Causality and the conformal boundary of AdS space in real time holography

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We consider the holographic prescription problem in a (Lorentzian) AdS background, deriving from first principles the explicit formulas that relate the field at infinity with the field in the bulk. In contrast with the previous studies of the ''real-time'' holography problem, our derivation uses purely classical arguments that involve causality, as in the usual treatment of the holographic prescription problem in Wick-rotated spaces of Euclidean signature. We show that there is a unique propagator that preserves causality and see that this provides a simple picture of the relationship between the bulk manifold and its conformal boundary.

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The gravity/gauge theory duality [1] is a conjectured duality between gravitational theories in certain background geometries and dual gauge theories (with no gravity) on their lower-dimensional boundary manifolds. This far-reaching set of ideas occupies a central position in theoretical physics, as this duality implies that strongly coupled gauge theories can be analyzed using gravity (thus opening up new perspectives in the subject) and, conversely, that spacetime geometry emerges from gauge-theoretical data.

Most authors analyze duality for theories in spaces of Euclidean signature, which are heuristically obtained from the Lorentzian ones through Wick rotation. While this ''imaginary time'' formulation certainly suffices for many purposes, there are also good reasons for developing the ''real time'' theory. In particular, it is well known that Wick rotation does not allow to faithfully capture the field dynamics, so a real-time analysis would be necessary, for example, for studying nonstationary spacetimes or time-dependent phenomena. An important example thereof arises in the holographic modeling of the quark-gluon plasma, which has actually motivated a significant portion of the recent work on real-time holography (cf. [2] and references therein).

Our goal in this paper is to present a first principles prescription for determining holographically the field in the bulk from a field on the boundary, and to analyze the implications of this prescription on the geometry of the problem. Real-time holography has been discussed in several papers, including [3] and, especially, [4,5], where real-time holographic prescriptions were given. We will characterize all the possible prescriptions (and therefore all the associated dynamics on the boundary) in terms of the data for a mixed initial-boundary value problem. The final formulas we obtain for the propagators are identical to those obtained in [4,5]; in particular, the AdS/CFT dictionary discussed in these references is still valid in the present context. What is very different, though, is both the picture that arises of the underlying boundary manifold and the arguments that lead to the prescription. In particular, and contrary to what happens in [5], we will consider neither several copies of the bulk manifold nor Euclidean caps. Among all possible prescriptions for the bulk-boundary propagator, we will single out the unique one for which the propagation is causal. As we will see, this prescription is entirely holographic.

One remarkable feature of the previous explorations of the holographic prescription in Lorentzian signature is that, even in the simplest cases, the discussions rely heavily on quantum field theoretic arguments. This is totally unlike what happens in its counterpart of Euclidean signature, that is, in the ball model of hyperbolic space, as considered e.g. in [6]. Hence in our discussion we will strive to make the discussion entirely ''classical.'' As we shall see, this is will also provide a much clearer picture of the role that the basic physical properties of the field equations (especially causality) play in the prescription.

As is customary, we shall present the prescription in the simplest case: a scalar, possibly massive field ϕ living in anti-de Sitter space AdS_{d+1}. For simplicity of notation, we will carry out the discussion in dimension $2+1$; the arguments extend almost verbatim to the higher-dimensional case. The metric can be written as

$$ds^2 = \frac{-dt^2 + dx^2 + \cos^2 x d\theta^2}{\sin^2 x},$$

where AdS_3 is covered by the coordinates (x, θ, t) whose respective ranges are the interval $(0, \frac{\pi}{2}]$, the circle $\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$ and the real line. (We are using the same coordinates as in [7], where the reader can find more detailed descriptions.) There is a coordinate singularity at the

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"origin" $x = \frac{\pi}{2}$, where the spacelike spheres $\{x\} \times \mathbb{S}^1$ contract to a point. We will identify the boundary of AdS_3 with the cylinder $\{x = 0\}$, so that it can be described by coordinates $(\theta, t) \in \mathbb{S}^1 \times \mathbb{R}$ and is naturally endowed with the conformal Lorentzian metric $-dt^2 + d\theta^2$.

