignored at lowest order. The role of the one pion exchange potential between the heavy mesons, and, in particular, the tensor force, is also analyzed. By using techniques developed in atomic physics for handling power-law singular potentials, which have been also successfully employed in nuclear physics, we determine the range of center-of-mass momenta for which the tensor piece of the one pion exchange potential is perturbative. In this momentum range, the one pion exchange potential can be considered a subleading order correction, leaving at lowest order a very simple effective field theory consisting only of contact-range interactions.

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I. INTRODUCTION

Heavy meson molecules are a long established theoretical prediction of hadronic physics [1–6]. The discovery of the $X(3872)$ by the Belle Collaboration [7], and the subsequent confirmation by CDF [8], D0 [9] and BABAR [10], has provided so far the strongest candidate for a bound state of heavy mesons. Owing to the closeness of the $X(3872)$ to the $D^{0*}\bar{D}^0$ threshold, an interpretation in terms of a bound state of these two mesons is both appealing and natural. In turn, the recent finding of the $Z_b(10610)$ and $Z_b(10650)$ resonances by Belle [11,12], which are located just a few MeV above the $B^*\bar{B}$ and $B^*\bar{B}^*$ thresholds, respectively, also calls for a molecular description. Moreover, from heavy quark spin symmetry [13,14] we should expect other low-lying states, the partners of the Z_b 's, in the bottom sector [15,16].

The shallow nature of the aforementioned candidates for molecular states, apart from being instrumental in their identification, also indicates the existence of a separation of scales between long- and short-range dynamics. The heavy mesons are far apart from each other and consequently are not able to resolve the details of the short-range interaction that may be ultimately responsible for binding them. We therefore expect that heavy meson molecules will be amenable to an effective field theory (EFT) description of their properties and decays [17–19].

In the EFT formalism, the long-range interaction between two heavy mesons (composed of a heavy and a light quark) is constrained by the low-energy symmetries of the system, in particular, chiral symmetry, which determines the pion exchange dynamics. In turn, the short-range interaction is mimicked by local, contact-range operators or counterterms. A second ingredient of the EFT description

is the existence of a power counting, that is, an organizational principle from which operators can be ordered from more to less relevant. From power counting we expect to be able to express operators and observables as a power series in terms of a small expansion parameter x_0 ,

$$O_{\text{EFT}} = \sum_{\nu=\nu_0}^{\nu_{\text{max}}} \hat{O}^{(\nu)} x_0^\nu + \mathcal{O}(x_0^{\nu_{\text{max}}+1}), \quad (1)$$

where ν_0 is the order at which the EFT expansion starts, ν_{max} the order at which we perform the calculation, and x_0 , the expansion parameter, can be written as the ratio

$$x_0 = \frac{Q}{\Lambda_0}, \quad (2)$$

with Q (Λ_0) the generic energy scale associated with the long- (short-)range physics. *A priori* power counting entails a remarkable advantage: the relative error of a calculation of order ν is known to be $x_0^{\nu+1}$. Of course, the realization of this promise of accuracy depends on the implementation details and on the unambiguous identification of the expansion parameter. This manuscript will try to deal with this problem.

At this point a natural question arises: how important are the pion exchanges in the description of the heavy meson molecules? Of course, having a bound state requires the nonperturbative treatment of a certain subset of the meson-antimeson interaction, but not necessarily the one pion exchange (OPE) potential. A nice illustration is provided by the $X(3872)$, in which the distance between the heavy meson and antimeson is so large ($\sqrt{\langle r^2 \rangle} \sim 10$ fm) that even pions may not be clearly distinguished. This means that we can use a contact-range, pionless EFT in which pions are already considered to be short-ranged [20,21]. In such a case we are left with a much simpler theory. Even though the hard scale is the pion mass, $\Lambda_0 = m_\pi$, which is an

*m.pavon.valderrama@ific.uv.es

extremely light scale in hadronic physics, the soft scale Q can be even lighter if the molecular state is close enough to threshold. In particular, Q can be identified with the wave number of the bound state, that is, $Q = \sqrt{2\mu_R B}$, where μ_R is the reduced mass of the two-particle system and B the binding energy. For the $D^{*0}\bar{D}^0$ mesons conforming to the $X(3872)$, Q is of the order of mere tens of MeV. Thus the EFT expansion is expected to converge fairly well, providing a motivation and explaining the success of contact-range descriptions of the $X(3872)$ [17,22–24].

However, the applicability of pionless EFT is subjected to limitations. In the $X(3872)$ we can list two: on the one hand, the pion exchanged between the D^{*0} and $\bar{D}^0$ meson is almost on the mass shell, and consequently its effects spread over a much larger range than expected. That is, the irrelevance of pion exchanges is not so evident as it appears at first sight. Luckily, as shown in X-EFT [19], pions are actually perturbative in the $X(3872)$. On the other, if we consider the charged $D^{*+}D^-$ component of the $X(3872)$, which is essential for explaining certain branching ratios [25,26], the associated soft scale is $Q \sim 125$ MeV, of the order of m_π . Thus the explicit inclusion of the charged channel may lie outside the range of applicability of a pionless EFT. If we now consider the bottom sector, the $Z_b(10610)$ and $Z_b(10650)$ resonances are a only few MeV away from the $B^*\bar{B}$ and $B^*\bar{B}^*$ threshold.¹ However, the large reduced mass of these systems implies that $Q \sim m_\pi$ corresponds to a binding energy of merely 4 MeV at which the wave function will start to probe the pions, requiring a nonperturbative theory [28].

In all the previous examples the inclusion of pion exchanges may be necessary for a proper EFT description of the molecular states. This can be rather cumbersome, owing to the rich angular momentum coupled-channel structure triggered by tensor forces, especially in the $D^*\bar{D}^*$ and $B^*\bar{B}^*$ cases. However, there exists a binding energy range in which the OPE potential is subleading with respect to the contact-range interactions and hence perturbative. In this energy window the pionful EFT will still be a contact-range theory at the lowest (or leading) order (LO), as pion exchanges will not enter until next-to-leading order (NLO). In the two-nucleon system the paradigmatic example of this kind of EFT is the Kaplan, Savage and Wise counting [29,30], which served as inspiration for X-EFT [19].

The question we want to answer is: where does the limit between perturbative and nonperturbative pions stand? If we take a second look to the deuteron in the two-nucleon system, which shares many similarities with the heavy

meson molecules, we see that the convergence of the EFT with perturbative pions is numerically marginal. As shown by Fleming, Mehen and Stewart (FMS) [31] by a thorough next-to-next-to-leading order (N²LO) calculation, the convergence of the Kaplan, Savage and Wise counting in the deuteron case is limited at most to $\Lambda_0 \sim 100$ MeV, a rather low figure (indeed smaller than m_π). Taking into account that in the deuteron the wave number is $\gamma = 45$ MeV, the previous breakdown scale translates into a rather slow convergence rate. In fact, the theory would not converge at all had the deuteron been bound by about 4 MeV or more. The purpose of this work is therefore to find the corresponding B_{max} below which pions are perturbative, as this limit will give us essential information about the convergence of the EFT for heavy meson molecules.

Of course, performing a N²LO calculation is beyond the scope of the present paper. Apart from that, the scarce experimental input available on heavy meson-antimeson systems makes the previous task impractical: we do not have the information required to fix the counterterms. Therefore, we need to resort to a more indirect path. In the two-nucleon system the solution was provided by Birse [32]. The idea is to consider the OPE in the chiral limit, in which it reduces to a pure power-law potential of the type $1/r^3$. The interesting thing here is that these kinds of potentials have been studied and analyzed in detail in the field of atomic physics. The long-range solutions of OPE in the chiral limit can be expressed in terms of a particular type of series, as shown by Cavagnero [33] and Gao [34]. Birse was able to show that one can extract the breakdown scale of a theory containing perturbative pions with the help of these techniques, and the results help to explain remarkably well the lengthier N²LO calculations of FMS [31]. In this work we will extend the observations of Birse to the peculiarities of the tensor force in heavy meson molecules.

The article is structured as follows: in Sec. II we present a brief overview of certain key EFT ideas that will help to put the results of this work into proper context. In Sec. III we will write the OPE and contact-range potentials between a heavy meson and antimeson, and in Sec. IV we will discuss the different power counting schemes that we can apply. In Sec. V we will extract the breakdown scale of perturbative pion theories, and finally, in Sec. VI we will discuss the results and their implications on the EFT treatment of heavy molecular states. We have also included an appendix containing the technical details involved in the derivation of the EFT potential at lowest order.

II. GENERAL CONSIDERATIONS

The purpose of this section is to provide a quick review of the EFT formalism for nonrelativistic two-body systems. The discussion is heavily based on the EFT formulation of the two-nucleon system (see Refs. [35–39] for

¹It is interesting to notice that even though the original experimental analysis by Belle [11,12] locates the two states above the $B^*\bar{B}$ and $B^*\bar{B}^*$ thresholds, this depends on the Breit-Wigner parametrization employed for the Z_b 's as stressed in Ref. [27]. This work also indicates that the two Z_b 's may be located below threshold and hence be bound states (instead of two-particle resonances) after all.

reviews), which can be trivially translated and applied to heavy meson molecules with minor modifications, as demonstrated in Ref. [28]. Thus, we begin with naive dimensional power counting, as originally proposed by Weinberg [40,41], and then explain the modifications that have been required to successfully formulate a nonrelativistic EFT to two-body systems forming shallow bound states [20,29,30,42,43]. As we are dealing with heavy meson molecules, we will include notation related to heavy quark symmetry. In what follows m_Q is the mass of the heavy quark conforming to a heavy meson, and $H = P, P^*$ is used to denote a generic heavy meson with orbital angular momentum $l = 0$ between the heavy quark and the light quark. The heavy meson $P (P^*)$ has total spin $s = 0$ ($s = 1$) and hence it is a pseudoscalar (vector) meson. If we are specifically dealing with the charm or bottom sector we will particularize the P, P^* heavy meson notation by D, D^* and B, B^* . We also use $D^{(*)}$ and $B^{(*)}$ as a generic for the pseudoscalar/vector cases. We will only consider the case in which the light quark is the u or d quark. The extension to the strange sector is straightforward.

A. Power counting

The formulation of EFT depends on the existence of a separation of scales: we can distinguish between Q , the low-energy scale that characterizes the physics we are interested in, and Λ_0 , the high-energy scale at which the effective description we are using stops being applicable. In the EFT framework, the two-body potential can be expanded as a power series on the ratio of these scales, leading to

$$V_{\text{EFT}} = \sum_{\nu=\nu_0}^{\nu_{\text{max}}} V^{(\nu)} + \mathcal{O}\left[\left(\frac{Q}{\Lambda_0}\right)^{\nu_{\text{max}}+1}\right], \quad (3)$$

where $\nu_0 \geq -1$ is the order at which the expansion begins and ν_{max} the order at which we perform the calculation. In chiral (nuclear [40,41] and heavy hadron [44]) EFT the generic scale Q usually includes the momenta of the two interacting particles and the mass of the pion, while Λ_0 refers to the mass of the rho meson or the momentum scale at which the internal structure of the two particles starts to be resolved.

As can be seen, we expect the theoretical error in the determination of the EFT potential to decrease, at least as long as $Q \leq \Lambda_0$. At this point we should take into account that the light scales include on the one hand the pion mass m_π , which does not change,² and on the other the momenta $Q \sim p, p'$ of the two heavy mesons, which can vary. This means that the expansion of the potential (and the scattering observables) is only valid for sufficiently small p, p' . To avoid the related divergences with taking $p, p' \geq \Lambda_0$ in loops we usually include a cutoff Λ that serves as an

²Unless we are considering chiral extrapolations.

intermediate scale between Q and Λ_0 (i.e., we take $Q \leq \Lambda \leq \Lambda_0$).

B. The scaling of operators

The power counting assignment of a certain operator tells us about its scaling properties. A contribution to the (momentum space) potential is assigned the order ν if it scales as

$$V^{(\nu)}(\lambda Q) = \lambda^\nu V^{(\nu)}(Q), \quad (4)$$

under the rescaling of all the light scales by a factor λ . This means that if we reduce the momentum or the pion mass by a factor $R = \frac{1}{\lambda}$, with $R \leq 1$, thus increasing the separation of scales by a factor of $\lambda = \frac{1}{R} \geq 1$, the size of the order ν contributions will decrease as R^ν . That is, such a contribution becomes smaller the larger the scale separation, justifying its power counting assignment.

Scaling is very interesting in the sense that it determines the behavior of the potential in coordinate space. If we consider the order ν contribution potential in momentum space, we simply have³

$$V^{(\nu)}(\lambda \vec{q}, \lambda Q) = \lambda^\nu V^{(\nu)}(\vec{q}, Q), \quad (5)$$

where we have now explicitly considered the dependence on the momentum exchanges between the two particles, $\vec{q} = \vec{p} - \vec{p}'$. After Fourier-transforming into coordinate space, the previous scaling translates to

$$V^{(\nu)}\left(\frac{\vec{r}}{\lambda}, \lambda Q\right) = \lambda^{3+\nu} V^{(\nu)}(\vec{r}, Q), \quad (6)$$

which admits two kind of general solutions, contact-range and finite-range. The contact-range solution is trivial to construct from the Dirac δ function and its derivatives, yielding

$$V_C^{(\nu)}(\vec{r}, Q) = C_\nu \partial^\nu \delta(\vec{r}), \quad (7)$$

where ∂ denotes a general derivative of the Dirac δ . Of course, parity constraints imply that this kind of contribution only appears at $\nu = 2n$. On the other hand, the finite-range solution must comply to the form

$$V^{(\nu)}(\vec{r}, Q) = \frac{F_\nu}{r^{3+\nu}} f^{(\nu)}(Qr), \quad (8)$$

with $f(x)$ an arbitrary (nonexclusively power-law) function, which decays exponentially at large distances as it stems from meson exchanges.

In the previous equations C_ν and F_ν are constants with dimensions of $1/[\text{energy}]^{\nu+2}$. From power counting we expect the related energy scale to be Λ_0 , that is,

³Notice that we have restricted ourselves to the local potential case. Nonlocalities only appear at high orders in the chiral expansion ($\nu = 4$ in the two-nucleon case) and in addition they are suppressed by the mass of the heavy mesons.

$$C_\nu \sim \frac{1}{\Lambda_0^{\nu+2}} \quad \text{and} \quad F_\nu \sim \frac{1}{\Lambda_0^{\nu+2}}. \quad (9)$$

However, as we will see in the following paragraphs, sometimes there is an unexpected contamination of C_ν and F_ν by a light scale. This will trigger a change in the power counting of the potential, promoting a certain contribution from a higher to a lower order. An illustration is given by the scaling arguments discussed in the introduction of Ref. [45].

C. Iteration

If we are considering scattering states, we should plug the potential into the Lippmann-Schwinger equation,

$$T = V + VG_0T, \quad (10)$$

where V is the EFT potential, T is the T matrix and $G_0 = 1/(E - H_0)$ the resolvent operator. Analogously, if we are considering heavy meson molecules we should iterate the potential in the bound state equation

$$|\Psi_B\rangle = G_0V|\Psi_B\rangle, \quad (11)$$

to obtain the binding energy and the wave function $|\Psi_B\rangle$.

