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Fermion production in a Coulomb field on a de Sitter universe

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Fermion production in an external Coulomb field on a de Sitter expanding universe is studied. The amplitude and probability of pair production in an external Coulomb field are computed and the cases of large/small values of the expansion factor comparatively with the particle mass are studied. We obtain from our calculations that the modulus of the momentum is no longer a conserved quantity. We find that in the de Sitter space there are probabilities for production processes where the helicity is no longer conserved. For a vanishing expansion factor, we recover the Minkowski limit where the amplitude of this process vanishes. The rate of pair production in a Coulomb field is found to be important in the early Universe when the expansion factor was large comparatively with the particle mass.

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I. INTRODUCTION

The problem of pair production in an external gravitational field was intensively studied in the literature beginning with the paper by Schrödinger [1], who first suggested that in an expanding universe particle production could arise. The next important step in this direction was taken by L. Parker [2–4] who understood quantitatively the phenomenon of pair production in a gravitational field. The main results of L. Parker's work are related to the computation of the number density of scalar particles and Dirac particles created in an expanding background. Also, the limit of zero mass and infinite mass was addressed and in addition, an estimation of the amount of particle production for the present-day expansion was made. Therefore, in the present paper we will compare our results with these important predictions made by L. Parker in [2–4], even if we will use another approach based on perturbations.

In this paper, we will follow the usual quantum field theory approach, where the transition amplitudes are defined as in [5] using perturbation methods. This approach was also used in [6–11], where transitions that generate particle production from vacuum were also studied on Friedmann-Robertson-Walker metrics. With this general approach, we can study the limit of these transition amplitudes calculated on de Sitter QED, for a vanishing expansion factor. This corresponds to the flat space limit, where the amplitude of this process vanishes and the connection with the Minkowski QED can be done. Also, from our general result, one could address the interesting case of large expansion, which corresponds to the early Universe where the rate of pair production becomes important. As it was well established in the literature, particle production could arise as the result of the fact that the in and out vacua are not the same [12–14], or because of the loss of the translational invariance with respect to time [5], which in loose terms refers to the well-known fact that in this geometry the law of conservation for energy is lost [2–5]. It is well known that in the Minkowski theory [15–18] the pair production in a Coulomb field could not arise as a perturbative phenomenon. Contrary to this, in de Sitter space the translational invariance with respect to time is lost and one should expect a nonvanishing probability of transition for the process: vacuum $\rightarrow e^- + e^+$ in the presence of a Coulomb potential. It is also important to mention that the electron-positron pair could also be absorbed by the Coulomb field in the de Sitter geometry, which represents the time-reversed process. The transition amplitudes will be calculated here using only the mode expansion in momentum basis, which have a defined momentum and helicity. The fermion production in a Coulomb field in de Sitter space was to the best of our knowledge not studied so far. We also specify that our results are in good agreement with the work previously done on the particle production in a pure gravitational field, which predicts important amounts of particle creation in the early stage of the Universe [2–4].

It is worth mentioning that important results were obtained in [12,13], where the production of fermionic particles and scalar particles in electric fields on de Sitter geometry was considered, using a nonperturbative approach. With this approach, one can obtain the density number of particles created in the far-out region and because there are some constraints related to the mass of the particle [13,14], the interesting case of large expansion from the early Universe is not properly addressed. For the Dirac field, the results presented in [12] show that there are nonvanishing rates for fermion production of mass zero and of finite mass in electric fields in de Sitter space. A few comments need to be made here related to the production of zero mass fermions. In [2–4] and more recently in [14], it was proven that because in the massless case the field is conformally coupled, there is no production of fermions with zero mass. It will be interesting to compare our result which was obtained using the exact solutions of the field equation with the above-mentioned results [2–4,12–14], in

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the massless case. This is because the results from [2–4,14] seem to be in contradiction with [12,13], in the null mass case.

From another point of view, it is important to directly prove that the transition amplitudes of pair creation in an external Coulomb field contain terms that break the conservation laws, which seems to have been known for a long time but was never proven in a direct calculation using the solutions of free field equations. For these reasons, we hope that the results presented here will be of interest for cosmology and quantum field theory on Friedmann-Robertson-Walker backgrounds.

The second section of our paper begins with a short reminder of the work that was done in the theory of free fields on de Sitter QED. In the third section, we present the main steps for calculating the transition amplitude corresponding to the process of pair production in the Coulomb potential in de Sitter geometry and we compute the probability for pair production. Physical consequences of our calculations are also analyzed in this section. The fourth section is dedicated to the study of limit cases as the expansion factor is small/large comparatively with the mass of the particle. Our conclusions are summarized in section five.

II. FREE FIELDS ON DE SITTER SPACE

The definition of transition amplitudes is expressed with the help of the exact solutions of the free Dirac and Maxwell equations on the de Sitter geometry, expanded in the momentum basis. For this reason, we make a short review of the results from free field theory in this background. The line element [19] for a de Sitter universe is

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x}^2, \tag{1}$$

where ω is the expansion factor and $\omega > 0$. Now, we know that for defining fields with half-integer spin on curved spacetime it is required to use the tetrad fields [19] $e_{\hat{\mu}}(x)$ and $\hat{e}^{\hat{\mu}}(x)$, fixing the local frames and corresponding coframes, which are labeled by the local indices $\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3$. The form of the line element allows us to choose the simple Cartesian gauge with the nonvanishing tetrad components

$$e^0_{\hat{0}} = e^{-\omega t}; \qquad e^i_{\hat{j}} = \delta^i_{\hat{j}} e^{-\omega t} \tag{2}$$

so that $e_{\hat{\mu}} = e^{\nu}_{\hat{\mu}} e_\nu$ and have the orthonormalization properties $e_{\hat{\mu}} e_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, $\hat{e}^{\hat{\mu}} e_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$ with respect to the Minkowski metric $\eta = \mathrm{diag}(1, -1, -1, -1)$.

Now, let us introduce unit normalized helicity spinors [16] for an arbitrary momentum vector $\vec{p}$ and denote them $\xi_\lambda(\vec{p})$ and $\eta_\sigma(\vec{p}) = i\sigma_2[\xi_\sigma(\vec{p})]^*$,

$$\vec{\sigma}\,\vec{p}\,\xi_\lambda(\vec{p}) = 2p\lambda\xi_\lambda(\vec{p}) \tag{3}$$

with $\lambda = \pm 1/2$ and where $\vec{\sigma}$ are the Pauli matrices and $p = |\,\vec{p}\,|$. The particle spinors have the form

$$\begin{aligned} \xi_{\frac{1}{2}}(\vec{p}) &= \sqrt{\frac{p_3 + p}{2p}} \begin{pmatrix} 1 \\ \frac{p_1 + ip_2}{p_3 + p} \end{pmatrix}, \\ \xi_{-\frac{1}{2}}(\vec{p}) &= \sqrt{\frac{p_3 + p}{2p}} \begin{pmatrix} \frac{-p_1 + ip_2}{p_3 + p} \\ 1 \end{pmatrix}. \end{aligned} \tag{4}$$