Mathematically, mapping a scalar field $\phi_0(\theta, t)$ in the boundary to a field $\phi(x, \theta, t)$ in the bulk is some kind of "boundary problem at infinity" for a wave equation in AdS_3, which is a rather nonstandard problem in partial differential equations problem that nevertheless arises naturally in the context of the gravity/gauge theory duality. The equation that governs a scalar field in AdS_3 is $\Box\phi = m^2\phi$, where $\Box$ is the wave operator and for simplicity we assume that the mass m is positive. In our coordinates, this equation reads

$$\partial_t^2\phi - \partial_x^2\phi - \frac{\partial_x\phi}{\sin x \cos x} - \frac{\partial_\theta^2\phi}{\cos^2 x} + \frac{m^2\phi}{\sin^2 x} = 0. \quad (1)$$

We will be interested in reasonably well-behaved fields that do not grow exponentially fast in time (in particular, for fixed x they define a tempered distribution), so we can Fourier-transform ϕ in θ and t. The resulting field will be denoted by $\hat{\phi}(x, k, \omega)$, with $k \in \mathbb{Z}$ and $\omega \in \mathbb{R}$. In the Fourier picture, Eq. (1) then reads

$$\left[\partial_x^2 + \frac{1}{\sin x \cos x}\partial_x - \frac{k^2}{\cos^2 x} - \frac{m^2}{\sin^2 x} + \omega^2\right]\hat{\phi}(x, k, \omega) = 0. \quad (2)$$

This is an ODE of hypergeometric type, whose general solution is the linear combination

$$\hat{\phi}(x, k, \omega) = \mathcal{C}(k, \omega)G(x, k, \omega) + \tilde{\mathcal{C}}(k, \omega)\tilde{G}(x, k, \omega),$$

where

$$G(x, k, \omega) := (\sin x)^{1+\nu}(\cos x)^{\sigma}F(\zeta^{\omega}_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}, 1 + \sigma; \cos^2 x),$$

$$\tilde{G}(x, k, \omega) := (\sin x)^{1+\nu}(\cos x)^{-\sigma}F(\zeta^{\omega}_{\nu,-\sigma}, \zeta^{-\omega}_{\nu,-\sigma}, 1 - \sigma; \cos^2 x),$$

$F(a, b, c; z)$ stands for the Gauss hypergeometric function, $\zeta^{\omega}_{\nu,\sigma} := (\nu + \sigma + \omega + 1)/2$ and we use the notation $\nu := (m^2 + 1)^{1/2}$ and $\sigma := (k^2 + \frac{1}{4})^{1/2}$. Since the solution $\tilde{G}$ does not lead to a well-behaved function at the point $x = \frac{\pi}{2}$, it follows that we must have

$$\hat{\phi}(x, k, \omega) = \mathcal{C}(k, \omega)G(x, k, \omega), \quad (3)$$

where the "constant" $\mathcal{C}(k, \omega)$ is, in general, a distribution in the variables (k, ω).

Our goal now is to ascertain to what extent the coefficients $\mathcal{C}(k, \omega)$ in Eq. (3) are determined by the condition that ϕ tends to ϕ_0, in an appropriate sense, at infinity. To this end, let us consider the behavior of the function $G(x, k, \omega)$ as $x \to 0$. For simplicity, we will make the technical assumption that ν is not an integer, which is true for almost all values of the mass m and implies that $\nu > 1$; the case of integer ν can be dealt with similarly. In this case, by the well-known formula [[8], 15.3.6] the function G can be decomposed as

$$G(x, k, \omega) = \alpha_-(k, \omega)G_-(x, k, \omega) + \alpha_+(k, \omega)G_+(x, k, \omega), \quad (4)$$