Within the EFT framework the previous equations are expected to be reexpanded according to the power counting of the potential. In this way, we are guaranteed to be able to estimate the error properly. For this, we need to take into account that the scaling of the resolvent operator is given by

$$G_0 \sim \mu_R Q, \quad (12)$$

where μ_R is the reduced mass of the two-body system, which for a nonrelativistic theory can be considered a high energy scale, $\mu_R \sim \Lambda_0$. From the point of view of power counting, the scaling of G_0 means that only contributions to the EFT potential such that $\mathcal{O}(G_0V) \sim 1$ should be iterated, i.e., the order Q^{-1} piece of the potential.

The existence of shallow bound states between heavy mesons implies that there is a contribution to the potential of order Q^{-1} . However, when we compute the EFT potential from the chiral Lagrangian we only find contributions of order Q^0 and higher. The OPE potential is order Q^0 , as can be trivially checked from

$$V_{\text{OPE}}(\vec{q}) \propto \frac{q^2}{q^2 + m_\pi^2}, \quad (13)$$

and it is also clear that contact-range potentials must be at least of order Q^0 . This means that naive dimensional analysis is not enough to determine accurately the order of an operator if there is a bound state. What is missing is the unexpected contribution from a light energy scale, as has been discussed extensively in nuclear EFT [20,29,30,42,43]. A way to see this is to consider the coupling C_0 of the lowest order contact operator: if there is a bound state, C_0 cannot be perturbative, that is, small.

We can account for the fact that C_0 is big by multiplying the naive expectation for its size by a big number, say, Λ_0/Q , yielding

$$C_0(Q) \sim \frac{1}{Q\Lambda_0}, \quad (14)$$

from which it is obvious that $C_0(Q)$ is of order Q^{-1} , promoting the lowest order contact-range potential from order Q^0 to Q^{-1} . A different way is to solve the bound state equation with a contact operator, in which case we obtain that C_0 should scale as

$$C_0(Q) \sim \frac{1}{\gamma\Lambda_0}, \quad (15)$$

with $\gamma = \sqrt{2\mu_R B}$ the wave number of the heavy meson molecule, and B the binding energy. For a shallow bound state we have $\gamma \sim Q$ and the wave number is the light scale contaminating C_0 .

Moreover, there is a second mechanism that can lead to the promotion of the potential, in particular, OPE, to order Q^{-1} . It is based on a well-known argument of Ericson and Karl [6] about the relative strength of OPE in heavy meson molecules. The idea is that the intrinsic strength of the OPE potential, i.e., F_0 as defined in Eq. (9), is roughly independent on the heavy quark mass.⁴ As a consequence, the ratio

$$\frac{V^{(0)}G_0V^{(0)}}{V^{(0)}} \sim \frac{\mu_R}{\Lambda_0} \frac{Q}{\Lambda_0} \quad (16)$$

is only small for the naive identification $\mu_R \sim \Lambda_0$, for which the ratio above scales as expected if the OPE potential is a Q^0 operator. However, while the chiral hard scale Λ_0 is fixed, the reduced mass of the two heavy meson systems scales as m_Q in the heavy quark limit. Eventually we can have $\mu_R \gg \Lambda_0$, enhancing the pion loops to the point of making the previous ratio to be of order Q^0 . In such a case the OPE potential will become nonperturbative, as expected from Ref. [6].

D. Coupled channels

If we consider heavy meson molecules, we expect from heavy quark symmetry that the energy gap between the thresholds of the three different combinations of pseudo-scalar and vector mesons, that is,

$$\text{HH} = \text{PP}, \text{PP}^*, \text{P}^*\text{P}^*, \quad (17)$$

will shrink as $\Delta_Q \sim 1/m_Q$ [46,47]. This indicates, at first sight, the necessity of a coupled-channel approach. In such a case, the Lippmann-Schwinger equation can be written as

$$T_{AB} = V_{AB} + V_{AC}G_0T_{CB}, \quad (18)$$

⁴Actually, it is proportional to the axial pion coupling g^2 , which contains a contribution of order m_Q^0 plus corrections of order m_Q^{-1} , m_Q^{-2} and so on.

where $A, B, C = \text{PP}, \text{PP}^*, \text{P}^*\text{P}^*$. If we are considering the scattering in the vicinity of a given threshold α , then we set $A = B = \alpha$ in the Lippmann-Schwinger equation. In addition, we set the center-of-mass energy to zero $E = 0$ at the α threshold. This means that we must include a proper energy shift for the resolvent operator if G_0 is acting on a channel $C \neq \alpha$, that is,

$$G_0^{-1}(E)|\vec{q}, C\rangle = \left(E - \frac{q^2}{2\mu_C} - \Delta_C^\alpha\right)|\vec{q}, C\rangle, \quad (19)$$

where we have considered the inverse of the G_0 operator for simplicity. In the expression above, μ_C is the reduced mass of channel C and the energy shift is given by $\Delta_C^\alpha = M_\alpha - M_C$, with M_α and M_C the total mass of channels α and C , respectively.

However, the counting of the G_0 operator is not necessarily Q on a channel $C \neq \alpha$. If we consider the ratio of the G_0 operator evaluated at the C and α channels, respectively, we find that

$$\frac{G_{0,C}^\alpha(E)}{G_{0,\alpha}^\alpha(E)} = \frac{2\mu_A E - q^2}{2\mu_C(E - \Delta_C^\alpha) - q^2} \sim \left(\frac{Q}{\Lambda_C}\right)^2, \quad (20)$$

where $Q \sim \sqrt{2\mu_R E} \sim q$ (we are assuming similar reduced masses), and $\Lambda_C = \sqrt{2\mu_C \Delta_C^\alpha}$ is the momentum scale related to the coupled-channel effects. We can distinguish here two possible situations: (i) $\Lambda_C \ll \Lambda_0$ and (ii) $\Lambda_C \sim \Lambda_0$. If $\Lambda_C \ll \Lambda_0$, we can consider the coupled-channel scale to be light, $\Lambda_C \sim Q$, and the resolvent operator will be of order Q , as expected (a more formal account can be consulted in Refs. [48,49]). On the contrary, if $\Lambda_C \sim \Lambda_0$, then the resolvent operator for the channel $C \neq \alpha$ will be suppressed by two powers of Q/Λ_C , that is, by two orders in the EFT expansion.

In the particular case of heavy mesons, $\Lambda_C = \sqrt{2\mu_{\text{HH}} \Delta_Q}$, where μ_{HH} is the reduced mass of the system and Δ_Q is the energy split between different pairs of heavy meson systems. For the charm sector, $\Lambda_C \simeq 520$ MeV for the $\text{DD}-\text{DD}^*$ and $\text{DD}^*-\text{D}^*\text{D}^*$ pairs and $\Lambda_C \simeq 740$ MeV for $\text{DD}-\text{D}^*\text{D}^*$. For the bottom sector, the coupled-channel scales are very similar, yielding $\Lambda_C \simeq 490$ MeV and $\Lambda_C \simeq 700$ MeV, respectively. The previous estimations indicate that Λ_C is similar to the hard scale of the theory, $\Lambda_0 \sim 0.5\text{--}1.0$ GeV, which indicates that coupled-channel effects are suppressed.

A surprising aspect about the coupled-channel effects is that the associated momentum scale is similar in the charm and bottom sectors. This can be easily understood if we consider the heavy quark limit, $m_Q \rightarrow \infty$, in which we expect $\mu_{\text{HH}} \sim m_Q$ and $\Delta_Q \sim 1/m_Q$, meaning that $\Lambda_C \sim m_Q^0$. That is, the suppression of the coupled-channel effects is basically independent of the heavy quark mass. Even though in the heavy quark limit the heavy mesons are degenerate, the energy split between them remains beyond

the scope of a nonrelativistic chiral EFT. The convergence of the EFT description for nonrelativistic particles depends on momentum scales instead of energy scales. However, from the scaling of the center-of-mass energy with respect to the heavy quark mass, i.e., $E \sim p^2/m_Q$, we see that the energy window in which the EFT is valid decreases as the heavy quark mass grows.

III. THE LEADING ORDER POTENTIAL

The lowest order potential in the effective field theory description of heavy meson molecules can be decomposed into a contact-range and a finite-range piece

$$V_{\text{HH}}^{(0)} = V_C^{(0)} + V_F^{(0)}, \quad (21)$$

where the subscripts C and F are used to denote the contact- and finite-range character of each contribution to the potential. In the equation above we have labeled the potential by the expected naive dimensional scaling, rather than the actual scaling that depends upon the particular power counting under consideration. Of course, the existence of a shallow molecular scale will imply that at least V_C is of order Q^{-1} . However, we will not consider the actual order of the different contributions of the potential (and their consequences) until the next section.

A. The contact-range potential

At LO we will naively expect a total of six counterterms in a given isospin channel. This number corresponds to the number of s -wave channels for the $\text{H}\bar{\text{H}}$ system: a 0^{++} state for $\text{P}\bar{\text{P}}$, two opposite C -parity states, 1^{++} and 1^{+-} for the $\text{P}^*\bar{\text{P}}/\text{P}\bar{\text{P}}^*$ system, and the three, 0^{++} , 1^{+-} and 2^{++} , $\text{P}^*\bar{\text{P}}^*$ states. However, the contact range is constrained by the requirements of heavy quark spin symmetry (HQSS) [13,14], which in turn implies that the six counterterm figure is reduced to only two independent counterterms [18]. We will call these counterterms C_{0a} and C_{0b} . Ignoring the coupled-channel effects, the potential reads (see the Appendix for details)

$$V_{C,\text{P}\bar{\text{P}}}^{(0)}(\vec{q}, 0^{++}) = C_{0a}, \quad (22)$$

$$V_{C,\text{P}^*\bar{\text{P}}/\text{P}\bar{\text{P}}^*}^{(0)}(\vec{q}, 1^{+-}) = C_{0a} - C_{0b}, \quad (23)$$

$$V_{C,\text{P}^*\bar{\text{P}}/\text{P}\bar{\text{P}}^*}^{(0)}(\vec{q}, 1^{++}) = C_{0a} + C_{0b}, \quad (24)$$

$$V_{C,\text{P}^*\bar{\text{P}}^*}^{(0)}(\vec{q}, 0^{++}) = C_{0a} - 2C_{0b}, \quad (25)$$

$$V_{C,\text{P}^*\bar{\text{P}}^*}^{(0)}(\vec{q}, 1^{+-}) = C_{0a} - C_{0b}, \quad (26)$$

$$V_{C,\text{P}^*\bar{\text{P}}^*}^{(0)}(\vec{q}, 2^{++}) = C_{0a} + C_{0b}, \quad (27)$$

depending on the J^{PC} quantum number and the specific $\text{H}\bar{\text{H}}$ system under consideration. The explicit representation of

these two counterterms in the coupled-channel basis of $H\bar{H}$ states can be consulted in Ref. [16].

B. The finite-range potential

The OPE potential between a heavy meson and antimeson is local, and hence it only depends on the exchanged momentum between the heavy mesons:

$$\langle \vec{p}' | V_F^{(0)} | \vec{p} \rangle = V_F^{(0)}(\vec{q}), \quad (28)$$

with $\vec{q} = \vec{p} - \vec{p}'$. The explicit consideration of the pseudoscalar/vector heavy meson channels allows us to write the potential as

$$V_{F,PP \rightarrow PP}^{(0)}(\vec{q}) = 0, \quad (29)$$

$$V_{F,P^*P \rightarrow PP^*}^{(0)}(\vec{q}) = -\frac{g^2}{2f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{\epsilon}_1 \cdot \vec{q} \vec{\epsilon}_2^* \cdot \vec{q}}{q^2 + \mu^2}, \quad (30)$$

$$V_{F,P^*P^* \rightarrow P^*P^*}^{(0)}(\vec{q}) = -\frac{g^2}{2f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q}}{q^2 + m_\pi^2}, \quad (31)$$

depending on the particular heavy meson channel under consideration. In the equations above g is the pion axial coupling, $f_\pi = 132$ MeV the pion decay constant, $m_\pi = 138$ MeV the pion mass, $\vec{\tau}_{1(2)}$ the isospin operators on the heavy meson $1(2)$, $\vec{\epsilon}_{1(2)}$ is the heavy vector meson polarization, and $\vec{S}_{1(2)}$ is the spin operator for intrinsic spin $S = 1$. For the $P^*P \rightarrow PP^*$ channel, we use μ instead of m_π : as a consequence of the different masses of the P and P^* mesons, the pion is emitted with the zero component of the momentum different from zero. In the static limit this is equivalent to changing the effective mass of the pion. We have that $\mu^2 = m_\pi^2 - \Delta_Q^2$, with Δ_Q the mass splitting between the P and P^* mesons. In the heavy quark limit, $m_Q \rightarrow \infty$, we obtain $\mu = m_\pi + \mathcal{O}(1/m_Q)$, and it would be practical to expand the potential in powers of $1/m_Q$. While this is the situation we find in the bottom sector ($B\bar{B}^*$) the charm one (the $D\bar{D}^*$ potential) is more complicated as we have $\mu^2 < 0$. The OPE potential thus acquires a small imaginary piece that is related to the probability loss induced by the open decay channel $D\bar{D}^*/\bar{D}D^* \rightarrow D\bar{D}\pi$. In this regard it is not clear up to what extent the static approximation to the potential holds. The rigorous theoretical treatment of this situation requires the inclusion of pions as dynamical degrees of freedom and the explicit consideration of the $D\bar{D}\pi$ three-body channel [50], which indicates a significant impact of the three-body dynamics on the $X(3872) \rightarrow D\bar{D}\pi$ decay rate, but only a mild effect on the $D\bar{D}^*$ wave functions [that is, the residue of the $X(3872)$ pole in the language of Ref. [50]]. This seems to indicate that, if we are only interested in the wave functions, we can simply ignore the imaginary piece of the OPE potential for $\mu^2 < 0$ and continue using the static limit.

Apart from the previous, there are additional pieces of the potential that mix the different heavy meson channels. However, as we have seen, coupled-channel effects can be safely disregarded (even in the heavy quark limit) as their size is similar to the short-range effects beyond chiral symmetry. Therefore we have decided to ignore the particle coupled-channel terms.

The previous potential has been computed for the meson-meson case. For obtaining the meson-antimeson potential in the isospin symmetric basis, we perform a G -parity transformation: this changes the sign of the P^*P^* potential, but leaves the P^*P/PP^* potential unchanged. However, the $P^*\bar{P}/P\bar{P}^*$ potential is better written in a definite C -parity basis, for which we employ

$$|P^*\bar{P}(\eta)\rangle = \frac{1}{\sqrt{2}}[|P^*\bar{P}\rangle - \eta|P\bar{P}^*\rangle], \quad (32)$$

where η is the intrinsic C parity of the meson-antimeson system. This translates into a factor of $-\eta$ for the $P^*\bar{P}/P\bar{P}^*$ potential when expressed in this basis. The final form of the meson-antimeson potentials is thus

$$V_{F,P^*\bar{P}(\eta)}^{(0)}(\vec{q}) = \eta \frac{g^2}{2f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{\epsilon}_1 \cdot \vec{q} \vec{\epsilon}_2^* \cdot \vec{q}}{q^2 + \mu^2}, \quad (33)$$

$$V_{F,P^*\bar{P}^*}^{(0)}(\vec{q}) = \frac{g^2}{2f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q}}{q^2 + m_\pi^2}. \quad (34)$$

The general form of the potential can be schematically written as

$$V_{F,H\bar{H}}^{(0)}(\vec{q}) = \eta \frac{g^2}{2f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{a}_1 \cdot \vec{q} \vec{b}_2 \cdot \vec{q}}{q^2 + \mu^2}, \quad (35)$$

where $\vec{a}_1, \vec{b}_2$ represent the particular spin operators and μ the particular value of the effective pion mass to be used in each case. In this form, the intrinsic C parity η is then to be taken $\eta = 1$ for the $P^*\bar{P}^*$ channel.