Then, the positive/negative frequency modes of momentum $\vec{p}$ and helicity λ that were derived in [20], assuming gamma matrices in Dirac representation, are

$$\begin{aligned} U_{\vec{p},\lambda}(t,\vec{x}) &= \frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}} \begin{pmatrix} \frac{1}{2} e^{\pi k/2} H^{(1)}_{\nu_-}(\frac{p}{\omega} e^{-\omega t}) \xi_\lambda(\vec{p}) \\ \lambda e^{-\pi k/2} H^{(1)}_{\nu_+}(\frac{p}{\omega} e^{-\omega t}) \xi_\lambda(\vec{p}) \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - 2\omega t}; \\ V_{\vec{p},\lambda}(t,\vec{x}) &= \frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}} \begin{pmatrix} -\lambda e^{-\pi k/2} H^{(2)}_{\nu_-}(\frac{p}{\omega} e^{-\omega t}) \eta_\lambda(\vec{p}) \\ \frac{1}{2} e^{\pi k/2} H^{(2)}_{\nu_+}(\frac{p}{\omega} e^{-\omega t}) \eta_\lambda(\vec{p}) \end{pmatrix} \\ &\quad \times e^{-i\vec{p}\cdot\vec{x} - 2\omega t}, \end{aligned} \tag{5}$$

where $H^{(1)}_\nu(z)$, $H^{(2)}_\nu(z)$ are Hankel functions of the first and second kind and $k = \frac{m}{\omega}$, $\nu_\pm = \frac{1}{2} \pm ik$.

Here, a few comments about the theory of free Dirac field on a de Sitter background need to be made. The mode functions for the Dirac field on de Sitter space were first derived exactly in [21], where also the Green function for the Dirac field was computed. It is worth mentioning that these solutions for the Dirac field in the Robertson-Walker metric were obtained also in [22], with the specification that the normalization constants were not determined. As was shown in [20,21], these constants depend on physical quantities such as momentum and play a central role in an exact calculation. In the present paper, we use the more recently derived solutions from [20].

These solutions are normalized such that [20]

$$\int d^3x (-g)^{1/2} \bar{U}_{\vec{p},\lambda}(x) \gamma^0 U_{\vec{p}',\lambda'}(x) = \delta_{\lambda\lambda'} \delta^3(\vec{p} - \vec{p}'), \tag{6}$$

with the specification that the V solutions obey similar relations and the two modes are orthogonal to each other. For a better explanation of what mean positive/negative frequencies we can use the expansion of the Hankel functions:

$$H^{(1,2)}_\nu(z) \simeq \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm i(z - \nu\pi/2 - \pi/4)}. \tag{7}$$

For $t \to -\infty$, the modes defined above behave as positive/negative frequency modes with respect to conformal time $t_c = -e^{-\omega t}/\omega$:

$$U_{\vec{p},\lambda}(t,\vec{x}) \sim e^{-ipt_c}; \qquad V_{\vec{p},\lambda}(t,\vec{x}) \sim e^{ipt_c}, \tag{8}$$

which is precisely the behavior in the infinite past for positive/negative frequency modes that define the Bunch-Davies vacuum [5].

As in Minkowski space, the negative frequency modes are obtained with the charge conjugation operation [15,17]

$$U_{\vec{p},\lambda}(x) \to V_{\vec{p},\lambda}(x) = i\gamma^2\gamma^0(\bar{U}_{\vec{p},\lambda}(x))^T, \tag{9}$$

because the charge conjugation symmetry remains valid in a curved background.

In our calculation, we need the form of the Coulomb potential on de Sitter spacetime, which is dependent on the line element. The situation here is simple if we recall the conformal invariance of Maxwell's equations [23]. The Coulomb field in Minkowski space is $A^0 = \frac{Ze}{|\vec{x}|}$. Then, one finds for the corresponding de Sitter potential the following formula:

$$A^{\hat{0}}(x) = \frac{Ze}{|\vec{x}|}e^{-\omega t}, \qquad A^{\hat{j}}(x) = 0, \tag{10}$$

where the hatted indices indicate that we refer to the components in the local Minkowski frames $A^{\hat{\mu}} = e^{\hat{\mu}}_{\nu}A^{\nu}$. We also observe that (10) is just the expression from flat space with spatial distances dilated/contracted by a factor $e^{-\omega t}$.

III. AMPLITUDE CALCULATION

In this section, we will calculate the transition amplitude for pair production in a Coulomb field and we will explore the physical consequences of our result. The form of the transition amplitude can be established using the same methods as in the flat space case, as was shown in [5,6,9,10]. For pair production in an external electromagnetic field, supposing that the fields are coupled by the elementary electric charge e, the expression of the transition amplitude is

$$\mathcal{A}_{e^-e^+} = -ie\int d^4x[-g(x)]^{1/2}\bar{U}_{\vec{p},\lambda}(x)\gamma_\mu A^{\hat{\mu}}(x)V_{\vec{p}',\lambda'}(x), \tag{11}$$

where e is the unit charge of the field and the fields $U_{\vec{p},\lambda}(x)$, $V_{\vec{p}',\lambda'}(x)$ are supposed to be exact solutions of the free Dirac equation in the momentum basis. The amplitude defined above for particle production in the potential (10) can now be expressed with the help of the solutions of free Dirac equation (5). The integral that results splits in a spatial integral and a temporal integral. The form of the spatial integral is better known from flat space theory [16] and this result is incorporated in the next formula for the amplitude:

$$\mathcal{A}_{e^-e^+} = i\frac{e^2Z\sqrt{pp'}}{8\pi|\vec{p}+\vec{p}'|^2}\left[-sgn(\lambda')\int_0^\infty dzzH^{(2)}_{\nu_+}(pz)H^{(2)}_{\nu_-}(p'z) + sgn(\lambda)\int_0^\infty dzzH^{(2)}_{\nu_-}(pz)H^{(2)}_{\nu_+}(p'z)\right]\xi^+_\lambda(\vec{p})\eta_{\lambda'}(\vec{p}'), \tag{12}$$

where the new variable of integration is $z = -t_c$ and sgn is the signum function. The temporal integral that contains the influence of the gravity in this process is not so simple. We shall give here only the main steps for solving the integrals. Using the relations between Hankel and Bessel functions J [24,25], all the integrals from the above amplitude can be expressed in the form

$$\int_0^\infty dzzJ_\mu(pz)J_\nu(p'z), \tag{13}$$

which are known as Weber-Schafheitlin integrals [24,26]. Some aspects related to these integrals are given in the Appendix.