where the functions

$$G_{\pm}(x, k, \omega) := (\sin x)^{1\pm\nu}(\cos x)^{\sigma} \times F\left(\zeta^{\omega}_{\pm\nu,\sigma}, \zeta^{-\omega}_{\pm\nu,\sigma}, \frac{1}{2} \pm \left(\nu - \frac{1}{2}\right); \sin^2 x\right)$$

respectively have the asymptotics $x^{1\pm\nu} + \mathcal{O}(x^{2\pm\nu})$ as $x \to 0$, and the coefficients

$$\alpha_{\pm}(k, \omega) := \frac{\Gamma(1 + \sigma)\Gamma(\mp\nu)}{\Gamma(\zeta^{\omega}_{\mp\nu,\sigma})\Gamma(\zeta^{-\omega}_{\mp\nu,\sigma})}$$

are well-behaved functions of k and ω.

Since G_- dominates G_+ at infinity, whenever the coefficient $\alpha_-(k, \omega)$ is nonzero, $G(x, k, \omega)$ then behaves as a multiple of $x^{1-\nu}$ for small x, so it is apparent that the assertion that ϕ tends to ϕ_0 at infinity should make precise the idea that $\phi(x, \theta, t)$ behaves as $x^{1-\nu}\phi_0(\theta, t)$ as $x \to 0$. A convenient way of making sense out of this intuition [9] is to *define* that the function $\phi(x, \theta, t)$ has (generally, a distribution) $\phi_0(\theta, t)$ as boundary value if for any test function $\psi(\theta, t)$

$$\lim_{x\to 0^+} x^{\nu-1}\int \phi(x, \theta, t)\psi(\theta, t)d\theta dt = \int \phi_0(\theta, t)\psi(\theta, t)d\theta dt,$$

where henceforth all integrals (or sums) are to be understood in the sense of distributions. Equivalently, we could have expressed this condition in terms of the Fourier representations of ϕ, ϕ_0 and ψ, so from Eqs. (3) and (4) and from the behavior of $G_{\pm}(x, k, \omega)$ as $x \to 0$ we infer that the Fourier transforms of the field in the bulk ϕ and the boundary field ϕ_0 are related via the distributional equation

$$\alpha_-(k, \omega)\mathcal{C}(k, \omega) = \hat{\phi}_0(k, \omega). \quad (5)$$

Our task is now to compute all solutions to the above equation, which in turn encode all possible prescriptions for the bulk field ϕ compatible with the boundary field ϕ_0. We will start by computing a particular solution to the inhomogeneous Eq. (5). The basic idea is that $\mathcal{C}(k, \omega)$ must be some regularization of $\hat{\phi}_0(k, \omega)/\alpha_-(k, \omega)$, so that by Eq. (3) the relation between ϕ and ϕ_0 would be essentially

$$\hat{\phi}(x, k, \omega) = H(x, k, \omega)\phi_0(k, \omega), \quad (6)$$

with

$$H(x, k, \omega) := G(x, k, \omega)/\alpha_-(k, \omega). \quad (7)$$

A problem with the above formula is that $\alpha_-(k, \omega)$ vanishes at the points $\omega = \pm\omega_{kj}$, with

$$\omega_{kj} := 1 + \sigma + \nu + 2j, \qquad j = 0, 1, \ldots,$$

so that the function $H(x, k, \omega)$ has poles at these points. It can be checked that these poles are simple. Following the treatment of the wave equation in Minkowski space [10], it is natural to take a small positive parameter ϵ and try to define a propagator as

$$\hat{K}(x, k, \omega) = \lim_{\epsilon \to 0^+} H(x, k, \omega - \mathrm{i}\epsilon). \tag{8}$$

This is analogous to the retarded propagator in Minkowski space. In fact, if one formally applies the well-known reasoning using complex integration and the residue theorem as in [[10], Sect. 1.3.1], one formally derives that the inverse Fourier transform of $\hat{K}(x, k, \omega)$ with respect to time, which we denote by $K(x, k, t)$, is identically zero for $t < 0$. Hence from (8) one derives, again formally, that

$$\phi(x, \theta, t) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{t} K(x, k, t - t') \mathrm{e}^{\mathrm{i}k(\theta - \theta')} \phi_0(\theta', t') \frac{d\theta' dt'}{2\pi}. \tag{9}$$

This formula, once it is proved to be valid, will make it manifest that this particular prescription for the bulk-boundary propagator is causal.