The coordinate space potential can be obtained by Fourier-transforming the momentum space one, in which case we obtain

$$V_{F,H\bar{H}}^{(0)}(\vec{r}) = \eta \vec{\tau}_1 \cdot \vec{\tau}_2 \vec{a}_1 \cdot \vec{b}_2 \frac{g^2}{6f_\pi^2} \delta(\vec{r}) - \eta \vec{\tau}_2 \cdot \vec{\tau}_1 [\vec{a}_1 \cdot \vec{b}_2 W_C(r) + (3\vec{a}_1 \cdot \hat{r} \vec{b}_2 \cdot \hat{r} - \vec{a}_1 \cdot \vec{b}_2) W_T(r)], \quad (36)$$

where $W_C(r)$ and $W_T(r)$ are the central⁵ and tensor pieces of the potential, which read

$$W_C(r) = \frac{g^2 \mu^3}{24\pi f_\pi^2} \frac{e^{-\mu r}}{\mu r}, \quad (37)$$

$$W_T(r) = \frac{g^2 \mu^3}{24\pi f_\pi^2} \frac{e^{-\mu r}}{\mu r} \left(1 + \frac{3}{\mu r} + \frac{3}{\mu^2 r^2} \right). \quad (38)$$

⁵Or, more properly, the spin-spin part of the potential.

The interesting feature of the tensor force is that it can mix channels with different angular momentum.

C. The partial wave decomposition

In this subsection we will show the explicit partial wave decomposition of the potential. We will use the spectroscopic notation $^{2S+1}L_J$ for characterizing a certain partial wave with spin S , orbital angular momentum L and total angular momentum J . If two or more partial waves are coupled we will indicate it with a dash, for example, $^3S_1-^3D_1$, $^5P_2-^5F_2$ or $^1F_3-^5P_3-^5F_3-^5H_3$.

There are certain rules on how partial waves coupled to each other. On the one hand, the central operator, which we define as

$$C_{12}^{\text{PP}^*}(\hat{r}) = \vec{\epsilon}_1 \cdot \vec{\epsilon}_2^*, \quad (39)$$

$$C_{12}^{\text{P}^*\bar{\text{P}}^*}(\hat{r}) = \vec{S}_1 \cdot \vec{S}_2, \quad (40)$$

conserves parity, C parity, spin, orbital and total angular momentum. Consequently it does not mix partial waves. On the other hand, the tensor operator, given by

$$S_{12}^{\text{PP}^*}(\hat{r}) = 3\vec{\epsilon}_1 \cdot \hat{r}\vec{\epsilon}_2^* \cdot \hat{r} - \vec{\epsilon}_1 \cdot \vec{\epsilon}_2^*, \quad (41)$$

$$S_{12}^{\text{P}^*\bar{\text{P}}^*}(\hat{r}) = 3\vec{S}_1 \cdot \hat{r}\vec{S}_2 \cdot \hat{r} - \vec{S}_1 \cdot \vec{S}_2, \quad (42)$$

conserves parity, C parity and total angular momentum, but neither spin nor orbital angular momentum. However, parity and C -parity conservation imply that the tensor force can only change the spin S and orbital angular momentum L by an even number of units. This in turn implies that we have four possibilities, which we define as follows:

$$^3U_J \equiv ^3J_J, \quad (43)$$

$$^3C_J \equiv ^3(J-1)_J - ^3(J+1)_J, \quad (44)$$

$$^5C_J \equiv ^5(J-1)_J - ^5(J+1)_J, \quad (45)$$

$$^{1-5}C_J \equiv ^1J_J - ^5(J-2)_J - ^5J_J - ^5(J+2)_J, \quad (46)$$

where $(U)C$ stands for (un)coupled. In this notation, 3C_1 is $^3S_1-^3D_1$, 5C_2 stands for $^5P_2-^5F_2$ and $^{1-5}C_3$ would be $^1F_3-^5P_3-^5F_3-^5H_3$. Of course, low J coupled channels can be uncoupled or have fewer members than expected. Examples are the $^3P_0(^3C_0)$ or the $^1S_0-^5D_0(^{1-5}C_0)$ partial waves.

The matrix elements of the central and tensor operators between different $^{2S+1}L_J$ partial wave are schematically calculated as

$$\begin{aligned} \langle b|\hat{O}_{12}|a\rangle &= \delta_{J_b J_a} \delta_{M_b M_a} \sum_{\{\mu\}} \int d^2\hat{r} \langle \mu_b | \hat{O}_{12}(\hat{r}) | \mu_a \rangle \\ &= \delta_{J_b J_a} \delta_{M_b M_a} O_{J(S_b S_a L_b L_a)}, \end{aligned} \quad (47)$$

where a, b are the initial and final partial wave, characterized by the vector $|(S_{a(b)}L_{a(b)})J_{a(b)}M_{a(b)}\rangle$, and $\{\mu\}$ is whatever internal angular momentum quantum numbers a and b have. The two Kronecker δ 's are a consequence of total angular momentum conservation. In the case of the central operator, which also conserves S and L , we simply have

$$C_{J(S_b S_a L_b L_a)} = C_{J(SL)} \delta_{S_b S_a} \delta_{L_b L_a}, \quad (48)$$

with

$$C_{J(1L)}^{\text{PP}^*} = 1, \quad (49)$$

$$C_{J(SL)}^{\text{P}^*\bar{\text{P}}^*} = \frac{1}{2}[S(S+1) - 4], \quad (50)$$

in the PP^* and $\text{P}^*\bar{\text{P}}^*$ cases, respectively.

We write the matrix elements of the tensor operator directly in one of the four coupled-channel bases previously defined. For the uncoupled-channel case we have

$$S_J^{\text{PP}^*}(^3U_J) = -1, \quad (51)$$

$$S_J^{\text{P}^*\bar{\text{P}}^*}(^3U_J) = +1. \quad (52)$$

In the 3C_J case we obtain

$$S_{J=0}^{\text{PP}^*}(^3C_J) = -2, \quad (53)$$

$$S_{J=0}^{\text{P}^*\bar{\text{P}}^*}(^3C_J) = +2, \quad (54)$$

$$S_{J \geq 1}^{\text{PP}^*}(^3C_J) = + \frac{1}{2J+1} \begin{pmatrix} J-1 & -3\sqrt{J(J+1)} \\ -3\sqrt{J(J+1)} & J+2 \end{pmatrix}, \quad (55)$$

$$S_{J \geq 1}^{\text{P}^*\bar{\text{P}}^*}(^3C_J) = - \frac{1}{2J+1} \begin{pmatrix} J-1 & -3\sqrt{J(J+1)} \\ -3\sqrt{J(J+1)} & J+2 \end{pmatrix}. \quad (56)$$

The other two sets of coupled channels, 5C_J and $^{1-5}C_J$, only happen for the $\text{P}^*\bar{\text{P}}^*$ system. In the 5C_J case, we obtain the matrices

$$S_{J=1}^{\text{P}^*\bar{\text{P}}^*}(^5C) = -1, \quad (57)$$

$$S_{J \geq 2}^{\text{P}^*\bar{\text{P}}^*}(^5C) = \frac{1}{2J+1} \begin{pmatrix} J+5 & 3\sqrt{J(J+1)-2} \\ 3\sqrt{J(J+1)-2} & J-4 \end{pmatrix}, \quad (58)$$

where there is no $J=0$ partial wave with fits into the 5C_J scheme (the 5P_0 wave is unphysical). In the $^{1-5}C_J$ case, we obtain

$$\mathbf{S}_{J=0}^{P^* \bar{P}^*} ({}^{1-5}C) = \begin{pmatrix} 0 & -\sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix}, \quad (59)$$

$$\mathbf{S}_{J=1}^{P^* \bar{P}^*} ({}^{1-5}C) = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & -\sqrt{\frac{6}{5}} \\ \frac{2}{\sqrt{5}} & -\frac{7}{5} & \frac{\sqrt{6}}{5} \\ -\sqrt{\frac{6}{5}} & \frac{\sqrt{6}}{5} & -\frac{8}{5} \end{pmatrix}, \quad (60)$$

$$\mathbf{S}_{J \geq 2}^{P^* \bar{P}^*} ({}^{1-5}C) = \begin{pmatrix} 0 & -\sqrt{\frac{3J(J-1)}{(2J+1)(2J-1)}} & \sqrt{\frac{2J(J+1)}{(2J+3)(2J-1)}} & -\sqrt{\frac{3(J+1)(J+2)}{(2J+3)(2J+1)}} \\ -\sqrt{\frac{3J(J-1)}{(2J+1)(2J-1)}} & -\frac{2J-4}{2J-1} & \sqrt{\frac{6(J+1)(J-1)(2J+3)}{(2J+1)(2J-1)^2}} & 0 \\ \sqrt{\frac{2J(J+1)}{(2J+3)(2J-1)}} & \sqrt{\frac{6(J-1)(J+1)(2J+3)}{(2J+1)(2J-1)^2}} & \frac{(2J+5)(2J-3)}{(2J+3)(2J-1)} & \sqrt{\frac{6J(J+2)(2J-1)}{(2J+3)^2(2J+1)}} \\ -\sqrt{\frac{3(J+1)(J+2)}{(2J+3)(2J+1)}} & 0 & \sqrt{\frac{6J(J+2)(2J-1)}{(2J+3)^2(2J+1)}} & -\frac{2J+6}{2J+3} \end{pmatrix}. \quad (61)$$

For $J = 0, 1$ the number of angular momentum channels is smaller than expected and the corresponding tensor matrices are written in the 1S_0 - 5D_0 basis for $J = 0$ and 1P_1 - 5P_1 - 5F_1 for $J = 1$.

IV. THE POWER COUNTING MAP FOR HEAVY MESON MOLECULES

The presence of a bound state between a heavy meson and antimeson indicates the necessity of promoting a piece of the EFT potential from order Q^0 to Q^{-1} . If the molecular state is shallow, as happens in the $X(3872)$ or the $Z_b(10610)$ and $Z_b(10650)$, probably it is enough to promote a contact-range interaction only. However, if the state is sufficiently deep (exactly how deep will be the subject of discussion of the next section), we are required to additionally promote the OPE potential as well. In this second case, the renormalizability of the theory will in turn generate important changes in the counterterm structure already at LO, as has been repeatedly discussed in the EFT description of nuclear forces [32,51–55]. Therefore we distinguish two cases: power counting with perturbative and nonperturbative pions.

A. Counting with perturbative pions

We begin by considering a theory in which there is at least a Q^{-1} contact operator but where pions are perturbative. In this case, pions represent a NLO correction. As there are two contact operators at LO, C_{0a} and C_{0b} , we have three options in what pertains to operator promotions, defining three power counting schemes:

- (a) C_{0a} is of order Q^{-1} ,
- (b) C_{0b} is of order Q^{-1} ,
- (c) both C_{0a} and C_{0b} are of order Q^{-1} .

Power countings (a), (b) and (c) are the EFT restatement of the observation made by Voloshin [15] about the heavy

quark spin structure of the Z_b resonances: depending on which piece of the heavy quark spin symmetric interaction is responsible for the appearance of the two low-lying Z_b states, we should expect a total of four or six molecular s -wave states of $B^{(*)}\bar{B}^{(*)}$. The latter situation, six states, corresponds to the promotion (a), while the former, giving four states, with (b)⁶ and (c). It is worth commenting that the third possibility (c) is quite general and can also accommodate other situations. For example, if $C_{0a} < 0$, $C_{0b} > 0$ and $C_{0a} + C_{0b}$ is negative and attractive enough as to bind, then there will be six states, but at most only two of them will be shallow (the 1^{++} and 2^{++}). However, this specific situation does not correspond to what we encounter in the $Z_b(10610)$ and $Z_b(10650)$ resonances.

Power countings (a) and (b/c) also differ with respect to the treatment of coupled-channel effects. The contact-range interaction responsible for mixing different particle channels is C_{0b} . In contrast, C_{0a} is diagonal in the $H\bar{H}$ space. This means that in the (b/c) counting the coupled channels effects are promoted by one order in the EFT expansion, and are therefore $\mathcal{O}(Q^1)$, instead of the naive estimate $\mathcal{O}(Q^2)$. In turn, the coupled-channel effects dependent on the OPE interaction will still be of order Q^2 .

In what regards the subleading contact operators, C_{2n} , their scaling is affected by the corresponding scaling of C_0 . By this we mean that if C_0 is $Q^{-1}(Q^0)$, then C_{2n} is $Q^{2n-2}(Q^{2n})$, as can be deduced from renormalization group analysis arguments [43]. Of course, only the C_{2n} operators with a similar HQSS structure as their Q^{-1} counterparts will be promoted, depending on the whether we are considering the (a), (b) or (c) countings. However,

⁶Strictly speaking in (b) we should only expect three low-lying states, the two 1^{+-} states and the $P^*\bar{P}^* 0^{++}$ state. However, if we recover the coupled-channel effects, a fourth $P\bar{P} 0^{++}$ state may be expected. In this regard, it is more natural to use the promotion (c) to obtain the four states.

the point is that, independently of the kind of promotion, we will have new free parameters at NLO. This also implies that without new data to fix the C_2 's we cannot proceed to NLO. Finally, the expected relative accuracy of a LO calculation is of order Q/Λ_0 .

There is still one contribution that we have not addressed: the effect of heavy quarkonia states with the adequate quantum numbers in the vicinity of any of the $H\bar{H}$ thresholds. This situation may happen in the $X(3872)$ if the $\chi_{c1}(2S)$ lies nearby, as has been discussed in Ref. [56]. However, the coupling to heavy quarkonia is surely going to represent a subleading contribution only. The natural expectation is that the counterterm mixing these two states is of order Q^0 . Therefore the coupled-channel effects will be of order Q^0 and hence subleading if the thresholds of the $X(3872)$ and the $\chi_{c1}(2S)$ are very close, or order Q^2 otherwise. This type of contribution, being subleading, does not really affect the LO calculation and its accuracy.

B. Counting with nonperturbative pions

As we will see in the next section, pion exchanges can eventually become nonperturbative above a certain critical energy in heavy meson-antimeson systems. Interestingly, the nonperturbative treatment of pion exchange will be able to significantly alter the power counting scheme, as happens in the two-nucleon system [32,51–55]. We notice that Ref. [28] already develops an EFT description of heavy meson molecules with nonperturbative OPE. However, the previous work only considers the particular case of the $P^*\bar{P}/P\bar{P}^*$ channels. In the following we will concentrate instead on the general HQSS structures relating the different $H\bar{H}$ particle channels.