After we complete the calculations, the final result for the transition amplitude is expressed in terms of Gauss hypergeometric functions ${}_2F_1$, Beta Euler functions B, and unit step functions θ as follows:

$$\begin{aligned}\mathcal{A}_{e^-e^+} = i\frac{e^2Z}{16\pi|\vec{p}+\vec{p}'|^2}\bigg[&-sgn(\lambda')\left(p^{-1}\theta(p-p')f_{-k}\left(\frac{p'}{p}\right) + p'^{-1}\theta(p'-p)f_k\left(\frac{p}{p'}\right)\right)\\ &+ sgn(\lambda)\left(p^{-1}\theta(p-p')f_k\left(\frac{p'}{p}\right) + p'^{-1}\theta(p'-p)f_{-k}\left(\frac{p}{p'}\right)\right)\bigg]\xi^+_\lambda(\vec{p})\eta_{\lambda'}(\vec{p}').\end{aligned} \tag{14}$$

We specify that in the integrals (12) for the case $p' > p$ (in the case $p' < p$ the analysis is similar) we let the momentum p' have a small imaginary part $p' \to p' + i\epsilon$, which assures the convergence of our integral and makes the unit step function and the $f_k(\frac{p}{p'})$, $f_k(\frac{p'}{p})$ functions well defined. In Eq. (14), the newly introduced functions $f_k(\frac{p}{p'})$, or $f_k(\frac{p'}{p})$, are defined as

$$\begin{aligned}f_k\left(\frac{p}{p'}\right) = {}& ie^{-\pi k}\frac{(\frac{p}{p'})^{1+ik}}{ch^2(\pi k)}\frac{{}_2F_1(\frac{3}{2}, 1+ik; \frac{3}{2}+ik; (\frac{p}{p'})^2 - i0)}{B(-\frac{1}{2}, \frac{3}{2}+ik)} + \frac{(\frac{p}{p'})^{1+ik}}{ch^2(\pi k)}\frac{{}_2F_1(\frac{3}{2}, 1+ik; \frac{3}{2}+ik; (\frac{p}{p'})^2 - i0)}{B(-ik, \frac{3}{2}+ik)} - \frac{(\frac{p}{p'})^{-ik}}{ch^2(\pi k)}\\ &\times\frac{{}_2F_1(\frac{1}{2}, 1-ik; \frac{1}{2}-ik; (\frac{p}{p'})^2 - i0)}{B(ik, \frac{1}{2}-ik)} + ie^{\pi k}\frac{(\frac{p}{p'})^{-ik}}{ch^2(\pi k)}\frac{{}_2F_1(\frac{1}{2}, 1-ik; \frac{1}{2}-ik; (\frac{p}{p'})^2 - i0)}{B(\frac{1}{2}, \frac{1}{2}-ik)},\end{aligned} \tag{15}$$

with the observation that $f_{-k}(\frac{p}{p'})$ is obtained when one makes the substitution $k \to -k$ and $f_k(\frac{p'}{p})$ is obtained when one makes the substitution $p \rightleftarrows p'$. To simplify our formulas, we can introduce the following notation: $\chi = p/p'(= p'/p)$. A notable observation is that the result can be expressed without letting the momentum have a small imaginary part, in which case the argument χ in the hypergeometric functions must be considered in the interval $0 \leq \chi < 1$. This is the domain of convergence of the hypergeometric functions, because in the limit $\chi \rightarrow 1$ the hypergeometric functions become divergent. In this case, one should naturally ask what happens with the result of our integrals with Bessel functions in the limit $\chi = 1$. In this limit, the result of the integrals can be expressed in terms of delta Dirac functions $\delta(p - p')$, however, when summed, these terms cancel out between them, for each of the two integrals with Hankel functions.

From the final result given in (14) and (15), it is obvious that the transition amplitude is nonvanishing only for $p \neq p'$ and from here we conclude that the law of conservation for the modulus of the momentum is lost for the process of pair production in a Coulomb field on a de Sitter space. Let us make some comments about this result. The de Sitter geometry (1) has spatial translation invariance as an exact symmetry, so the momentum is conserved. The thing that breaks this symmetry is the Coulomb field, not the de Sitter background. From here, we can conclude that the background that enables the breaking of momentum conservation law is the external Coulomb field, not the de Sitter geometry. If one had just the de Sitter background, without the Coulomb field, there would still be certain types of particle production, but no violation of momentum conservation law [6–8].

It is also more intuitive to imagine the following situation. We know that the energy and momenta are conserved for a particle that interacts with a constant external field. So, in our case it is not surprising that the momentum conservation is broken due to the fact that the perturbation potential acts as an external time-dependent source. Thereby, we conclude that the Coulomb field will favor production from vacuum of electrons and positrons with different modulus of momenta $p' \neq p$.

It is clear that the contribution that is specific to the de Sitter geometry in our amplitude is contained in the function $f_k(\chi)$. This quantity encodes via the dependence of the parameter $k = m/\omega$, the effect of the expansion of space on the pair production process, and it is the quantity that enters in our probability and help us to take a closer look at the limit cases. For studying the validity of our result, in the sense that the functions defined in (15) are not divergent, we plot the real part (Re[f]) and imaginary part (Im[f]) of $f_k(\chi)$ as a function of the parameter k for different values of $\chi = p/p'(p'/p)$.

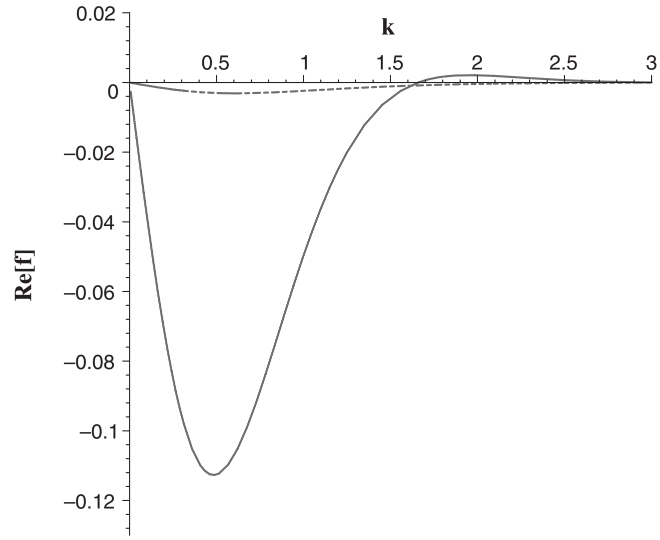

FIG. 1. The real part of $f_k(\chi)$ as a function of k. The solid line is for $\chi = 0.1$ and the dashed line for $\chi = 0.9$.

Our graphs show that both the real part and the imaginary part of the function $f_k(\chi)$ are finite in origin and are very convergent for large values of k (see Figs. 1–4). As χ takes small values, we observe that these functions become oscillatory (see Figs. 3 and 4). This oscillatory behavior is the result of the behavior of Gauss hypergeometric functions as their algebraic argument χ^2 approaches zero, combined with the oscillatory factors $\chi^{\pm ik}$.

Because we use here the methods based on perturbations, the outcome of our calculations must be the probabilities of transitions. Our results for the amplitude are used in defining the probabilities of fermion production in de Sitter space. The main quantity that enters in the probability is $|\mathcal{A}_{e^-e^+}|^2$, so we will focus on studying its properties. We can write down the expression of the probability of transition by squaring the amplitude:

$$\begin{aligned}\mathcal{P}_{e^-e^+} &= \frac{1}{2}\sum_{\lambda\lambda'}|\mathcal{A}_{e^-e^+}|^2\\ &= \frac{1}{2}\sum_{\lambda\lambda'}\frac{e^4Z^2}{256\pi^2|\vec{p}+\vec{p}\,'|^4}|\xi^+_\lambda(\vec{p})\eta_{\lambda'}(\vec{p}\,')|^2\bigg\{p'^{-2}\theta(p'-p)\bigg[\bigg|f_k\bigg(\frac{p}{p'}\bigg)\bigg|^2+\bigg|f_{-k}\bigg(\frac{p}{p'}\bigg)\bigg|^2-sgn(\lambda)sgn(\lambda')\\ &\quad\times\bigg(f_k\bigg(\frac{p}{p'}\bigg)f^*_{-k}\bigg(\frac{p}{p'}\bigg)+f^*_k\bigg(\frac{p}{p'}\bigg)f_{-k}\bigg(\frac{p}{p'}\bigg)\bigg)\bigg]+p^{-2}\theta(p-p')\bigg[\bigg|f_k\bigg(\frac{p'}{p}\bigg)\bigg|^2+\bigg|f_{-k}\bigg(\frac{p'}{p}\bigg)\bigg|^2-sgn(\lambda)sgn(\lambda')\\ &\quad\times\bigg(f_k\bigg(\frac{p'}{p}\bigg)f^*_{-k}\bigg(\frac{p'}{p}\bigg)+f^*_k\bigg(\frac{p'}{p}\bigg)f_{-k}\bigg(\frac{p'}{p}\bigg)\bigg)\bigg]\bigg\}.\end{aligned} \tag{16}$$