Unfortunately, making sense out of this formula is considerably harder than it is for its counterpart in Minkowski space. In Minkowski space, the issue is settled by introducing an additional exponentially decaying factor of the form $\mathrm{e}^{-\eta|t|}$ in the integrand to obtain converging Fourier integrals and then taking the limit $\eta \to 0^+$. This is not as straightforward to implement in anti-de Sitter space because one does not know *a priori* the behavior of the function $H(x, k, \omega + \mathrm{i}\epsilon)$ for large values of ω (so we do not even know if its inverse Fourier transform is well defined as a distribution) or for large negative values of ϵ (which is crucial in the contour integration argument [10]). It should be noted that these problems arise when considering $H(x, k, \omega - \mathrm{i}\epsilon)$ for any fixed ϵ, and not just only in the limit. For these reasons, we will sketch at the end of this paper an argument showing that the limit (8) exists and that the formula (9) is therefore rigorously justified.

We now parametrize all the different possible prescriptions for the field ϕ. According to Eq. (3), the difference between the constant $\mathcal{C}$ characterizing this solution and the prescription (9) (which corresponds to a particular solution of this equation) will be characterized by a solution $\mathcal{C}_{\mathrm{hom}}(k, \omega)$ to the homogeneous equation $\alpha_- \mathcal{C}_{\mathrm{hom}} = 0$. A solution of this equation must be supported on the zero set of the function $\alpha_-(k, \omega)$, so it can be readily checked using [[11], Th. V.11] that $\mathcal{C}_{\mathrm{hom}}$ must be a sum of delta functions:

$$\mathcal{C}_{\mathrm{hom}}(k, \omega) = \sum_{-\infty}^{\infty} (a_{kj}^{+} \delta(\omega - \omega_{kj}) + a_{kj}^{-} \delta(\omega + \omega_{kj})).$$

Here, $a_{kj}^{\pm}$ are complex constants. Using Eq. (3) and the fact that $G(x, k, \omega) = G(x, k, -\omega)$, we can write the bulk field corresponding to $\mathcal{C}_{\mathrm{hom}}$ as

$$\phi_{\mathrm{hom}}(x, \theta, t) = \sum_{k=-\infty}^{\infty} (a_{kj}^{+} \mathrm{e}^{\mathrm{i}\omega_{kj} t} + a_{kj}^{-} \mathrm{e}^{-\mathrm{i}\omega_{kj} t}) \mathrm{e}^{\mathrm{i}k\theta} G(x, k, \omega_{kj}). \tag{10}$$

The content of Eq. (3) is that any prescription for the bulk field ϕ differs from the particular solution (9) by a function ϕ_{hom} of the form (10) and, conversely, any such linear combination is a valid prescription for the field ϕ.

We can now give an interpretation for the complex coefficients $a_{kj}^{\pm}$ parametrizing all the possible prescriptions through the following basic idea. Let us assume, for the sake of simplicity, that the boundary field ϕ_0 is smooth and has very good decay properties; more concretely, let us assume that it is compactly supported in time and that its angular Fourier transform only has a finite number of frequencies (functions of this type are dense e.g. in the space of L^2 functions of θ and t). Hence, we can take a large negative time T such that $\phi_0(\theta, t)$ is zero for all $t \leq T$. The particular bulk field given by (9) is then obviously zero at time T, so that the value at time T of any bulk field ϕ associated with this boundary condition will coincide with that of the homogeneous field (10). Now, by separation of variables and Sturm–Liouville theory [11], the functions $\psi_{kj}(x, \theta) := \mathrm{e}^{ik\theta} G(x, k, \omega_{kj})$ form an orthogonal basis of the spatial L^2 space. Hence, one can expand the value of the field ϕ and its time derivative $\partial_t \phi$ at time T in this basis and find out that the coefficients in this expansion are in a one-to-one correspondence with the coefficients $a_{kj}^{\pm}$. More concretely, if we denote, respectively, by A_{kj} and B_{kj} the components of the field and its velocity at time T in the basis ψ_{kj}, one readily finds that they determine the coefficients $a_{kj}^{\pm}$ characterizing the holographic prescription through the relation