We must begin with distinguishing which piece of the OPE potential is to be iterated, the central or the tensor. If only the central pieces are nonperturbative, the changes in the power counting are minimal with respect to the theories in which we already iterate the contact-range operators, as demonstrated by Barford and Birse [57]. The reason is that the $1/r$ singularity of central OPE is not enough to change the power-law behavior of the wave function at short distances, which is intimately related to the scaling of the counterterms (see, for example, the detailed discussion of Ref. [53]). In such a case, we simply refer to the results of the previous section.

On the contrary, the nonperturbative treatment of tensor OPE entails significant changes in the scaling of the contact-range operators, as has been extensively discussed in the context of nuclear EFT [32,51–55,58,59]. The reason is that tensor OPE is a singular potential behaving as $1/r^3$ at distances below the pion Compton wavelength. The most evident effect of a singular potential is on the LO counterterms: if the singular potential is attractive, the renormalizability of the theory requires the inclusion of a contact interaction to stabilize the cutoff dependence [58,59]. This can be appreciated if we consider the solution

of the reduced Schrödinger equation for a $-1/r^3$ potential, which for short enough distances reads [58,59]

$$u_k(r) \rightarrow \left(\frac{r}{a_3}\right)^{3/4} \sin\left[\frac{a_3}{r} + \varphi\right]. \quad (62)$$

The problem with the wave function above is that it is always regular at the origin, no matter what the values of the semiclassical phase φ is. If we try to regularize the potential by cutting it off at short distances, i.e., $V(r; r_c) = V(r)\theta(r - r_c)$, we do obtain a value of $\varphi(r_c)$. However, there is no well-defined $r_c \rightarrow 0$ limit for $\varphi(r_c)$: the semiclassical phase simply oscillates faster and faster on its way to the origin. The solution is to make the counterterm oscillate [60],

$$\frac{\mu_R C_0(r_c)}{2\pi r_c^2} \rightarrow -\frac{2}{a_3} \left(\frac{r}{a_3}\right)^{3/4} \cot\left[\frac{a_3}{r} + \varphi\right], \quad (63)$$

so φ remains a constant. In contrast, if the potential is repulsive, we have a unique and regular solution at short distances,

$$u_k(r) \rightarrow \left(\frac{r}{a_3}\right)^{3/4} \exp\left[-\frac{a_3}{r}\right], \quad (64)$$

and no counterterm is required [58,59].

The (angular momentum) coupled-channel case can be a bit more complicated however. In this case the important factor is the eigenvalues of the potential matrix in coupled-channel space: if there are n attractive (i.e., negative) eigenvalues, we will need $n(n+1)/2$ counterterms [58,59]. For the tensor matrices we have calculated in the previous section, the eigenvalues are

$$\lambda_{P\bar{P}}(0^{++}) = 0, \quad (65)$$

$$\lambda_{P\bar{P}^*}(1^{++}) = \tau\{1, -2\}, \quad (66)$$

$$\lambda_{P\bar{P}^*}(1^{+-}) = \tau\{2, -1\}, \quad (67)$$

$$\lambda_{P^*\bar{P}^*}(0^{++}) = \tau\{1 + \sqrt{3}, 1 - \sqrt{3}\}, \quad (68)$$

$$\lambda_{P^*\bar{P}^*}(1^{+-}) = \tau\{2, -1\}, \quad (69)$$

$$\lambda_{P^*\bar{P}^*}(2^{++}) = \tau\{1, -2, 1 + \sqrt{3}, 1 - \sqrt{3}\}, \quad (70)$$

with $\tau = \vec{\tau}_1 \cdot \vec{\tau}_2$; that is, we have included the $-\eta \vec{\tau}_1 \cdot \vec{\tau}_2$ factor multiplying the potential of Eq. (36) for obtaining the overall sign right. From the eigenvalues above it is apparent that the C_{0a} and C_{0b} counterterms are not enough as to renormalize the scattering amplitude if tensor OPE is nonperturbative.

Each distinct negative eigenvalue requires a different, independent counterterm. However, if two different channels share a negative eigenvalue, a common counterterm will be able to renormalize the two channels [60]. In principle we can count a total of eight counterterms for

renormalizing the six s -wave states (one per channel, with the exception of the 2^{++} that requires three counterterms). This figure is then reduced by counting the shared eigenvalues, that is, (i) the $1^{+-} P^* \bar{P}$ and $P^* \bar{P}^*$ molecules can be renormalized with the same counterterm, (ii) the 0^{++} and $2^{++} P^* \bar{P}^*$ share another eigenvalue and finally (iii) another common contact operator between the 1^{++} and 2^{++} channels. In total we end up with five independent counterterms.

All this makes the theory with nonperturbative pions rather cumbersome. In addition, with five contact interactions HQSS, although still there, is not so manifest as in the pionless theory.⁷ It is interesting to notice though that the $1^{+-} P^* \bar{P}$ and $P^* \bar{P}^*$ states are still expected to be degenerate: this situation corresponds to the quantum numbers of the $Z_b(10610)$ and $Z_b(10650)$ resonances. Luckily, as we will see in the next section, nonperturbative tensor forces are only expected in the isoscalar $B^{(*)} \bar{B}^{(*)}$ states.

However, the nonperturbative OPE counting has certain advantages too. The momentum dependent C_2 operators are not expected until order $Q^{3/2}$ at least (or even Q^2 : this is still an unresolved issue in nuclear EFT [53]). The lowest subleading order corrections are expected to happen at order Q^1 : they are subleading contributions to the C_0 operators that are needed for absorbing the expected divergences in (particle) coupled channels, owing to the $1/r^3$ singularity of the tensor force. Thus the expected relative error of a LO calculation is now $(Q/\Lambda_0)^2$, representing an improvement over the (Q/Λ_0) error in the perturbative pion case.

V. THE PERTURBATIVE TREATMENT OF ONE PION EXCHANGE

A. Central one pion exchange

The perturbative character of the central piece of the OPE potential can be determined by a direct comparison of the diagrams of Fig. 1. This corresponds to computing the ratio of the matrix elements of the operators $V^{(0)} G_0 V^{(0)}$ and $V^{(0)}$, which in terms of power counting is expected to scale as

$$\frac{\langle p | V^{(0)} G_0 V^{(0)} | p \rangle}{\langle p | V^{(0)} | p \rangle} \sim \frac{Q}{\Lambda_C}, \quad (71)$$

indicating the breakdown scale of a theory with central perturbative pions, where it is understood that we only consider the central piece of $V^{(0)}$. The evaluation is trivial

⁷What is actually happening is that certain s - to d -wave operators are being promoted from Q^2 to Q^{-1} . However, this may require the additional promotion of all (s - to s -wave) counterterms with two derivatives. Moreover, nonperturbative pions can change the bound state spectrum considerably from the expectations based on the HQSS of the potential; see Ref. [61] for an explicit example. A further issue, which we do not discuss in this work, is the possibility to go beyond the additivity of the long- and short-range forces; see footnote 6 of Ref. [62].

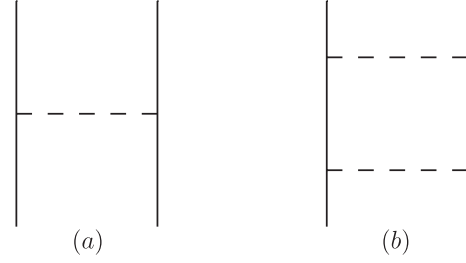


FIG. 1. Feynman diagrams corresponding to (a) the OPE potential between two heavy mesons and (b) the first iteration of the OPE potential. The ratio of diagram (b) over (a) can be used to determine the energy range in which OPE is perturbative.

for s waves and $p = 0$, for which the generic low-energy scale Q can only be identified with the pion mass m_π , leading to [31]

$$\Lambda_C = \frac{1}{|\sigma\tau|} \frac{24\pi f_\pi^2}{\mu_{\text{H}\bar{\text{H}}} g^2} \quad (72)$$

where $\sigma = \frac{1}{2}S(S+1) - 2$ and $\tau = 2I(I+1) - 3$ with S and I the total spin and isospin of the system. In general, this number is quite large for isovector channels ($\Lambda_C \gtrsim 1$ GeV), so the central piece of the OPE potential can always be treated as a perturbation in these cases. In contrast, for the $S = 0$ isoscalar case, corresponding to a $I^G(J^{PC}) = 0^+(0^{++}) P^* \bar{P}^*$ molecule, the central piece can be quite important, but not so much as the tensor piece, as we will see later.

B. Tensor one pion exchange

The determination of the breakdown scale for a power counting in which tensor OPE is order Q^0 and hence perturbative is not trivial. The reason is that the argument of comparing the relative size of diagrams does not work when considering the tensor piece of OPE. In the first place, tensor OPE does not directly act on s waves, but only indirectly owing to transitions to intermediate d -wave states. This means that we should consider second and third order perturbation theory to obtain the breakdown scale

$$\frac{\langle V_{sd}^{(0)} G_0 V_{dd}^{(0)} G_0 V_{ds}^{(0)} \rangle}{\langle V_{sd}^{(0)} G_0 V_{ds}^{(0)} \rangle} \sim \frac{Q}{\Lambda_T}, \quad (73)$$

where by $V^{(0)}$ we refer to the tensor piece only. However, there is a serious problem at this point: in the equation above the numerator and denominator are divergent. The reason lies in the $1/r^3$ divergent behavior of the tensor force at distances below the Compton wavelength of the pion, which induces a linear (quadratic) divergence in the denominator (numerator); see, for example, Ref. [63] for the details of this kind of calculation. From the power counting point of view, $V^{(0)} G_0 V^{(0)}$ requires the inclusion of the order Q correction to the C_0 counterterm, while $V^{(0)} G_0 V^{(0)} G_0 V^{(0)}$ involves in addition two new order Q^2

counterterms, a purely s -wave one and another one connecting the s - and d -wave channels for the result to be properly renormalized.⁸ The divergence spoils the evaluation of the previous ratio. If we include counterterms, the ratio is again finite, but is also contaminated by whatever we assume to be the hard scale of the counterterms. In principle it is still possible to find the value of Λ_T *a posteriori* by means of a full Q^2 calculation as in the nuclear case [31], but that requires plenty of experimental information about the low-energy scattering of heavy mesons to fix the counterterms. Therefore we need to resort to other kind of arguments.

The failure of standard perturbation theory for inverse power-law potentials ($1/r^n$) is well-known in atomic physics, where techniques have been developed to handle this type of potential. Of particular interest in this regard is the work of Cavagnero [33], who pointed out that the divergences of the perturbative series are analogous to the role of *secular* perturbations in classical mechanics, that is, small perturbations that however end up diverging at large enough time scales. The solution is to formulate a secular series in which we redefine (or renormalize) some quantity, in this case the angular momentum, in order to obtain finite results at arbitrary order. By this we specifically mean that the zeroth order perturbative approximation to the l -wave radial wave function is taken to be

$$\Psi_l(r; k) = \frac{J_{\nu(ka)}(kr)}{\sqrt{r}} + \mathcal{O}(g), \quad (74)$$

instead of the customary solution $r^{-1/2}J_{l+(1/2)}(kr)$, where $\nu(ka)$ is the renormalized angular momentum, k the center-of-mass momentum, a is the coupling constant and $J_n(x)$ the Bessel function of order n . The interesting feature is that at low energies the secular expansion can be reinterpreted as a particular resummation of the perturbative expansion, i.e., the renormalized angular momentum $\nu(ka)$ can be expanded as a power series in a . This observation, if translated to the EFT potential between heavy mesons, indicates the existence of an energy range for which the inverse power-law potential can be treated perturbatively at long distances.⁹

For the particular inverse power-law potential of interest in this work, the $1/r^3$ potential (corresponding to the form of the tensor force for $m_\pi r < 1$), the secular series for the wave function have been studied and analyzed in detail by Gao [34] for the uncoupled-channel case. These techniques were extended by Birse [32] to the coupled-channel case,

with a particular emphasis on the applications to nucleon-nucleon scattering. In the present work, we particularize the results of Birse [32] to the specific type of coupled channels appearing in two heavy meson systems.

In agreement with EFT expectations, the secular expansion of the wave function converges fast at long distances/low energies. Meanwhile, at high enough energies a very interesting thing happens: the secular expansion becomes a nonconverging series in powers of the coupling constant above a certain critical value of the center-of-mass momentum.¹⁰ That is, the secular expansion can no longer be considered as a perturbative series in disguise. From the EFT viewpoint this critical momentum is to be identified with the hard scale of the theory: if we map the secular series of the wave function at low energies onto the EFT expansion

$$\Psi_{\text{HH}} = \sum_{\nu=-1}^{\infty} \hat{\Psi}_{\text{HH}}^{(\nu)} \left(\frac{Q}{\Lambda_T} \right)^\nu, \quad (75)$$

it is apparent that the mapping will stop being valid at $Q = \Lambda_T$, where Q is to be understood as the momentum (as the pion mass is fixed). This means that Λ_T is the tensor breakdown scale we were looking for. Above this critical value of the momentum the EFT description of the two heavy meson system will require the full iteration of the $1/r^3$ tensor force.

In the following we will consider the OPE potential in the $\mu \rightarrow 0$ limit, for which it simplifies to a pure power-law potential of the type $1/r^3$:

$$2\mu_{\text{HH}} \mathbf{V}_{\text{F}}^{(0)}(r) = \frac{a_3}{r^3} \mathbf{S}_j, \quad (76)$$

where $\mathbf{S}_j$ is the tensor force matrix, a_3 is a length scale related to the strength of the tensor force, and the subscript HH denotes any combination of heavy meson and antimeson for which the OPE potential is nonzero. As can be appreciated, only the tensor piece of the OPE potential survives in this limit. Even though $\mu = 0$ can be physically interpreted as taking the chiral and heavy quark limit simultaneously ($m_\pi \rightarrow 0$ and $m_Q \rightarrow \infty$), here we simply consider the $\mu \rightarrow 0$ limit as a mathematical convenience that allows us to directly apply the results of Refs. [32,34]. Deviations from $\mu = 0$ will in general raise the value of the critical momentum, as they will reduce the strength of the potential at distances $\mu r \geq 1$ and therefore increase the perturbative character of the interaction at long distances. Analogously, the finite size of the heavy mesons is expected to weaken the potential at short distances. In this regard, the $\mu = 0$ (plus $\Lambda_0 = \infty$) determination is merely a lower bound: the real critical momenta will be higher. We will estimate the finite pion mass corrections to the critical momenta at the end of this section.

⁸Of course, this is assuming that the leading counterterm appears at order Q^0 . If we are considering instead a two heavy meson system with a shallow bound state, then the leading counterterm is order Q^{-1} and the C_2/C_4 of order Q^0/Q^2 .

⁹In this statement we are implicitly assuming the presence of exponential suppression (e^{-mr}) for standard perturbation theory to make sense at long distances and of suitable counterterms to absorb the divergences we commented on previously.

¹⁰Specifically $\nu(x)$ encounters a nonanalyticity in $x = ka$.

Curiously there is a case for which the condition $\mu = 0$ is almost fulfilled: the $X(3872)$, in which m_π and Δ_Q are very similar. In such a case the critical momenta calculated in this work will really represent the actual boundary to the perturbative treatment of tensor OPE.