The total probability of electron-positron pair production in a Coulomb field will be

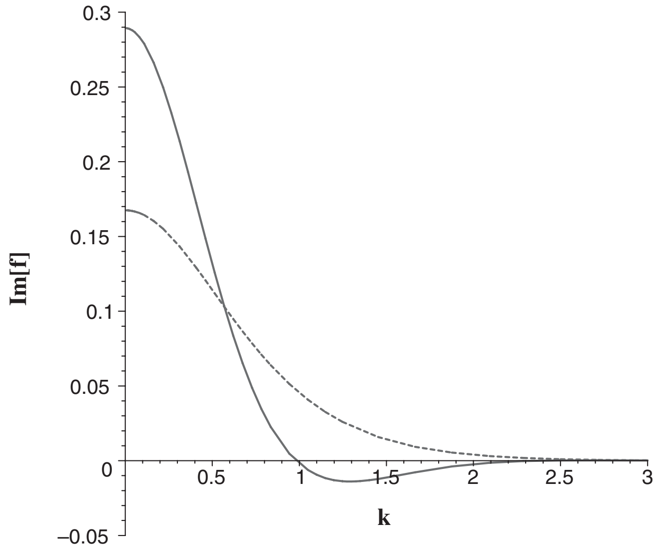

FIG. 2. The imaginary part of $f_k(\chi)$ as a function of k. The solid line is for $\chi = 0.1$ and the dashed line for $\chi = 0.9$.

$$\mathcal{P}^{\text{tot}}_{e^-e^+} = \int \mathcal{P}_{e^-e^+} \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3}, \tag{17}$$

and via the squared amplitude $|\mathcal{A}_{e^-e^+}|^2$ it is also determined by the factor $k = m/\omega$. Carrying out the integration over p, p' is a complicated task. The form of the integrals that need to be solved is not known in the literature and only a numerical estimation can be made. However, some physical consequences can be obtained from (16) by drawing the graph of probability of transition $\mathcal{P}_{e^-e^+}$ as a function of k, for different values of the ratio $\chi = p/p'(p'/p)$. As we know from Minkowski QED, the helicity conservation is a property that becomes important in the ultrarelativistic regime. Let us see what happens in the not-so-simple de Sitter case. From the above formula for probability of transition, we observe an interesting property. In the process of pair production in Coulomb field on de Sitter space, there are nonvanishing probabilities for both production processes that conserve or do not conserve the helicity. In the process under study, it is clear that helicity is conserved when $\lambda = -\lambda'$, so that the initial zero helicity equals the sum of the final helicities $\lambda + \lambda' = 0$. Contrary, the process in which helicity is not conserved is obtained when $\lambda = \lambda'$. It is interesting to obtain from here some quantitative results from the probability formula (16). These results will help us understand whether or not, in a Coulomb field, pair production processes with conserved/nonconserved helicity will be favored.

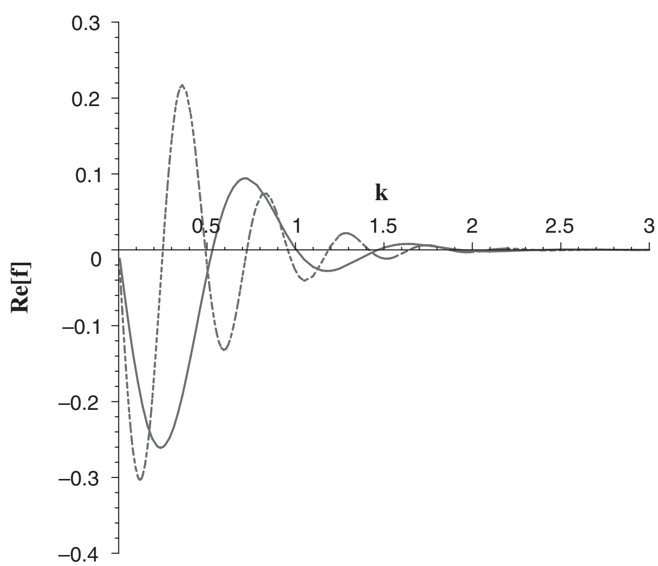

FIG. 3. The real part of $f_k(\chi)$ as a function of k. The solid line is for $\chi = 0.001$ and the dashed line for $\chi = 0.000\,001$.

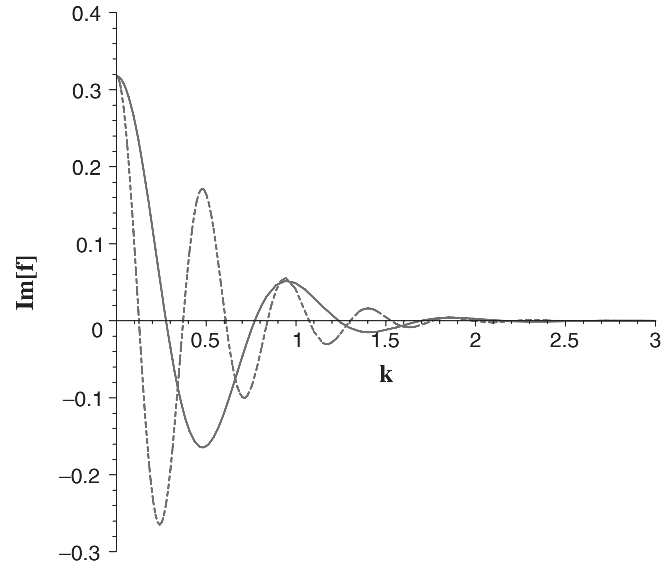

FIG. 4. The imaginary part of $f_k(\chi)$ as a function of k. The solid line is for $\chi = 0.001$ and the dashed line for $\chi = 0.000\,001$.

Let us make some comments about breaking the helicity conservation law. This can only happen for nonzero mass, so it is the mass, not the de Sitter background, that breaks the helicity conservation. One could object that there would not even be any particle production on flat space, and that would be correct, but this does not change the fact that the nonconservation of helicity is driven by the fermion mass and not by the de Sitter geometry.

First, we represent our probability of transition (16) as a function of $k = m/\omega$, for $\chi = 0.1$ and $\chi = 0.4$, both corresponding to close values of the modulus for the two momenta $p' \sim p$, but not equal. For $\chi = 0.9$, our graphical analysis shows that the probability for a helicity nonconserving process is zero, and for this reason we begin our graphical representation with $\chi = 0.4$.