$$a_{kj}^{\pm} = \frac{\mathrm{e}^{\mp \mathrm{i}\omega_{kj} T}}{2} \left(A_{kj} \pm \frac{B_{kj}}{\mathrm{i}\omega_{kj}} \right). \tag{11}$$

Equation (11) simply asserts that any choice of bulk-boundary propagator is completely specified by a choice of "initial conditions," that is, of the choice of the value of the field and its time derivative at any fixed time T. From this it follows that the only choice for the bulk field ϕ that is causal corresponds to $\phi_{\mathrm{hom}} = 0$, i.e., is given by (9). This is because, when the boundary field is compactly supported, causality demands that $(\phi, \partial_t \phi)$ should vanish for sufficiently large negative times T, essentially because the boundary condition has not yet been "switched on." But by Eq. (11) this is only possible when all the coefficients $a_{kj}^{\pm}$ are zero, which yields $\phi_{\mathrm{hom}} = 0$, and proves the uniqueness of the causal propagator.

Some remarks are now in order. The main result in this paper is the derivation of the formula (9) for a holographic prescription in an AdS background (with its Lorentzian signature), which we have obtained from first principles without resorting to any quantum field theory methods.

This is considerably harder than in the Euclidean case [6], since one needs to deal with the nonuniqueness of solutions to the prescription problem, an issue that arises from the absence of ellipticity and manifests itself in the existence of multiple solutions to Eq. (5). All possible solutions to the prescription problem in AdS geometry have been characterized in this paper in terms of data corresponding to a mixed boundary-initial value problem.

It should be stressed that, from the point of view adopted in this paper, the causal prescription (9) is fully holographic in the sense that, as it corresponds to choosing zero Cauchy data below the support of the boundary field (or, to put it differently, at "$t = -\infty$"), it describes the case where the whole bulk field is radiated from the boundary. In our approach, it is the only prescription for the propagator with this property.

It is worth pointing out that nearly all our computations so far are purely distributional (the only but important exception is the justification of the validity of the formula (9), which we shall present below). We also note that the use of energy estimates will lead to existence results in more general spacetimes but not to the explicit representation (9) for the causal propagator (and actually for any other propagator (10)). We will address this problem in a future work. The regularity conditions we have imposed on the boundary field are inessential: one can easily deal with more general conditions through a standard approximation argument.

We should remark at this point the picture of the conformal boundary of AdS that emerges from our treatment is rather different from what has been previously suggested. The most comprehensive treatment to date of the real-time holographic prescription problem is surely that of Skenderis and van Rees [5]. We have seen that our boundary manifold is a cylinder, which is consistent with the fact that AdS (our bulk manifold) is topologically a solid cylinder. Furthermore, the boundary manifold inherits a conformal Lorentzian metric, so its associated gauge theories are obviously expected (and can be shown) to be governed by wave-type equations, as befits a Lorentzian background. In Skenderis and van Rees's picture, the boundary manifold consists of several components, the actual number of which depends on the choice of the bulk-to-boundary propagator. It involves several copies of the Lorentzian cylinder that appears in our picture plus several spherical caps, which do not carry a Lorentzian metric but a Euclidean one. The bulk manifold that is holographically filled by the datum on the boundary in their picture consists of several copies of AdS and the (Euclidean) hyperbolic half-space associated with each cap. The holographic prescription is finally obtained by an ingenious argument which posits that the fields in the different components of the bulk should satisfy certain matching conditions.