1. The uncoupled-channel case

We start with the uncoupled-channel case, for which the tensor operator $\mathbf{S}_j = S_j$ is simply a number (instead of a matrix in the coupled-channel basis). In this case, the reduced Schrödinger equation reads

$$-u''_{k,j} + \left[S_j \frac{a_3}{r^3} + \frac{l_j(l_j + 1)}{r^2} \right] u_{k,j}(r) = k^2 u_{k,j}(r), \quad (77)$$

where $u_{k,j}$ is the reduced wave function, l_j is the angular momentum of the two-body system, which for the uncoupled-channel case coincides with the total angular momentum of the system ($l_j = j$), and k is the center-of-mass momentum. We have $|S_j| = 1$ in all the $\text{H}\bar{\text{H}}$ uncoupled channels, with the exception of the 3P_0 partial wave for which $|S_j| = 2$.

The general solution of this equation (for $k \neq 0$ ¹¹) can be written as the linear combination

$$u_{k,j}(r) = \alpha \xi_j(r; k) + \beta \eta_j(r; k), \quad (78)$$

where $\xi_{k,j}(r)$ and $\eta_{k,j}(r)$ are expressible as series in terms of Bessel functions

$$\xi_j(r; k) = \sum_{m=-\infty}^{+\infty} b_m \sqrt{r} J_{m+\nu}(kr), \quad (79)$$

$$\eta_j(r; k) = \sum_{m=-\infty}^{+\infty} (-1)^m b_m \sqrt{r} J_{-m-\nu}(kr). \quad (80)$$

The shift ν is a function of the dimensionless parameter $\kappa = ka_3$. The b_m coefficients can be expressed in terms of a recurrence relation that admits an explicit solution [34].

The dependence of the renormalized angular momentum ν in terms of κ is determined by the condition that ν is the zero of a characteristic function

$$F_l(\nu, \kappa) = 0, \quad (81)$$

where the function $F_l(\nu, \kappa)$ is in turn defined by

$$F_l(\nu, \kappa) \equiv (\nu^2 - \nu_0^2) - \frac{\kappa^2}{\nu} [R_0(\nu) - R_0(-\nu)], \quad (82)$$

where $\nu_0 = (l_j + \frac{1}{2})$. In the equation above, $R_0(\nu)$ can be expressed again in terms of a recursive relation

¹¹For simplicity we are expressing $u_{k,j}$ in terms of wave functions which do not have a well-defined $k \rightarrow 0$ limit. This limitation can be easily overcome by adding energy-dependent normalization coefficients; see Ref. [34] for the details of such a normalization.

$$R_n(\nu) = \frac{1}{(\nu + 1)[(\nu + 1)^2 - \nu_0^2] - S_j^2 \kappa^2 R_{n+1}(\nu)}. \quad (83)$$

Birse [32] notices that between 20 and 30 iterations are enough as to obtain $\nu(\kappa)$ with six significant digits. As already commented, the analytical properties of the function $\nu(\kappa)$ are the essential ingredient for determining the range of applicability of the perturbative reexpansion of the wave function defined in Eqs. (78)–(80).

The behavior of the solutions $\nu = \nu(\kappa)$ can be described as follows: for $\kappa = 0$ (plus the condition $k \neq 0$) we have the Schrödinger equation for free waves with angular momentum l . In such a case, the solution of Eq. (77) is the spherical Bessel functions, which is equivalent to having

$$\nu(\kappa = 0) = \nu_0 = l_j + \frac{1}{2}. \quad (84)$$

If κ is smaller than a critical value κ_c , the shift can be written as

$$\nu(\kappa < \kappa_c) = \nu_0 - \delta\nu, \quad (85)$$

with $0 < \delta\nu < \frac{1}{2}$. For sufficiently small κ the $\delta\nu$ correction is of order $|\kappa|$ (κ^2) for $l_j = 0$ ($l_j \neq 0$), where the exact expression can be consulted in Ref. [32]. For larger values of κ , $\delta\nu$ is still a power series in κ and therefore perturbative. However, this situation changes once $|\kappa|$ is large enough for the shift to reach the value $\nu = l_j$: above κ_c the shift ν splits into two imaginary solutions of the type

$$\nu(\kappa > \kappa_c) = l_j \pm i\rho(\kappa). \quad (86)$$

Beyond this point, the renormalized angular momentum $\nu(\kappa)$ cannot be expressed as a power series in terms of κ , as a consequence of the nonanalytical character of the split. Consequently, we expect the standard perturbative treatment to fail above this critical value κ_c . The critical reduced momenta for the uncoupled channels can be consulted in Table I.

2. The coupled-channel case

The (angular momentum) coupled-channel case was worked out in detail by Birse [32]. The extension is trivial and only requires the following definitions. In the first place, the Schrödinger equation reads

$$-\mathbf{u}''_{k,j} + \left[\mathbf{S}_j \frac{a_3}{r^3} + \frac{\mathbf{L}_j^2}{r^2} \right] \mathbf{u}_{k,j}(r) = k^2 \mathbf{u}_{k,j}(r), \quad (87)$$

where there are N angular momentum channels, $\mathbf{S}_j$ is the tensor matrix, and $\mathbf{L}_j$ is a diagonal matrix containing the value of the angular momenta

$$\mathbf{L}_j^2 = \text{diag}(l_1(l_1 + 1), \dots, l_N(l_N + 1)). \quad (88)$$

As in the previous case, the general solution of the Schrödinger equation is a linear combination of the functions

TABLE I. Reduced critical momenta κ_c for the different types of coupled channels appearing in the $H\bar{H}$ system ($j \leq 3$).

3U_J	κ_c	3C_J	κ_c	5C_J	κ_c	${}^{1-5}C_J$	κ_c
...	...	3P_0	1.259	...	...	${}^1S_0-{}^5D_0$	1.328
3P_1	2.518	${}^3S_1-{}^3D_1$	1.367	5D_1	8.333	${}^1P_1-{}^5P_1-{}^5F_1$	4.314
3D_2	8.333	${}^3P_2-{}^3F_2$	3.237	${}^5P_2-{}^5F_2$	1.644	${}^1D_2-{}^5S_2-{}^5D_2-{}^5G_2$	1.089
3F_3	19.70	${}^3D_3-{}^3G_3$	7.913	${}^5D_3-{}^5G_3$	5.594	${}^1F_3-{}^5P_3-{}^5F_3-{}^5H_3$	2.615

$$\mathbf{u}_{k,j}(r) = \sum_{\{l_j\}} [\alpha_{l_j} \boldsymbol{\xi}_{l_j}(r; k) + \beta_{l_j} \boldsymbol{\eta}_{l_j}(r; k)], \quad (89)$$

where we sum over the possible values of the angular momenta. In this case $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are N -component vectors that can be expressed as sums of Bessel functions

$$\boldsymbol{\xi}_{l_j}(r; k) = \sum_{m=-\infty}^{\infty} \mathbf{b}_m(\nu_{l_j}) \sqrt{r} J_{m+\nu_{l_j}}(kr), \quad (90)$$

$$\boldsymbol{\eta}_{l_j}(r; k) = \sum_{m=-\infty}^{\infty} (-1)^m \mathbf{b}_{-m}(\nu_{l_j}) \sqrt{r} J_{-m-\nu_{l_j}}(kr), \quad (91)$$

where we have labeled the different solutions by the subscript l_j , which indicates that for $\kappa = 0$ the solutions will behave as free waves with angular momentum l_j . The $\mathbf{b}_m(\nu_{l_j})$ coefficients are now vectors satisfying a certain recursive relation; see Ref. [32] for details.

Of course, what is important are the analytical properties of $\nu(\kappa)$. The functional dependence $\nu = \nu(\kappa)$ is determined by finding the zeros of

$$\det(\mathbf{F}_j(\nu, \kappa)) = 0, \quad (92)$$

where the characteristic function $\mathbf{F}_j(\nu, \kappa)$ is a $N \times N$ matrix. The equation above admits N solutions, one for each value of the angular momentum, which can be labeled by their value for $\kappa = 0$, that is,

$$\nu_{l_i}(\kappa = 0) = l_i + \frac{1}{2}, \quad (93)$$

with $i = 1, \dots, N$. We define $\mathbf{F}_j(\nu, \kappa)$ as follows:

$$\mathbf{F}_j(\nu, \kappa) \equiv \mathbf{f}_j(\nu) - \frac{\kappa^2}{\nu} [\mathbf{R}_0(\nu) - \mathbf{R}_0(-\nu)], \quad (94)$$

where $\mathbf{f}_j(\nu)$ is a diagonal matrix defined by

$$\mathbf{f}_j(\nu) = \text{diag}(\nu^2 - (l_1 + \frac{1}{2})^2, \dots, \nu^2 - (l_N + \frac{1}{2})^2). \quad (95)$$

In turn, $\mathbf{R}_0(\nu)$ can be expressed in terms of a recursive relation

$$\mathbf{R}_n(\nu) = [\mathbf{f}_j(\nu) - \mathbf{S}_j \kappa^2 \mathbf{S}_j \mathbf{R}_{n+1}(\nu)]^{-1}. \quad (96)$$

As in the uncoupled-channel case, between 20 and 30 iterations are more than enough to obtain a sufficiently accurate result.

The analytical properties of the shift $\nu_{l_i}(\kappa)$ are analogous to what happened in the uncoupled-channel case: the shift begins its trajectory at $\nu_{l_i} = l_i + \frac{1}{2}$ for $\kappa = 0$ and

moves slowly downwards. Once the shift reaches the value $\nu_{l_i} = l_i$ at the critical value of the coupling $\kappa = \kappa_c$, the shift splits into two complex conjugate solutions of the type $\nu_{l_i}(\kappa) = l_i \pm i\rho_{l_i}(\kappa)$, signaling the breakdown of a long-range perturbative expansion of the wave function.

In general, the first shift to split into the complex plane is the one corresponding to the smallest angular momentum. For that reason we will simply study the critical value of κ corresponding to the lowest angular momentum in the coupled channel. The results for the 3C_J , 5C_J and ${}^{1-5}C_J$ families of coupled channels with $j \leq 3$ are listed in Table I. While the critical momenta for the 3C_J type of coupled channels were already determined by Birse in Ref. [32], the results for the 5C_J and ${}^{1-5}C_J$ coupled channels are new.

In general, for the 3C_J channels we obtain values of κ_c that are half the values obtained in Ref. [32]. The reason is simple: the tensor operator matrix elements in the 3C_J channels for heavy meson molecules is half of the value of the corresponding matrix in the two-nucleon system. Usually the critical reduced momenta κ_c grow rapidly with the angular momentum of the system, with the exception of the 3P_0 partial wave for which a relatively low value of κ_c is obtained. If we assume similar values of a_3 , the tensor length scale, the channel less likely to be perturbative is the ${}^1D_2-{}^5S_2-{}^5D_2-{}^5G_2$ partial wave, corresponding to a $J^{PC} = 2^{++} P^* \bar{P}^*$ meson molecule.

C. Finite pion mass effects

In the previous paragraphs we have calculated the critical reduced momenta κ_c for a pure $1/r^3$ potential, corresponding to the form of OPE for $\mu = 0$. For $\mu > 0$ we expect the values of κ_c to rise by a certain amount, as the finite pion mass $e^{-\mu r}$ will decrease the effective strength of the potential at large distances. Curiously, we will see that the dependence of κ_c on μ is actually triggered by the far from perfect separation of scales in the chiral EFT, rather than from the finite pion mass alone.

The argument is the following: we begin by solving the Schrödinger equation stepwise; that is, we divide the possible radii $0 < r < \infty$ into several regions of a given size Δr . In particular, if we consider the region defined by the condition

$$R - \frac{1}{2}\Delta r < r < R + \frac{1}{2}\Delta r, \quad (97)$$

then for $\mu\Delta r < 1$ we can approximate the finite pion effects by the substitution

$$\frac{a_3}{r^3} e^{-\mu r} \simeq \frac{a_3}{r^3} e^{-\mu R}, \quad (98)$$

where we have ignored the $1/r$ and $1/r^2$ components of the tensor force for simplicity.¹² At this point it is worth noticing that the explicit solutions of the Schrödinger equation we have written in the preceding subsections can also be applied in a specific subset of the real axis $0 < r < \infty$. That is, if the $1/r^3$ potential is only valid in the region defined by Eq. (97),

$$V(r) = \frac{a_3}{r^3} \theta\left(\left|R + \frac{\Delta r}{2}\right| - r\right) \theta\left(r - \left|R - \frac{\Delta r}{2}\right|\right), \quad (99)$$

then Eqs. (78)–(80) represent the full solution of the Schrödinger equation in this region. Moreover, the critical reduced momenta κ_c calculated for $0 < r < \infty$ are still valid in $R - \Delta r/2 < r < R + \Delta r/2$.

The consequence for a decaying $e^{-\mu r}/r^3$ potential is that, by making Δr sufficiently small, we can define a reduced critical momentum

$$\kappa_c(\mu, R) = \kappa_c e^{\mu R}, \quad (100)$$

which applies in the vicinity of R . Therefore, the real $\kappa_c(\mu)$ setting the actual limits of perturbative tensor OPE is the minimum of $\kappa_c(\mu, R)$, which is in turn determined by the smallest radius for which the OPE potential is valid. In chiral EFT we expect this radius to be proportional to the inverse of the breakdown scale, $R_0 \propto 1/\Lambda_0$. Naively, we expect R_0 to lie around 0.5 fm, but we cannot discard larger (or smaller) values. In particular, a recent analysis of the convergence of the two-nucleon potential in nuclear EFT [64] suggests values around 0.8 fm (and even higher). Of course, the extrapolation of this latter value to the heavy meson-antimeson potential is to be taken with a grain of salt, but we will nevertheless use the 0.8 fm figure only as an upper bound for R_0 . For $\mu = m_\pi$, the $R_0 = 0.5$ –0.8 fm window yields

$$\kappa(m_\pi) = \kappa_c(m_\pi, R_0) \simeq (\sqrt{2} - \sqrt{3})\kappa_c, \quad (101)$$

where we have approximated $e^{m_\pi R_0}$ by $\sqrt{2}(\sqrt{3})$ for $R_0 = 0.5(0.8)$ fm. In the nucleon-nucleon case the $\sqrt{2}$ factor approximately corresponds to the observed mismatch between the critical momentum obtained by Birse for the 3S_1 – 3D_1 channel ($p_c = 66$ MeV) [32] and the momentum at which the FMS calculations [31] fail ($p_c \sim 100$ MeV). However, a more recent formulation of nuclear EFT with perturbative OPE [65] suggests a larger $p_c \geq 150$ MeV, requiring $R_0 \geq 1.0$ fm at least.

¹²These components can be implicitly taken into account by a redefinition of a_3 . However, provided R is sufficiently small, they are suppressed by a factor of μR .