Our numerical and graphical analyses show that the probability for pair production with opposite helicities $\lambda = -\lambda'$ (helicity conservation) is sensibly bigger than the

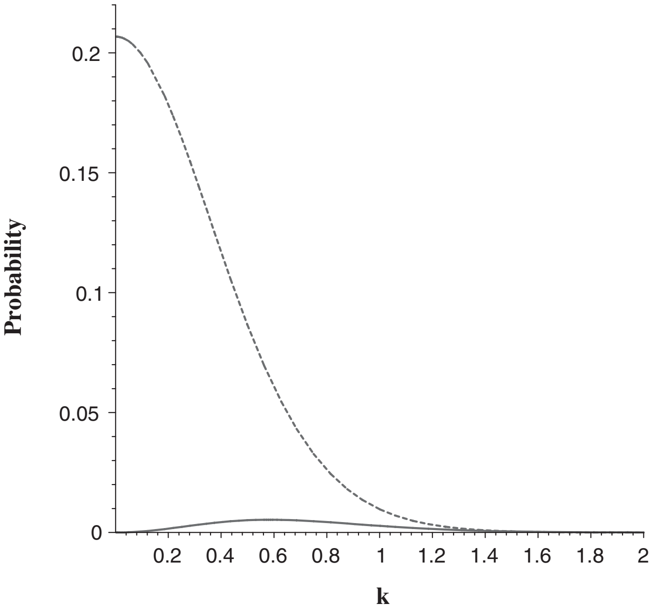

FIG. 5. $\mathcal{P}_{e^-e^+}$ as a function of k for $\chi = 0.4$. The dashed line represents the case of helicity conservation and the solid line the case when helicity is not conserved.

probability for production of pair with equal helicities $\lambda = \lambda'$ (helicity nonconserving case) in the case $\omega > m$ (see Figs. 5 and 6). For $m = 0$, the probability of pair production in the helicity nonconserving case is zero, while the probability for a conserving helicity process is finite. We note that the zero probability of production of fermions with zero mass in the helicity nonconserving case is

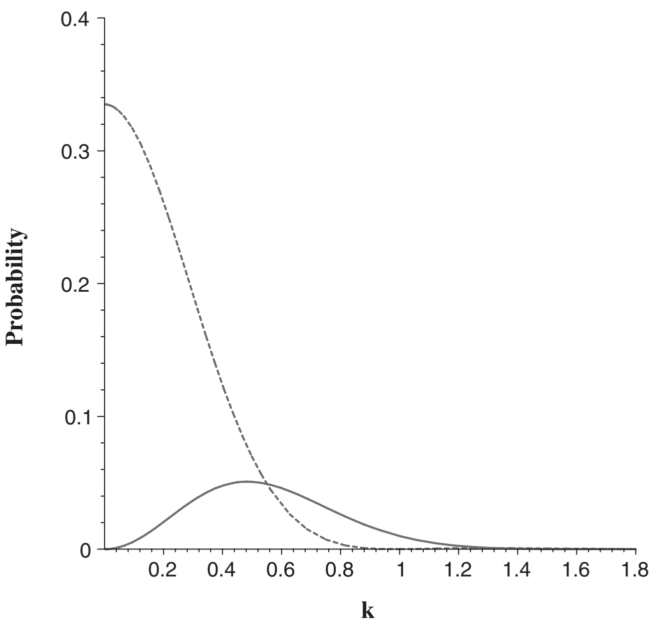

FIG. 6. $\mathcal{P}_{e^-e^+}$ as a function of k for $\chi = 0.1$. The dashed line represents the case of helicity conservation and the solid line represents the case when helicity is not conserved.

consistent with the identically null amplitude in the case when helicity is not conserved due to conformal invariance. Also, the graphs (see Figs. 5 and 6), show that when m/ω increases the probability of pair production with opposite helicities decreases and the probability of production of fermions with equal helicities becomes finite. Summarizing our result for $\omega > m$, if the two momenta are close as moduli $p' \sim p$ but not equal, the production processes that preserve the helicity conservation law will be favored. This result is in good agreement with [27], where the helicity conservation was discussed in the Coulomb scattering on de Sitter space. The result from [27] proves that helicity conservation in Coulomb scattering is preserved as long as we deal with equal initial and final momenta $p_i = p_f$.

We also see from Figs. 5 and 6, that our probabilities in both conserving/nonconserving helicity cases become negligible for large m/ω, and from here we conclude that the production rate was important only in the inflationary regime, when this ratio was very small. Looking at the helicity nonconserving case ($\lambda = \lambda'$), we obtain from our graphical analysis that for $\chi = 0.4$, or equivalently as χ is closer to unity, the probability for a process where the helicity is not conserved becomes very small. The conclusion is that the helicity conservation seems to be a property that is preserved as long as we deal with pair production that has the ratio of the momenta p/p' closer to unity.

Let us consider now the case when χ is very small. This case corresponds to the situation in which, for example, $p' \gg p$ (for $p' \ll p$ the analysis is similar), so that the momentum of the particle p is much smaller than the momentum of antiparticle p'. We must specify that large momenta does not refer here to a relativistic momentum, but rather to the situation when one of the momenta is almost zero and the other has a value much smaller comparatively with the relativistic case, i.e, $p \in (10^{-7}, 10^{-2})$ and $p' \in (1, 5)$. Plotting our probability (16) as a function of m/ω, we obtain the results shown in Figs. 7–9.

The first observation in this case is that the probability for a production process where helicity is not conserved becomes important as $\chi \ll 1$ (Fig. 9). We see that in this limit the pair production processes that break the helicity conservation law become important as the two probabilities that correspond to the helicity conservation/nonconservation processes become approximatively equal (Fig. 9) for $\omega > m$. Another interesting result is that the probabilities for pair production in both conserving/nonconserving helicity cases have a series of maxima (resonances) and minima (Figs. 7–9) with the observation that the amplitudes of these resonances decrease as the ratio m/ω increases. This oscillatory behavior of the probability becomes more pronounced as χ approaches zero. The limit $\chi \to 0$ could help us understand the case in which one of the produced particles has a small momentum close to zero comparatively with momentum of the other particle. Let us

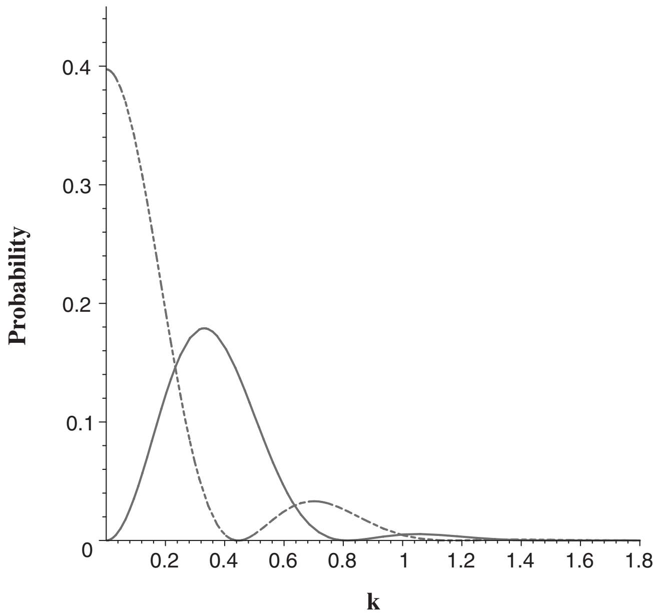

FIG. 7. $\mathcal{P}_{e^-e^+}$ as a function of k for $\chi = 0.01$. The dashed line represents the case of helicity conservation and the solid line represents the case when helicity is not conserved.