To conclude, let us sketch the argument that allows us to prove the existence of the causal propagator of kernel $K(x, \theta, t)$. The first thing one has to prove is that the inverse time Fourier transform of $H(x, k, \omega - i\epsilon)$ exists as a distribution, when ϵ is a positive number. For this, it suffices to show that the absolute value of H is bounded by a power of ω, i.e., that is does not grow exponentially fast for large $|\omega|$ (this shows that H is a tempered distribution in ω, which is the most convenient setting for Fourier transforms [11]). For this, let us start by recalling (see (7)) that H is of the form

$$\frac{H(x,k,\omega)}{A_1(x,k)} = \Gamma\left(\frac{a+\omega}{2}\right)\Gamma\left(\frac{a-\omega}{2}\right)F\left(\frac{b+\omega}{2}, \frac{b-\omega}{2}, c; \cos^2 x\right)$$

where $A_1(x, k)$, a, b and c are quantities that depend on k but not on ω (they can be easily read off (3) and (5)), and where we consider any complex value of ω. Using the reflection formula [[8], 6.1.17] and the asymptotics of the Gamma function [[8], 6.1.47], we readily derive that

$$\Gamma\left(\frac{a+\omega}{2}\right)\Gamma\left(\frac{a-\omega}{2}\right) = \frac{C\Gamma(\frac{\omega+a}{2})\Gamma(\frac{\omega-a}{2})^{-1}}{(\omega-a)\sin\frac{\pi(\omega-a)}{2}} = \frac{C'\omega^{a-1} + \text{l.o.t.}}{\sin\frac{\pi(\omega-a)}{2}}$$

when $|\omega|$ is large for a nonzero, k-dependent constant C'. For the hypergeometric function, there is a uniform asymptotic formula due to Jones [12] which ensures that

$$F\left(\frac{b+\omega}{2}, \frac{b-\omega}{2}, c; \cos^2 x\right) = \frac{A_2(x,k)J_{c-1}(\omega(\frac{\pi}{2}-x))}{\omega^{c-1}} + \text{l.o.t.}$$

Here, $A_2(x, k)$ is independent of ω and J_{c-1} denotes the Bessel function, which admits the asymptotic expansion [[8], 9.2.1]

$$J_{c-1}\left(\omega\left(\frac{\pi}{2}-x\right)\right) = \frac{A_3(x,k)}{\omega^{1/2}}\sin\left(\omega\left(\frac{\pi}{2}-x\right) + A_3(x,k)\right) + \text{l.o.t.},$$

where $A_3(x, k)$ is independent of ω. Put together, these formulas imply that for any nonzero ϵ and large real values of $|\omega|$ one has the power-type bound $|H(x, k, \omega - i\epsilon)| \le A_4(x, k, \epsilon)|\omega|^{b-c-(1/2)}$, which guarantees that these functions have a well defined Fourier transform. Here, A_4 is independent of ω and blows up only for $\epsilon \to 0$.

In order to use the contour integration argument to show that the inverse time Fourier transform of $H(x, k, \omega - i\epsilon)$ (for a fixed positive ϵ) is zero for $t < 0$, one needs to show that one can close the contour [10], which amounts to showing that for $t < 0$ and frequencies ω with very large and negative imaginary part, the contribution to the integral becomes negligible. Since $|\sin z| = \frac{1}{2} e^{|\mathrm{Im} z|} + \text{l.o.t.}$, this follows from the above asymptotic formulas without much effort. The fact the distributional limit (8) exists also follows from the asymptotics, as $H(x, k, \omega)$ can the be decomposed as a regular polynomially bounded function plus a sum of terms of the form $c_{kj}(\omega \pm \omega_{kj} - i\epsilon)^{-1}$ admitting simple poles, with polynomially bounded coefficients, that can be dealt with using a standard argument [[11], Exercise V.22].

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