VI. DISCUSSION AND CONCLUSIONS

The critical values κ_c can be converted into critical momenta by dividing by the tensor length scale a_3 ,

$$a_3 = |\tau| \frac{\mu_{\text{H}\bar{\text{H}}} g^2}{4\pi f_\pi^2}. \quad (102)$$

In the equation above, $\tau = 2I(I + 1) - 3$ is the eigenvalue of the isospin operator $\vec{\tau}_1 \cdot \vec{\tau}_2$ and $\mu_{\text{H}\bar{\text{H}}}$ the reduced mass of the heavy meson molecule under consideration. For the axial coupling constant g , we take $g = 0.6 \pm 0.1$ in the charm sector [$D^{(*)}\bar{D}^{(*)}$] and $g = 0.5 \pm 0.1$ in the bottom one [$B^{(*)}\bar{B}^{(*)}$]. For the charm mesons, the value of g is known from the $D^* \rightarrow D\pi$ and $D^* \rightarrow D\gamma$ decays [66,67]. This yields $g = 0.59 \pm 0.01 \pm 0.07$, which we round to $g = 0.6 \pm 0.1$. For the bottom mesons, g is experimentally unknown, even though from heavy quark symmetry we should expect a value similar to the charm mesons one. There are however theoretical determinations, for example, from lattice QCD [68–73] or from the QCD Dyson-Schwinger equations [74] to name just a few of them, usually ranging from 0.3 to 0.7 [75]. The value we have chosen, $g = 0.5 \pm 0.1$, represents a compromise between the previous determinations, but it should be kept in mind that the uncertainties for the bottom sector may be much larger.

The critical momentum (including finite pion mass corrections) is defined as

$$p_{\text{crit}}(\mu) = \frac{\kappa_c}{a_3} e^{+\mu R_0}, \quad (103)$$

with $R_0 = 0.65 \pm 0.15$ fm (i.e., $R_0 = 0.5$ –0.8 fm), and κ_c to be taken from Table I. We can also translate $p_{\text{crit}}(\mu)$ into critical binding energies by

$$B_{\text{crit}}(\mu) = \frac{p_{\text{crit}}^2(\mu)}{2\mu_{\text{H}\bar{\text{H}}}}, \quad (104)$$

where we can make the approximation $B_{\text{crit}}(m_\pi) \simeq (2.5 \pm 0.5)B_{\text{crit}}(0)$. It is also interesting that the critical binding energy scales as $1/g^4$. This dependence will generate remarkable uncertainties on the value of B_{crit} for bottom meson molecules.

In the charm isovector sector, that is, the $D^{(*)}\bar{D}^{(*)}$ molecules with $I = 1$ and $\tau = 1$, the critical momenta are of the order of 0.8–1.1 GeV, similar to the chiral breakdown scale $\Lambda_0 \sim 0.5$ –1.0 GeV. In this case, central and tensor OPE are expected to be a perturbation in all the range of validity of the EFT. On the other extreme we have the bottom isoscalar sector, for which the critical momenta are of the order of m_π (~ 200 MeV for $\mu = m_\pi$). For these systems the onset of nonperturbative OPE starts at around 4 MeV in the chiral limit, or around 10 MeV if we take into account the finite pion mass effects. It should be noted that the previous limits could be more stringent if g is larger than the expected 0.5, as the critical binding depends on $1/g^4$.

TABLE II. Critical value of the momenta for the tensor piece of OPE for the s -wave isoscalar $D\bar{D}^*$ and $D^*\bar{D}^*$ and isovector $B\bar{B}^*$ and $B^*\bar{B}^*$ systems. We compute the critical value both in the $\mu = 0$ limit and in the finite μ limit. In the second case, we usually have $\mu = m_\pi$, with the exception of the $D\bar{D}^*$ ($B\bar{B}^*$) case in which $|\mu| = 31$ MeV ($\mu = 131$ MeV), generating no significant deviation from $\mu = 0$ ($\mu = m_\pi$). The respective value of the critical momenta and maximum binding energies for the isovector $D\bar{D}^*$ and $D^*\bar{D}^*$ molecules is three (nine) times larger than for their isoscalar counterparts. The contrary happens with the isoscalar $B\bar{B}^*$ and $B^*\bar{B}^*$ molecules, for which the critical momenta are 1/3 of the isovector case and the maximum binding energies 1/9.

$I^G(J^{PC})$	$2S+1L_J$	$H\bar{H}$	$p_{\text{crit}}(0)$	$B_{\text{crit}}(0)$	$p_{\text{crit}}(\mu)$	$B_{\text{crit}}(\mu)$
$0^\pm(1^\pm)$	$^3S_1-^3D_1$	$D\bar{D}^*$	290^{+120}_{-80}	42^{+46}_{-19}	290^{+120}_{-80}	42^{+46}_{-19}
$0^+(0^{++})$	$^1S_0-^5D_0$	$D^*\bar{D}^*$	270^{+120}_{-70}	36^{+38}_{-17}	420^{+190}_{-120}	89^{+97}_{-41}
$0^-(1^{+-})$	$^3S_1-^3D_1$	$D^*\bar{D}^*$	280^{+120}_{-80}	38^{+41}_{-18}	440^{+200}_{-120}	90^{+100}_{-50}
$0^+(2^{++})$	$^1D_2-^5S_2-^5D_2-^5G_2$	$D^*\bar{D}^*$	220^{+100}_{-60}	24^{+26}_{-11}	350^{+160}_{-100}	60^{+65}_{-30}
$1^\mp(1^\pm)$	$^3S_1-^3D_1$	$B\bar{B}^*$	450^{+260}_{-140}	38^{+56}_{-19}	690^{+400}_{-220}	90^{+130}_{-50}
$1^-(0^{++})$	$^1S_0-^5D_0$	$B^*\bar{B}^*$	440^{+240}_{-140}	36^{+51}_{-19}	690^{+390}_{-220}	90^{+130}_{-50}
$1^+(1^{+-})$	$^3S_1-^3D_1$	$B^*\bar{B}^*$	450^{+250}_{-140}	38^{+55}_{-20}	710^{+410}_{-230}	90^{+140}_{-50}
$1^-(2^{++})$	$^1D_2-^5S_2-^5D_2-^5G_2$	$B^*\bar{B}^*$	360^{+200}_{-110}	24^{+35}_{-12}	560^{+320}_{-170}	60^{+87}_{-33}

However, the bottom line is that in these heavy meson molecules OPE is expected to be nonperturbative unless the bound states are genuinely shallow. The corresponding EFT treatment is the one employed in Ref. [28] for exploring prospective $B^*\bar{B}$ states.

The two interesting cases are the isovector charm and isoscalar bottom meson molecules: in these two systems the values of the critical momenta lie between m_π and Λ_0 , defining a specific window in which OPE is perturbative. The critical momenta for the charm and bottom sectors can be found in Table II. For the charm isoscalar the critical binding energy lies around 25–40 MeV (60–90 MeV) for $\mu = 0$ ($\mu = m_\pi$). In this regard, the $D^*\bar{D}/D\bar{D}^*$ systems are especially interesting in the sense that they naturally fulfill the condition $|\mu| \sim 0$.¹³ This means that the $X(3872)$ is well within the energy range for which the LO EFT description consists only of contact-range interactions. In particular, the critical momentum of 290 MeV implies that the charged $D^{*+}D^-/D^{*-}D^+$ component of the $X(3872)$ can also be accounted for without the explicit inclusion of pion exchanges at the lowest order. This represents a remarkable simplification that allows, for example, reinterpreting the explanation of Gamermann *et al.* [25,26] for the branching ratio

$$\frac{\Gamma[X(3872) \rightarrow J/\psi \pi^+ \pi^- \pi^0]}{\Gamma[X(3872) \rightarrow J/\psi \pi^+ \pi^-]}, \quad (105)$$

as a result within a specific EFT framework, or to extend X-EFT [19] to the charged channels of the $X(3872)$.

In the isovector bottom sector, to which the recently discovered $Z_b(10610)$ and $Z_b(10650)$ belong, the critical energies range from 60 to 90 MeV, although they greatly depend on the axial coupling g , whose value is not that well-known. This energy is in fact of the order of the

splitting between the different channels ($\Delta_B = 46$ MeV/92 MeV), indicating that nonperturbative OPE is probably not needed until we reach binding energies such that coupled-channel effects can no longer be ignored. In this regard, the pionless EFT exploration of Mehen and Powell [16] probably provides a perfectly good LO description of the prospective HQSS partners of the Z_b 's as far as their binding energies are less than 40–50 MeV. Moreover, the inclusion of perturbative pions in the sub-leading orders of the aforementioned EFT will be a direct extension of X-EFT [19] to the bottom meson-antimeson system. We can call this theory Z-EFT.

Of course, the previous results are to be held in proper context: they refer to the EFT framework rather than to a more fundamental or phenomenological level of description of heavy meson molecules. In phenomenological approaches nonperturbative pion exchanges and coupled-channel effects may be important in suggesting the possibility or explaining the location of heavy molecular states [61,76], in particular, if the states are shallow, in which case a fine-tuning takes place between the different long- and short-range effects that are explicitly included. In contrast, in the EFT approach these fine-tunings are absorbed in the counterterms, which are set to reproduce the already known position of a certain state. This in turn stabilizes the calculations and reduces the impact of new physical effects we may subsequently incorporate in the EFT description as they are expected to be constrained by power counting. The price to pay is the inability of the EFT framework to predict the positions of these states from first principles.

To summarize, we have found that pion exchange effects are perturbative in most types of heavy meson molecules over the expected range of applicability of heavy meson EFT. The exception is the isoscalar bottom meson molecules, for which OPE is particularly strong and presumably nonperturbative. To a lesser extent, in the isoscalar charm sector nonperturbative OPE may be required if the

¹³Strictly speaking $|\mu| = 31$ MeV $\ll m_\pi$, which is small enough as to consider $\mu \sim 0$ as a pretty good approximation.

molecular states are deep enough (around 80 MeV or more). At lowest order, below the critical binding energies we have computed, the EFT description is a contact-range theory. Surprisingly, this description holds for momenta much larger than naively expected.

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APPENDIX: DERIVATION OF THE LEADING ORDER POTENTIAL IN HEAVY HADRON CHIRAL PERTURBATION THEORY

In this appendix, we derive the LO potential between a heavy meson and antimeson within heavy hadron chiral perturbation theory [41]. The LO potential can be decomposed into a contact-range and a finite-range piece. The latter can be identified with the well-known OPE potential. We also compute the complete partial wave projection of the LO potential.

1. The effective Lagrangian at leading order

We define the heavy meson (antimeson) fields in terms of the pseudoscalar and vector meson (antimeson) fields in the following way:

$$H^Q = \frac{1 + \not{v}}{2} [P_v^{*\mu} \gamma_\mu - P_v \gamma^5], \quad (A1)$$

$$\bar{H}^Q = \gamma_0 H^{Q\dagger} \gamma_0, \quad (A2)$$

$$H^{\bar{Q}} = [\bar{P}_v^{*\mu} \gamma_\mu - \bar{P}_v \gamma^5] \frac{1 - \not{v}}{2}, \quad (A3)$$

$$\bar{H}^{\bar{Q}} = \gamma_0 H^{\bar{Q}\dagger} \gamma_0, \quad (A4)$$

where γ_μ and γ_5 are the Dirac gamma matrices, v is the velocity parameter and $\not{v} = v^\mu \gamma_\mu$. The field $P_v^{(*\mu)}$ ($\bar{P}_v^{(*\mu)}$) destroys a heavy meson (antimeson) of velocity v and polarization μ for the vector meson. The subscript v also indicates that the normalization of the one-particle states is given by

$$\langle P_v(\vec{k}) | P_{v'}(\vec{k}') \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'), \quad (A5)$$

$$\langle P_v^{*\mu}(\vec{k}) | P_{v'}^{*\nu}(\vec{k}') \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \delta_{\mu\nu}, \quad (A6)$$

depending on whether we are considering the pseudoscalar or vector meson case.

The heavy hadron chiral perturbation theory Lagrangian at lowest order (Q^0) can be decomposed as the sum of two different contributions:

$$\mathcal{L}^{(0)} = \mathcal{L}_{4H}^{(0)} + \mathcal{L}_{\pi HH}^{(0)}, \quad (A7)$$

where the first term represents a 4-meson interaction vertex and the second the meson-pion vertex. The contact interaction term reads [18]

$$\begin{aligned} \mathcal{L}_{4H}^{(0)} = & D_{0a} \text{Tr}[\bar{H}^Q H^Q \gamma_\mu] \text{Tr}[H^{\bar{Q}} \bar{H}^{\bar{Q}} \gamma^\mu] \\ & + D_{0b} \text{Tr}[\bar{H}^Q H^Q \gamma_\mu \gamma_5] \text{Tr}[H^{\bar{Q}} \bar{H}^{\bar{Q}} \gamma^\mu \gamma_5] \\ & + E_{0a} \text{Tr}[\bar{H}^Q \tau_i H^Q \gamma_\mu] \text{Tr}[H^{\bar{Q}} \tau_i \bar{H}^{\bar{Q}} \gamma^\mu] \\ & + E_{0b} \text{Tr}[\bar{H}^Q \tau_i H^Q \gamma_\mu \gamma_5] \text{Tr}[H^{\bar{Q}} \bar{H}^{\bar{Q}} \tau_i \gamma^\mu \gamma_5], \end{aligned} \quad (A8)$$

where τ_i are the isospin matrices (i.e., the Pauli matrices). That is, in the previous Lagrangian D_{0a} and D_{0b} are isospin independent, while E_{0a} and E_{0b} are isospin dependent. In general we will be only considering a specific isospin channel, in which case we will generically define

$$C_{0a} = D_{0a} + \vec{\tau}_1 \cdot \vec{\tau}_2 E_{0a}, \quad (A9)$$

$$C_{0b} = D_{0b} + \vec{\tau}_1 \cdot \vec{\tau}_2 E_{0b}. \quad (A10)$$

For the Lagrangian describing the pion-meson vertex we have [44]

$$\begin{aligned} \mathcal{L}_{\pi H \bar{H}}^{(0)} = & i \frac{g}{2} \text{Tr}[\bar{H}^Q H^Q \gamma_\mu \gamma_5 (\xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger)] \\ & - i \frac{g}{2} \text{Tr}[H^{\bar{Q}} \bar{H}^{\bar{Q}} \gamma_\mu \gamma_5 (\xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger)], \end{aligned} \quad (A11)$$

where ξ is defined as

$$\xi = e^{(i/f_\pi)M}, \quad (A12)$$

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \vec{\tau} \cdot \vec{\pi}, \quad (A13)$$

with π the pion fields and $f_\pi = 132$ MeV. The isospin indices convention for the heavy meson and antimeson fields is

$$P_v^{(*\mu)} = \begin{pmatrix} P_{v,+}^{(*\mu)} \\ P_{v,0}^{(*\mu)} \end{pmatrix} \quad \text{and} \quad \bar{P}_v^{(*\mu)} = \begin{pmatrix} -\bar{P}_{v,0}^{(*\mu)} \\ \bar{P}_{v,-}^{(*\mu)} \end{pmatrix}. \quad (A14)$$

This convention explains the minus sign in the second line of Eq. (A11). That is, we have performed a G -parity transformation to obtain the chiral Lagrangian for the heavy antimeson fields from the heavy meson one.