make a more detailed analysis of this situation. It is known that in an expanding de Sitter universe the following first-order processes could arise as the result of the loss of energy law [6–10]: the electron or positron can absorb or can emit photons ($e^{\mp} \leftrightarrows e^{\mp} + \gamma$), one photon electron-positron pair creation or annihilation of the pair in a single photon ($\gamma \leftrightarrows e^- + e^+$), and that electron-positron and photon triplet could be produced or absorbed by the de Sitter vacuum, all effects arising as perturbative phenomena. The case $\chi \ll 1$ could be now explained in terms of the above-mentioned reactions. One of the Dirac particles arising from vacuum in a Coulomb field could absorb one photon, and if we take into consideration that this photon can be produced from vacuum in the above-mentioned reactions, we can resume that the Dirac particle absorbs energy and momenta from vacuum. Opposite to this situation, the emission of one photon is responsible for the small momentum of the other Dirac particle. The emitted photon could end up in another electron-positron pair that could annihilate in vacuum, resulting that the momentum of the particle can be absorbed by the vacuum. Our remarks also help us to understand why the helicity is no longer conserved in the case $\chi \ll 1$. If we start, for example, with an electron, it could emit or absorb one photon and the spin in this reaction is no longer conserved. As a final conclusion, in the limit of small χ we establish that helicity conservation is no longer a rule and the processes where helicity is not conserved become equal in importance.

It is also important to mention that our graphical results for probability (Figs. 5–9) are valid for small momenta (p, p'). This is because the factor $|\vec{p} + \vec{p}'|^{-4}$ cancels our probability in the limit of large momenta. Thus, we can conclude that particles with low momenta were produced in an external Coulomb field, due to the early expansion of the space. This is in agreement with the formula for physical momentum $p_{\rm ph} = p_c e^{-\omega t}$ (p_c being the comoving momentum), from which we see that for large expansion factor ω the physical momentum becomes small.

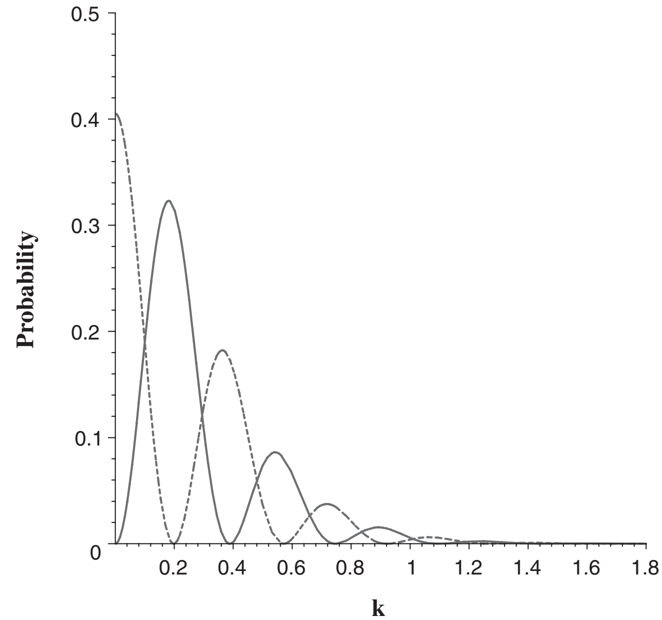

FIG. 8. $\mathcal{P}_{e^-e^+}$ as a function of k for $\chi = 0.0001$. The dashed line represents the case of helicity conservation and the solid line represents the case when helicity is not conserved.

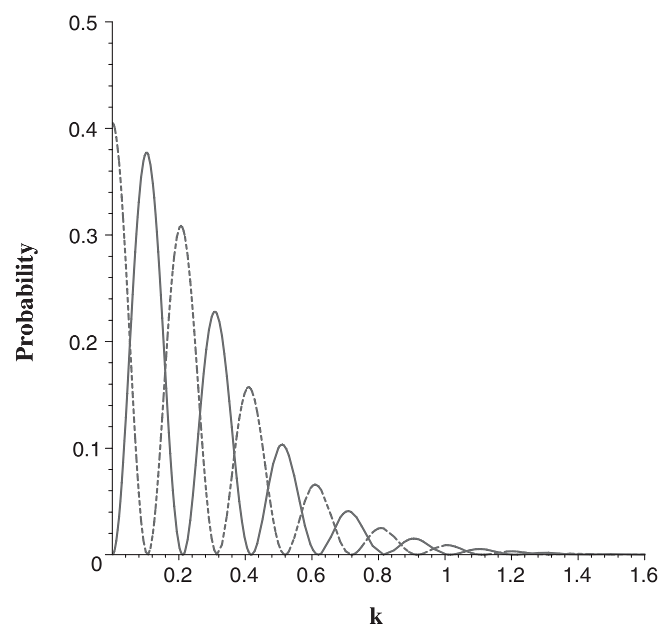

FIG. 9. $\mathcal{P}_{e^-e^+}$ as a function of k for $\chi = 0.000\,000\,1$. The dashed line represents the case of helicity conservation and the solid line represents the case when helicity is not conserved.

As a general observation from all the graphs of probability as a function of $k = m/\omega$, it is clear that the amount of particle production was important only for small k. This corresponds to the early Universe, when the expansion was large comparatively with particle mass ($\omega \gg m$). This result is in good agreement with the previously obtained results [2–4], which also predict that the amount of particle production in the early Universe was important.

Taking into consideration that the particle production rate was important for small values of $k \ll 1$, it will be interesting to discuss this case. This is the case of a large expansion factor, with nonzero mass. The amplitude and probability formulas in this section are used in our analysis and, in addition, we have at our disposal the graphs for probabilities. We specify that the two cases for conserving/nonconserving helicity are considered. For our considerations, it will be sufficient to restrict ourselves, for example, to the case $p < p'$. We use helicity bispinors, for $\lambda = \lambda'$ and $\lambda = -\lambda'$, and restrict the analysis only to the square of our amplitude. In an orthogonal local frame $\{\vec{e}_i\}$, we take the electron and positron momenta in the plane (1, 2), denoting their spherical coordinates as $\vec{p} = (p, \alpha, \beta)$ and $\vec{p}\,' = (p', \gamma, \varphi)$, where $\alpha, \gamma \in (0, \pi)$; $\beta, \varphi \in (0, 2\pi)$. Then, for $\beta = \pi$, $\varphi = 0$, and $p < p'$ the final result for the square amplitude is

$$|\mathcal{A}_{e^-e^+}|^2 = \frac{e^4 Z^2}{256\pi^2 |\vec{p} + \vec{p}\,'|^4} \frac{1}{p'^2} \theta(p' - p) \left[\left| f_k\left(\frac{p}{p'}\right) \right|^2 + \left| f_{-k}\left(\frac{p}{p'}\right) \right|^2 \pm f_k\left(\frac{p}{p'}\right) f^*_{-k}\left(\frac{p}{p'}\right) \pm f^*_k\left(\frac{p}{p'}\right) f_{-k}\left(\frac{p}{p'}\right) \right] \times \begin{cases} \cos^2\left(\frac{\alpha+\gamma}{2}\right) & \text{for } \lambda = -\lambda' \\ \sin^2\left(\frac{\alpha+\gamma}{2}\right) & \text{for } \lambda = \lambda'. \end{cases} \tag{18}$$

From this formula, if we set $\alpha = \pi$, $\gamma = 0$, corresponding to the case when the momenta of the electron and positron are parallel but move in opposite directions, we obtain zero probability for a helicity conservation process, while the probability for a process where helicity is not conserved becomes maximum. All the particular cases where the momenta are oriented on different directions can now be obtained from (18). We can conclude that for $k \ll 1$, the helicity nonconserving processes in which the electrons and positrons have parallel momenta and move in the opposite directions will be favored. This result shows that only in the helicity nonconserving processes the chances of separation between electrons and positrons in the early Universe become important. Thus, it seems that for $\alpha = 0$, $\gamma = 0$ the situation is opposite, conserving processes being dominant, with the specification that this case corresponds to the pair having parallel momenta and moving in the same direction, which increases the probability of annihilation of the pair.

In the next section, we shall pay closer attention to the analysis of this probability in the limit cases of the parameter k. It will be interesting to study the probability in the limit of large k to make a connection with the well-established results from literature, where the transition from vacuum to nonvacuum states is studied using the Bogoliubov transformations. This is because we expect the behavior of our amplitude/probability for large k to somehow reproduce up to some factors, the results obtained when one uses the asymptotic solutions of the field equations.

IV. LIMIT CASES

The de Sitter metric reduces to the Minkowski one when the expansion factor of the space vanishes, $\omega = 0$. In this limit, the free field equations (Dirac and Maxwell equations) also become the field equations from flat space, so we expect our amplitude to reduce in this limit to the corresponding amplitude from Minkowski space, which is zero. In the flat space, this process of pair creation from a Coulomb field is forbidden by the laws of conservation for energy and momentum. The Minkowski limit corresponds in our case to $k = \infty$, which is the ratio (mass of particle/expansion factor).

Let us see first what happens for large values of the parameter k. In this case, one can observe from (15) that the second and the third arguments in the hypergeometric functions ${}_2F_1(a, b; c; z)$ become approximatively equal ($b \simeq c$) and we can use ${}_2F_1(a, b; b; z) = (1 - z)^{-a}$ to simplify these functions. For Beta Euler functions, one can apply the Stirling formula to obtain an approximation in the limit of large k: $\Gamma(z) = z^{z-1/2} e^{-z} \sqrt{2\pi}(1 + O(z^{-1}))$, $|z| \to \infty$. Finally, one finds for large k (only the terms proportional with $e^{\pi k}$ are considered in the functions f_k, f_{-k})

$$f_k(\chi) \simeq \sqrt{\frac{k}{\pi}} \frac{i e^{-\pi k} \chi^{-ik}}{(1 - \chi^2)^{1/2}}, \qquad f_{-k}(\chi) \simeq \sqrt{\frac{1}{\pi k}} \frac{i e^{-\pi k} \chi^{1-ik}}{2(1 - \chi^2)^{3/2}}, \tag{19}$$

with the specification that the term $e^{-\pi k}$ originates from $ch^{-2}(\pi k)$. We see that these functions are highly convergent and vanish as

$$f_{\pm k}(\chi) \sim e^{-\pi k} \tag{20}$$

for $k \to \infty$. An important observation is that the factor $e^{-\pi k}$ is used in the literature [5,12–14] for characterizing the vacuum to nonvacuum transitions in this geometry. As noted before, the well-established results from the literature use the asymptotic behavior of Hankel functions to establish the positive and negative modes and then with the help of the Bogoliubov coefficient β_p, one can obtain the number density of particles per volume $|\beta_p|^2 \frac{d^3p}{(2\pi)^3}$. For the above-mentioned vacuum–nonvacuum transitions [5,12–14], the Bogoliubov coefficient is proportional to $|\beta_p|^2 \sim e^{-2\pi m/\omega}$. This exponential factor is also found in our probability for large k. Using Eq. (19), one can obtain the probability of pair production in a Coulomb field for $k \gg 1$ (or $m \gg \omega$). The cases $p < p'$ and $p > p'$ are considered in our analysis, with the observation that we no longer need to use step functions. Using the spherical coordinate of $\vec{p} = (p, \alpha, \pi)$ and $\vec{p}\,' = (p', \gamma, 0)$, the final result for the probability (conserving and nonconserving helicity processes are included), in the case $m \gg \omega$, is

$$\mathcal{P}_{e^-e^+} = \frac{e^4 Z^2 e^{-2\pi m/\omega}}{256\pi^3|\vec{p} + \vec{p}\,'|^4} \times \frac{pp'}{(p'^2 - p^2)^2} \begin{cases} \cos^2\left(\frac{\alpha+\gamma}{2}\right) & \text{for } \lambda = -\lambda' \\ \sin^2\left(\frac{\alpha+\gamma}{2}\right) & \text{for } \lambda = \lambda'. \end{cases} \tag{21}$$

Recently, in [14], it was shown that the condition $m \gg \omega$ must be imposed in order to obtain production of a thermal spectrum of radiation, as calculated in the work done by Gibbons and Hawking [28]. The exponential in the probability equation (21) has the form of a Boltzmann factor, which means that the spectrum of particles created in our case for $m \gg \omega$ is thermal, having the Gibbons-Hawking temperature $T = \omega/2\pi$. Also, it is remarkable that our probability is proportional with $e^{-2\pi m/\omega}$, which was found in [5,12–14], to be the factor that characterizes the probability of particle production from vacuum for $m \gg \omega$.

Now, let us make an estimation of this probability for the present-day expansion of the Universe. The last astronomical observations give $\omega = 2,\ 3 \times 10^{-18}\ \mathrm{s}^{-1}$, for the Hubble constant. Then, from the ratio $mc^2/\hbar\omega \sim 5 \times 10^{37}$, (where m is the electron mass) one can see that the large mass approximation is good in considering the pair production at present. It is immediate from (21), in which we replace the above-mentioned ratio, that the probability of pair production at present is very small, and we can say that there is no fermion production of finite mass in a Coulomb field due to present-day expansion, a result which is in good agreement with the conclusion from [2–4].

We can now address the problem of the Minkowski limit. The graphs (Figs. 1–4) from the previous section, for functions $f_k(\chi)$, invite some comments. First, we see that in the limit $k \to \infty$, the functions $f_k(\chi)$ vanish, which is in agreement with the result obtained analytically, $f_\infty(\chi) = 0$. From here, we recover the Minkowski limit, where particle production in a Coulomb field is forbidden by the laws of conservation for energy and momentum. As k goes to large values, the de Sitter amplitude equals the Minkowski one $\mathcal{A}_{e^-e^+}|_{k=\infty} = \mathcal{A}_{e^-e^+}(\text{flat}) = 0$, which corresponds to the fact that in an asymptotic flat universe these amplitudes vanish. Also, it is worth mentioning that for large momenta, $p,\ p' \to \infty$, the amplitude (14) and probability (16) vanish and the recovery of Minkowski results would probably not be surprising.