2. The nonrelativistic normalization

In the heavy quark limit, $m_Q \rightarrow \infty$, the static potential between two heavy mesons is a well-defined object. Relativistic effects are suppressed, and the two heavy meson system can be effectively described in terms of nonrelativistic quantum mechanics. For making the nonrelativistic transition we specify the velocity parameter to be $v = (1, \vec{0})$ and employ a new normalization of the heavy meson fields defined by

$$P^{(*j)} = \sqrt{2}P_v^{(*j)}, \quad (\text{A15})$$

where we have substituted the Greek index μ by the Latin index j , as the $\mu = 0$ polarization component is completely irrelevant for $v = (1, \vec{0})$. In this heavy meson field normalization, the one-particle states have the usual non-relativistic normalization

$$\langle P(\vec{k}) | P(\vec{k}') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'), \quad (\text{A16})$$

$$\langle P^{*i}(\vec{k}) | P^{*j}(\vec{k}') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \delta_{ij}, \quad (\text{A17})$$

which is more convenient for the definition of the quantum mechanical potential.

3. The contact-range potential

In the nonrelativistic normalization, after expanding the $H^{(Q)}$ and $\bar{H}^{(Q)}$ fields, the contact-range Lagrangian can be rewritten as

$$V_C(\vec{q}) = C_{0a} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \vec{\epsilon}_{1'}^* \cdot \vec{\epsilon}_1 & 0 & 0 \\ 0 & 0 & \vec{\epsilon}_{2'}^* \cdot \vec{\epsilon}_2 & 0 \\ 0 & 0 & 0 & \vec{\epsilon}_{1'}^* \cdot \vec{\epsilon}_1 \vec{\epsilon}_{2'}^* \cdot \vec{\epsilon}_2 \end{pmatrix} + C_{0b} \begin{pmatrix} 0 & 0 & 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{\epsilon}_{2'}^* \\ 0 & 0 & -\vec{\epsilon}_1 \cdot \vec{\epsilon}_{2'}^* & +\vec{S}_1 \cdot \vec{\epsilon}_{2'}^* \\ 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{\epsilon}_2 & 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{S}_2 \\ -\vec{\epsilon}_1 \cdot \vec{\epsilon}_2 & +\vec{S}_1 \cdot \vec{\epsilon}_2 & -\vec{\epsilon}_1 \cdot \vec{S}_2 & +\vec{S}_1 \cdot \vec{S}_2 \end{pmatrix}. \quad (\text{A20})$$

In the equation above, $\vec{\epsilon}_{1(2)}/\vec{\epsilon}_{1'(2')}$ represents the polarization of the incoming particles 1(2) or the outgoing particles 1'(2'). In turn, the matrix elements of the spin-1 operator $\vec{S}_{1(2)}$ are equivalent to the vector product of the polarization wave functions, that is,

$$i\langle 1\lambda' | \vec{S} | 1\lambda \rangle = (\vec{\epsilon}_{\lambda'}^* \times \vec{\epsilon}_{\lambda}), \quad (\text{A21})$$

where $|1\lambda^{(\prime)}\rangle$ is the vector corresponding to the polarization wave function $\vec{\epsilon}_{\lambda^{(\prime)}}$.

4. The finite-range OPE potential

For obtaining the finite-range piece of the potential, we start by considering the $HH\pi$ vertex in the nonrelativistic normalization, for which the Lagrangian reads

$$\begin{aligned} \mathcal{L}_{\pi HH} = & \frac{g}{\sqrt{2}f_\pi} [(P^* P^\dagger + P^{*\dagger} P) + i(P^{*\dagger} \times P^*)] \vec{\tau} \cdot \partial \vec{\pi} \\ & + \frac{g}{\sqrt{2}f_\pi} [(\bar{P}^* \bar{P}^\dagger + \bar{P}^{*\dagger} \bar{P}) - i(\bar{P}^{*\dagger} \times \bar{P}^*)] \vec{\tau} \cdot \partial \vec{\pi}, \end{aligned} \quad (\text{A22})$$

where the polarization indices are implicit. If we define the nonrelativistic amplitude as follows:

$$\mathcal{A}(H \rightarrow H\pi^a) = i\langle H | \mathcal{L}_{\pi HH} | H\pi^a \rangle, \quad (\text{A23})$$

we find

$$\begin{aligned} \mathcal{L}_{4H(a)} = & -C_{0a}(P^\dagger P + P^{*\dagger} P^*)(\bar{P} \bar{P}^\dagger + \bar{P}^* \bar{P}^{*\dagger}) \\ & + C_{0b}(P^* P^\dagger + P P^{*\dagger})(\bar{P}^* \bar{P}^\dagger + \bar{P} \bar{P}^{*\dagger}) \\ & - iC_{0b}[(P^* P^\dagger + P P^{*\dagger})(\bar{P}^{*\dagger} \times \bar{P}^*) \\ & - (P^{*\dagger} \times P^*)(\bar{P}^* \bar{P}^\dagger + \bar{P} \bar{P}^{*\dagger})] \\ & + C_{0b}(P^{*\dagger} \times P^*)(\bar{P}^{*\dagger} \times \bar{P}^*). \end{aligned} \quad (\text{A18})$$

In the expressions above, the polarization of the vector mesons has been made implicit to simplify the notation. The potential corresponding to the Lagrangian above can mix different particle channels. Thus, it is convenient to write the potential in the coupled-channel basis

$$\mathcal{B}_{H\bar{H}} = \{|P\bar{P}\rangle, |P^*\bar{P}\rangle, |P\bar{P}^*\rangle, |P^*\bar{P}^*\rangle\}, \quad (\text{A19})$$

for which we have (in momentum space)

$$\mathcal{A}(P \rightarrow P\pi^a) = 0, \quad (\text{A24})$$

$$\mathcal{A}(P^{*i} \rightarrow P\pi^a) = \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{\epsilon}_i \cdot \vec{q}, \quad (\text{A25})$$

$$\mathcal{A}(P^{*i} \rightarrow P^{*j}\pi^a) = -i \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{S} \cdot \vec{q}, \quad (\text{A26})$$

where $\vec{q}$ is the momentum of the outgoing pion, a the isospin index of the pion and $\vec{\epsilon}_i$ the polarization wave function of the heavy vector meson P^{*i} . The amplitudes with heavy antimesons can be obtained either from the Lagrangian or from a G -parity transformation of the expressions above, yielding

$$\mathcal{A}(\bar{P} \rightarrow \bar{P}\pi^a) = 0, \quad (\text{A27})$$

$$\mathcal{A}(\bar{P}^{*i} \rightarrow \bar{P}\pi^a) = \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{\epsilon}_i \cdot \vec{q}, \quad (\text{A28})$$

$$\mathcal{A}(\bar{P}^{*i} \rightarrow \bar{P}^{*j}\pi^a) = i \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{S} \cdot \vec{q}, \quad (\text{A29})$$

where only the $\bar{P}^* \bar{P}^* \pi$ vertex changes sign.

From the nonrelativistic amplitudes above, the OPE potential in the heavy quark limit is given by

$$V_{F,\text{HH}}^{(0)}(\vec{q}) = \frac{g^2}{2f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{1}{\vec{q}^2 + m_\pi^2} \begin{pmatrix} 0 & 0 & 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{q} \vec{\epsilon}_{2'}^* \cdot \vec{q} \\ 0 & 0 & -\vec{\epsilon}_1 \cdot \vec{q} \vec{\epsilon}_{2'}^* \cdot \vec{q} & +\vec{S}_1 \cdot \vec{q} \vec{\epsilon}_{2'}^* \cdot \vec{q} \\ 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{q} \vec{\epsilon}_2 \cdot \vec{q} & 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{q} \vec{S}_2 \cdot \vec{q} \\ -\vec{\epsilon}_1 \cdot \vec{q} \vec{\epsilon}_2 \cdot \vec{q} & +\vec{S}_1 \cdot \vec{q} \vec{\epsilon}_2 \cdot \vec{q} & -\vec{\epsilon}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} & +\vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} \end{pmatrix} + \mathcal{O}\left(\frac{1}{m_Q}\right), \quad (\text{A30})$$

where we have already particularized for the heavy meson-antimeson case. If we consider the effect of the splitting between the masses of the $\text{P}\bar{\text{P}}$, $\text{P}\bar{\text{P}}^*/\text{P}^*\bar{\text{P}}$, and $\text{P}^*\bar{\text{P}}^*$ systems, then we should substitute the pion propagator by

$$\frac{1}{\vec{q}^2 + m_\pi^2} \rightarrow \frac{1}{\vec{q}^2 + \mu^2} \quad (\text{A31})$$

in the nondiagonal terms of the potential matrix, where the effective pion mass is given by $\mu^2 = m_\pi^2 - \Delta_Q^2$, with Δ_Q the energy splitting. These corrections are fundamental in the charm sector, where Δ_Q is of the order of m_π . A further problem arises in the $\text{P}\bar{\text{P}} \rightarrow \text{P}^*\bar{\text{P}}^*$ and $\text{P}\bar{\text{P}}^*/\text{P}^*\bar{\text{P}} \rightarrow \text{P}^*\bar{\text{P}}^*$ transitions, where either the initial or final heavy meson states will be off-shell. In this case it is not entirely clear whether the static approximation can be employed to define a transition potential between the different heavy meson states in the charm sector. Luckily this problem is not present in the heavy quark limit, in which the possible off-shell effects are strongly suppressed.

We can calculate the coordinate space representation of the OPE potential by Fourier-transforming the momentum space potential

$$V_{F,\text{HH}}^{(0)}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} V_{F,\text{HH}}^{(0)}(\vec{q}) e^{-i\vec{q} \cdot \vec{r}}, \quad (\text{A32})$$

from which we obtain

$$V_{F,\text{HH}}^{(0)}(\vec{r}) = \vec{\tau}_2 \cdot \vec{\tau}_1 \text{C}_{12}(\hat{r}) \frac{g^2}{6f_\pi^2} \delta(\vec{r}) - \vec{\tau}_2 \cdot \vec{\tau}_1 [\text{C}_{12}(\hat{r}) W_C(r) + \text{S}_{12}(\hat{r}) W_T(r)] + \mathcal{O}\left(\frac{1}{m_Q}\right), \quad (\text{A33})$$

with $W_C(r)$ and $W_T(r)$ defined in Eqs. (37) and (37). The central and tensor operators $\text{C}_{12}(\hat{r})$ and $\text{S}_{12}(\hat{r})$ are 4×4 matrices in the particle channel basis $\mathcal{B}_{\text{HH}}$. Their explicit representation is

$$\text{C}_{12}(\hat{r}) = \begin{pmatrix} 0 & 0 & 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{\epsilon}_{2'}^* \\ 0 & 0 & -\vec{\epsilon}_1 \cdot \vec{\epsilon}_{2'}^* & +\vec{S}_1 \cdot \vec{\epsilon}_{2'}^* \\ 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{\epsilon}_2 & 0 & -\vec{\epsilon}_{1'}^* \cdot \vec{S}_2 \\ -\vec{\epsilon}_1 \cdot \vec{\epsilon}_2 & +\vec{S}_1 \cdot \vec{\epsilon}_2 & -\vec{\epsilon}_1 \cdot \vec{S}_2 & +\vec{S}_1 \cdot \vec{S}_2 \end{pmatrix}, \quad (\text{A34})$$

$$\text{S}_{12}(\hat{r}) = \begin{pmatrix} 0 & 0 & 0 & -S_{12}(\vec{\epsilon}_{1'}^*, \vec{\epsilon}_{2'}^*, \hat{r}) \\ 0 & 0 & -S_{12}(\vec{\epsilon}_1, \vec{\epsilon}_{2'}^*, \hat{r}) & +S_{12}(\vec{S}_1, \vec{\epsilon}_{2'}^*, \hat{r}) \\ 0 & -S_{12}(\vec{\epsilon}_{1'}^*, \vec{\epsilon}_2, \hat{r}) & 0 & -S_{12}(\vec{\epsilon}_{1'}^*, \vec{S}_2, \hat{r}) \\ -S_{12}(\vec{\epsilon}_1, \vec{\epsilon}_2, \hat{r}) & +S_{12}(\vec{S}_1, \vec{\epsilon}_2, \hat{r}) & -S_{12}(\vec{\epsilon}_1, \vec{S}_2, \hat{r}) & +S_{12}(\vec{S}_1, \vec{S}_2, \hat{r}) \end{pmatrix}, \quad (\text{A35})$$

where S_{12} is defined as

$$S_{12}(\vec{a}_1, \vec{b}_2, \hat{r}) = 3\vec{a}_1 \cdot \hat{r} \vec{b}_2 \cdot \hat{r} - \vec{a}_1 \cdot \vec{b}_2. \quad (\text{A36})$$

5. Partial wave projection

We can simplify the representation of the OPE potential by projecting into $\text{H}\bar{\text{H}}$ states with well-defined J^{PC} quantum numbers. This task is greatly simplified in coordinate space, for which only the operators $\text{C}_{12}(\hat{r})$ and $\text{S}_{12}(\hat{r})$ are to be projected. In the $\text{P}\bar{\text{P}}$ case, we consider the partial waves

$$|\text{P}\bar{\text{P}}(ljm)\rangle = \delta_{jl} Y_{lm}(\hat{r}), \quad (\text{A37})$$

where $Y_{lm}(\hat{r})$ is a spherical harmonic. The parity and C parity of this partial wave are $P = C = (-1)^l$. The $\text{P}\bar{\text{P}}^*/\text{P}^*\bar{\text{P}}$ cases are a bit more involved owing to the polarization of the $\text{P}^*/\bar{\text{P}}^*$ vector meson. The partial waves with well-defined orbital and total angular momentum l and j are defined as

$$|\text{P}\bar{\text{P}}^*(ljm)\rangle = \sum_{m_l, \nu} Y_{lm_l}(\hat{r}) |1\nu\rangle \langle lm_l 1\nu | jm\rangle, \quad (\text{A38})$$

$$|\text{P}^*\bar{\text{P}}(ljm)\rangle = \sum_{m_l, \mu} Y_{lm_l}(\hat{r}) |1\mu\rangle \langle lm_l 1\mu | jm\rangle, \quad (\text{A39})$$

where $|1\mu\rangle$, $|1\nu\rangle$ are the polarization vectors of particle 1 and 2 and $\langle lm_l 1\mu | jm\rangle$ a Clebsch-Gordan coefficient. The polarization vectors are to be understood as giving matrix elements of the type

$$\langle 1\mu' | \vec{\epsilon}_{1'}^* \cdot \vec{\epsilon}_1 | 1\mu \rangle = \hat{e}_{\mu'}^* \cdot \hat{e}_\mu, \quad (\text{A40})$$

$$\langle 1\nu' | \vec{\epsilon}_{2'}^* \cdot \vec{\epsilon}_2 | 1\mu \rangle = \hat{e}_{\nu'}^* \cdot \hat{e}_\mu, \quad (\text{A41})$$

and so on, where we have added primes to denote final states and with $\hat{e}_\mu$ the unit vector in the spherical basis. A problem with the two partial waves we have written above is that they do not have a well-defined C parity. Thus, we define the linear combination

$$|P\bar{P}^*(\eta)(ljm)\rangle = \frac{1}{\sqrt{2}}[|P\bar{P}^*(ljm)\rangle - \eta|P^*\bar{P}(ljm)\rangle], \quad (\text{A42})$$

with intrinsic C parity η and total C parity $C = (-1)^l \eta$. Finally, the $P^*\bar{P}^*$ partial wave with intrinsic, orbital and total angular momentum s , l and j is given by

$$|P^*\bar{P}^*(sljm)\rangle = \sum_{m_l, m_s} Y_{lm_l}(\hat{r}) |sm_s\rangle \langle lm_l sm_s | jm\rangle, \quad (\text{A43})$$

where the spin state $|sm_s\rangle$ is

$$|sm_s\rangle = \sum_{\mu, \nu} |1\mu\rangle |1\nu\rangle \langle 1\mu 1\nu | sm_s\rangle. \quad (\text{A44})$$

The C parity of these states is $C = (-1)^{l+s}$. Alternatively we can also use the spectroscopic notation $^{2s+1}l_j$ instead of $(s)lj$ for denoting the partial waves.

From the previous definitions we can compute the matrix elements of the central and tensor operators as

$$\begin{aligned} \langle (s')l'j'm' | \hat{\mathbf{O}}_{12} | (s)ljm \rangle &= \int d^2\hat{r} \langle (s')l'j'm' | \hat{\mathbf{O}}_{12}(\hat{r}) | (s)ljm \rangle \\ &= \delta_{jj'} \delta_{mm'} \mathbf{O}_{(s'l')l'}^j \end{aligned} \quad (\text{A45})$$

where $\hat{\mathbf{O}}_{12}(\hat{r})$ either represents $\hat{\mathbf{C}}_{12}(\hat{r})$ or $\hat{\mathbf{S}}_{12}(\hat{r})$. The particle channels have been made implicit owing to the matrix notation. The central and tensor operators preserve the J^{PC} quantum numbers, which for the particular case of the tensor operator means that the orbital angular momentum can only change by an even number of units. In addition, if we only consider the $P^*\bar{P}^*$ particle channel, the intrinsic angular momentum is also subjected to the restriction of even $|s - s'|$. However, terms mixing the $P^*\bar{P}^*$ and $P\bar{P}^*/P^*\bar{P}$ channels can change s by one unit.

Thus the most compact way to write the matrix elements of the $\hat{\mathbf{C}}_{12}(\hat{r})$ or $\hat{\mathbf{S}}_{12}(\hat{r})$ operators is to consider a particle/partial wave basis with well-defined J^{PC} . Examples are the following:

$$\mathcal{B}_{\text{HH}}(0^{++}) = \{|^1S_0(P\bar{P})\rangle, |^1S_0(P^*\bar{P}^*)\rangle, |^5D_0(P^*\bar{P}^*)\rangle\}, \quad (\text{A46})$$

$$\begin{aligned} \mathcal{B}_{\text{HH}}(1^{+-}) &= \{|^3S_1(P\bar{P}^* + P^*\bar{P})\rangle, |^3D_1(P\bar{P}^* + P^*\bar{P})\rangle, \\ &\quad |^3S_1(P^*\bar{P}^*)\rangle, |^3D_1(P^*\bar{P}^*)\rangle\}, \end{aligned} \quad (\text{A47})$$

$$\begin{aligned} \mathcal{B}_{\text{HH}}(2^{++}) &= \{|^1D_2(P\bar{P})\rangle, |^3D_2(P\bar{P}^* - P^*\bar{P})\rangle, |^1D_2(P^*\bar{P}^*)\rangle, \\ &\quad |^5S_2(P^*\bar{P}^*)\rangle, |^5D_2(P^*\bar{P}^*)\rangle, |^5F_2(P^*\bar{P}^*)\rangle\}, \end{aligned} \quad (\text{A48})$$

for which we obtain the central matrices

$$\mathbf{C}_{12}(0^{++}) = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{A49})$$

$$\mathbf{C}_{12}(1^{+-}) = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}, \quad (\text{A50})$$

$$\mathbf{C}_{12}(2^{++}) = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{A51})$$

and the tensor matrices

$$\mathbf{S}_{12}(0^{++}) = \begin{pmatrix} 0 & 0 & -\sqrt{6} \\ 0 & 0 & -\sqrt{2} \\ -\sqrt{6} & -\sqrt{2} & -2 \end{pmatrix}, \quad (\text{A52})$$

$$\mathbf{S}_{12}(1^{+-}) = \begin{pmatrix} 0 & \sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & -1 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & -1 & \sqrt{2} & -1 \end{pmatrix}, \quad (\text{A53})$$

$$\mathbf{S}_{12}(2^{++}) = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{\frac{6}{5}} & 2\sqrt{\frac{3}{7}} & -6\sqrt{\frac{3}{35}} \\ 0 & -1 & 0 & 3\sqrt{\frac{2}{5}} & -\frac{3}{\sqrt{7}} & -\frac{12}{\sqrt{35}} \\ 0 & 0 & 0 & -\sqrt{\frac{2}{5}} & \frac{2}{\sqrt{7}} & -\frac{6}{\sqrt{35}} \\ -\sqrt{\frac{6}{5}} & 3\sqrt{\frac{2}{5}} & -\sqrt{\frac{2}{5}} & 0 & \sqrt{\frac{14}{5}} & 0 \\ 2\sqrt{\frac{3}{7}} & -\frac{3}{\sqrt{7}} & \frac{2}{\sqrt{7}} & \sqrt{\frac{14}{5}} & \frac{3}{7} & \frac{12}{7\sqrt{5}} \\ -6\sqrt{\frac{3}{35}} & \frac{12}{\sqrt{35}} & -\frac{6}{\sqrt{35}} & 0 & \frac{12}{7\sqrt{5}} & -\frac{10}{7} \end{pmatrix}. \quad (\text{A54})$$

Other J^{PC} possibilities are to be computed as these three examples. As can be seen, the resulting coupled-channel structure can be quite complex: in certain cases we can have up to six coupled channels. The simplifying feature is that particle coupled-channel effects are suppressed by two orders in the chiral expansion of the potential, which

means that in low order calculations we can concentrate in a specific particle channel and ignore the others. In this case, we only need to consider the diagonal terms of the central operator and, with regard to the tensor operator, we end up with the matrices shown from Eq. (51) to (61) with at most four coupled channels.

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- [1] M. Voloshin and L. Okun, *JETP Lett.* **23**, 333 (1976).
 - [2] A. De Rujula, H. Georgi, and S. Glashow, *Phys. Rev. Lett.* **38**, 317 (1977).
 - [3] N. A. Tornqvist, *Phys. Rev. Lett.* **67**, 556 (1991).
 - [4] N. A. Tornqvist, *Z. Phys. C* **61**, 525 (1994).
 - [5] A. V. Manohar and M. B. Wise, *Nucl. Phys.* **B399**, 17 (1993).
 - [6] T. E. O. Ericson and G. Karl, *Phys. Lett. B* **309**, 426 (1993).
 - [7] S. K. Choi *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **91**, 262001 (2003).
 - [8] D. E. Acosta *et al.* (CDF II Collaboration), *Phys. Rev. Lett.* **93**, 072001 (2004).
 - [9] V. M. Abazov *et al.* (D0 Collaboration), *Phys. Rev. Lett.* **93**, 162002 (2004).
 - [10] B. Aubert *et al.* (BABAR Collaboration), *Phys. Rev. D* **71**, 071103 (2005).
 - [11] I. Adachi *et al.* (Belle Collaboration), [arXiv:1105.4583](https://arxiv.org/abs/1105.4583).
 - [12] A. Bondar *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **108**, 122001 (2012).
 - [13] N. Isgur and M. B. Wise, *Phys. Lett. B* **232**, 113 (1989).
 - [14] N. Isgur and M. B. Wise, *Phys. Lett. B* **237**, 527 (1990).
 - [15] M. Voloshin, *Phys. Rev. D* **84**, 031502 (2011).
 - [16] T. Mehen and J. W. Powell, *Phys. Rev. D* **84**, 114013 (2011).
 - [17] E. Braaten and M. Kusunoki, *Phys. Rev. D* **69**, 074005 (2004).
 - [18] M. T. AlFiky, F. Gabbiani, and A. A. Petrov, *Phys. Lett. B* **640**, 238 (2006).
 - [19] S. Fleming, M. Kusunoki, T. Mehen, and U. van Kolck, *Phys. Rev. D* **76**, 034006 (2007).
 - [20] U. van Kolck, *Nucl. Phys.* **A645**, 273 (1999).
 - [21] J.-W. Chen, G. Rupak, and M. J. Savage, *Nucl. Phys.* **A653**, 386 (1999).
 - [22] F. E. Close and P. R. Page, *Phys. Lett. B* **578**, 119 (2004).
 - [23] M. Voloshin, *Phys. Lett. B* **579**, 316 (2004).
 - [24] M. Voloshin, *Int. J. Mod. Phys. A* **21**, 1239 (2006).
 - [25] D. Gamermann and E. Oset, *Phys. Rev. D* **80**, 014003 (2009).
 - [26] D. Gamermann, J. Nieves, E. Oset, and E. Ruiz Arriola, *Phys. Rev. D* **81**, 014029 (2010).
 - [27] M. Cleven, F.-K. Guo, C. Hanhart, and U.-G. Meissner, *Eur. Phys. J. A* **47**, 120 (2011).
 - [28] J. Nieves and M. Valderrama, *Phys. Rev. D* **84**, 056015 (2011).
 - [29] D. B. Kaplan, M. J. Savage, and M. B. Wise, *Phys. Lett. B* **424**, 390 (1998).
 - [30] D. B. Kaplan, M. J. Savage, and M. B. Wise, *Nucl. Phys.* **B534**, 329 (1998).
 - [31] S. Fleming, T. Mehen, and I. W. Stewart, *Nucl. Phys.* **A677**, 313 (2000).
 - [32] M. C. Birse, *Phys. Rev. C* **74**, 014003 (2006).
 - [33] M. Cavagnero, *Phys. Rev. A* **50**, 2841 (1994).
 - [34] B. Gao, *Phys. Rev. A* **59**, 2778 (1999).
 - [35] S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips, and M. J. Savage, *At the Frontiers of Particle Physics: Handbook of QCD* (World Scientific Publishing Company, Singapore, 2002), Vol. 4.
 - [36] P. F. Bedaque and U. van Kolck, *Annu. Rev. Nucl. Part. Sci.* **52**, 339 (2002).
 - [37] E. Epelbaum, *Prog. Part. Nucl. Phys.* **57**, 654 (2006).
 - [38] E. Epelbaum, H.-W. Hammer, and U.-G. Meissner, *Rev. Mod. Phys.* **81**, 1773 (2009).
 - [39] R. Machleidt and D. Entem, *Phys. Rep.* **503**, 1 (2011).
 - [40] S. Weinberg, *Phys. Lett. B* **251**, 288 (1990).
 - [41] S. Weinberg, *Nucl. Phys.* **B363**, 3 (1991).
 - [42] J. Gegelia, *Phys. Lett. B* **429**, 227 (1998).
 - [43] M. C. Birse, J. A. McGovern, and K. G. Richardson, *Phys. Lett. B* **464**, 169 (1999).
 - [44] M. B. Wise, *Phys. Rev. D* **45**, R2188 (1992).
 - [45] M. Pavon Valderrama and E. Ruiz Arriola, *Ann. Phys. (N.Y.)* **323**, 1037 (2008).
 - [46] M. Neubert, *Phys. Rep.* **245**, 259 (1994).
 - [47] A. V. Manohar and M. B. Wise, *Cambridge Monogr. Part. Phys., Nucl. Phys., Cosmol.* **10**, 1 (2000).
 - [48] T. D. Cohen, B. A. Gelman, and U. van Kolck, *Phys. Lett. B* **588**, 57 (2004).
 - [49] V. Lensky and M. C. Birse, *Eur. Phys. J. A* **47**, 142 (2011).
 - [50] V. Baru, A. Filin, C. Hanhart, Y. Kalashnikova, A. Kudryavtsev, and A. V. Nefediev, *Phys. Rev. D* **84**, 074029 (2011).
 - [51] A. Nogga, R. G. E. Timmermans, and U. van Kolck, *Phys. Rev. C* **72**, 054006 (2005).
 - [52] M. Pavon Valderrama, *Phys. Rev. C* **83**, 024003 (2011).
 - [53] M. Pavon Valderrama, *Phys. Rev. C* **84**, 064002 (2011).
 - [54] B. Long and C. Yang, *Phys. Rev. C* **84**, 057001 (2011).
 - [55] B. Long and C. Yang, *Phys. Rev. C* **85**, 034002 (2012).
 - [56] E. Braaten, H.-W. Hammer, and T. Mehen, *Phys. Rev. D* **82**, 034018 (2010).
 - [57] T. Barford and M. C. Birse, *Phys. Rev. C* **67**, 064006 (2003).
 - [58] M. Pavon Valderrama and E. Ruiz Arriola, *Phys. Rev. C* **74**, 064004 (2006).

- [59] M. Pavon Valderrama and E. R. Arriola, [Phys. Rev. C **74**, 054001 \(2006\)](#).
- [60] M. Pavon Valderrama and E. Ruiz Arriola, [Phys. Rev. C **83**, 044002 \(2011\)](#).
- [61] S. Ohkoda, Y. Yamaguchi, S. Yasui, K. Sudoh, and A. Hosaka, [arXiv:1111.2921](#).
- [62] D. Entem, E. Ruiz Arriola, M. Pavon Valderrama, and R. Machleidt, [Phys. Rev. C **77**, 044006 \(2008\)](#).
- [63] M. Pavon Valderrama and E. Ruiz Arriola, [Phys. Rev. C **72**, 054002 \(2005\)](#).
- [64] V. Baru, E. Epelbaum, C. Hanhart, M. Hoferichter, A. Kudryavtsev *et al.*, [arXiv:1202.0208](#).
- [65] S. R. Beane, D. B. Kaplan, and A. Vuorinen, [Phys. Rev. C **80**, 011001 \(2009\)](#).
- [66] S. Ahmed *et al.* (CLEO Collaboration), [Phys. Rev. Lett. **87**, 251801 \(2001\)](#).
- [67] A. Anastassov *et al.* (CLEO Collaboration), [Phys. Rev. D **65**, 032003 \(2002\)](#).
- [68] A. Abada, D. Bećirević, P. Boucaud, G. Herdoiza, J. P. Leroy, A. Le Yaouanc, and O. Pène, [J. High Energy Phys. **02** \(2004\) 016](#).
- [69] S. Negishi, H. Matsufuru, and T. Onogi, [Prog. Theor. Phys. **117**, 275 \(2007\)](#).
- [70] W. Detmold, K. Orginos, and M. J. Savage, [Phys. Rev. D **76**, 114503 \(2007\)](#).
- [71] D. Becirevic, B. Blossier, E. Chang, and B. Haas, [Phys. Lett. B **679**, 231 \(2009\)](#).
- [72] W. Detmold, C.-J.D. Lin, and S. Meinel, [Phys. Rev. Lett. **108**, 172003 \(2012\)](#).
- [73] W. Detmold, C.-J.D. Lin, and S. Meinel, [arXiv:1203.3378](#).
- [74] B. El-Bennich, M. A. Ivanov, and C. D. Roberts, [Phys. Rev. C **83**, 025205 \(2011\)](#).
- [75] X.-Q. Li, F. Su, and Y.-D. Yang, [Phys. Rev. D **83**, 054019 \(2011\)](#).
- [76] Z.-F. Sun, J. He, X. Liu, Z.-G. Luo, and S.-L. Zhu, [Phys. Rev. D **84**, 054002 \(2011\)](#).