Secondly, it is obvious from our graphs (Figs. 1–4) that our formula for functions $f_k(\chi)$ becomes important when the expansion factor is larger than the mass of the particle. The null mass limit corresponds to the early stage inflation of the Universe, when the expansion factor was large compared to the mass of the particle $\omega \to \infty$. The $f_k(\chi)$ function simplifies in this case (i.e., ${}_2F_1(1, 1/2; 1/2; z) = {}_2F_1(1, 3/2; 3/2; z) \simeq (1-z)^{-1}$) and for $k = 0$ we obtain

$$f_0(\chi) = \frac{i}{\pi}(1+\chi)^{-1}. \tag{22}$$

Also, it is immediate from (14) that for $k = 0$ the amplitude of transition is nonvanishing only for $\lambda = -\lambda'$. This is in agreement with the observations made in the previous section, that the most probable transitions in the null mass case are those in which the helicity is conserved. Replacing (22) in the transition amplitude (14) and observing that in this case we no longer need to use unit step functions, the final result is

$$\mathcal{A}_{e^-e^+}|_{(k=0)} = \frac{e^2 Z}{4\pi^2|\vec{p} + \vec{p}\,'|^2} \frac{\delta_{\lambda,-\lambda}}{p + p' - i0} \xi^+_\lambda(\vec{p})\eta_{-\lambda}(\vec{p}\,'). \tag{23}$$

It is remarkable that the amplitude of transition in this limit depends only on the modulus of two momenta $p,\ p'$. The same results are obtained from our integrals in (12) if we set $k = 0$, in which case the Hankel functions simplify to

$$H^{(2)}_{1/2}(z) = i\left(\frac{2}{\pi z}\right)^{1/2} e^{-iz}, \tag{24}$$

and the integrals from (12) can be evaluated adding a factor $e^{-\epsilon z}$, where the parameter ϵ is positive, to assure the convergence in the limit $z \to \infty$. Note also that the amplitude in the zero mass limit vanishes if the particle and antiparticle have the same helicities, which corresponds to the nonconserving helicity in the process presented above, which also confirm our graphical results for probabilities and our observation that the nonzero mass break helicity conservation. This is an expected result since the Dirac field for a null fermion mass is conformally invariant and the de Sitter metric is conformal with the Minkowski one.

One can also see that for a very large expansion factor but nonzero mass, the exponential factor $e^{-\omega t}$, from the Coulomb potential, implies that only negative large time will be relevant, which means that we can consider $z \simeq e^{-\omega t}$ large everywhere in the integrals (12). Translated in our calculations, this implies that we can use the expansion

$$H_{\mu}^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\mu\pi/2-\pi/4)}, \qquad z \to \infty, \tag{25}$$

which differs from $H_{1/2}^{(2)}(z)$ only by the phase factor $e^{i(\mu\pi/2+\pi/4)}$. The conclusion is that for very large expansion factors, the amplitude of this process will be mainly determined by the contributions from the distant past when space was contracted. It is from here immediate that the frequency blueshift in the distant past makes the mass irrelevant, a fact that leads back to a Minkowski conformal theory.

Further, let us consider a closer look at our amplitude in the null mass case (23). As before, kinetic parameters can be introduced in the orthogonal local frame $\{\vec{e}_i\}$. We take the particle and antiparticle momenta in the plane (1, 2) denoting their spherical coordinates by $\vec{p} = (p, \alpha, \beta)$ and $\vec{p}' = (p', \gamma, \varphi)$ where α, $\gamma \in (0, \pi)$ and β, $\varphi \in (0, 2\pi)$. Then, using the helicity bispinors, one can obtain for $\lambda = -\lambda' = \frac{1}{2}$ the final expression of the transition amplitude:

$$\mathcal{A}_{e^-e^+}|_{(k=0)} = \frac{e^2 Z}{4\pi^2|\vec{p}+\vec{p}'|^2}\frac{1}{p+p'-i0}\left[\cos\left(\frac{\alpha}{2}\right)\times\cos\left(\frac{\gamma}{2}\right) + e^{i(\varphi-\beta)}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right)\right], \tag{26}$$

where the terms in the square brackets were obtained from $\xi^+_{1/2}(\vec{p})\eta_{-1/2}(\vec{p}')$. From the above formula, one can obtain all the particular cases of interest in which the electron and positron move in the same direction or move in opposite directions. The probability of production of null mass fermions, if we set $\beta = 0$, $\varphi = \pi$, will be

$$\mathcal{P}_{e^-e^+(k=0)} = \frac{e^4 Z^2}{32\pi^4|\vec{p}+\vec{p}'|^4}\times\frac{1}{|p+p'-i0|^2}\cos^2\left(\frac{\alpha+\gamma}{2}\right). \tag{27}$$

From the above formula, we see that for $\alpha = \pi$, $\gamma = 0$ our probability vanishes. This corresponds to the case in which the particle and antiparticle have parallel momenta and move in opposite directions. Now, if we set $\alpha = 0$, $\gamma = 0$ the probability takes its maximum value, and this is the case in which the particle and antiparticle momenta are parallel and they move in the same direction, increasing the probability of annihilation of the pair. The obvious conclusion is that if the null mass fermions are produced they will rather annihilate each other than become separate. From here, we conclude that there are small chances of production of null mass fermions in the early Universe. As a final remark, one can see that our perturbational calculations in the null mass case are close to those from the literature [2–4,14], which establishes that there is no production of zero-mass fermions in the early Universe.

V. CONCLUSIONS

Fermion pair production in a Coulomb field in a de Sitter expanding universe was considered in our paper. We found the expression of the differential probability for the transition vacuum $\rightarrow e^-e^+$ in the potential $A^{\hat{0}}$. We considered the initial and final states of the field as exact solutions of the free Dirac equation in de Sitter space with a defined momentum and helicity. We also found that the amplitude of pair production depends, in an essential way, on the parameter $k = m/\omega$. For a vanishing k, we recover the expected result due to the conformal invariance of the theory in the massless case. The final result proves that the law of conservation for the momentum is lost in the process of pair production in a Coulomb field on a de Sitter background. From our amplitude formula, we recover the Minkowski limit where there is no particle production in the Coulomb field. Our final results related to helicities show that the helicity conservation processes seem to be dominant in the case when the momenta of the electron and positron are close in modulus ($\chi \rightarrow 1$). When the ratio of the two momenta is small, we found that the processes where helicity is not conserved become important. Our study also led to an important result related to the mechanism of separation between matter and antimatter in the early Universe, namely, that only in the helicity nonconserving case there seems that are nonvanishing probabilities for producing pairs that move in opposite directions, increasing in this way the possibility of separation between particles and antiparticles.

A notable result is that only particles/antiparticles with small momenta were produced in a Coulomb field in the early Universe.

Another result is that for the present-day expansion, the amount of fermion production in a Coulomb field is negligible. Contrary to this, our calculations and graphs show that in the early stage of the Universe the amount of pair production was important. This confirms the results from literature related to the phenomenon of pair production in the early Universe.

Our study does not take into account the problem of wave packets in the de Sitter geometry. This problem could be important in studying the effects of pair production, and receives little attention in the literature. However, the study from [29] is a first step in this direction, where the construction and propagation of Gaussian wave packets on de Sitter geometry was discussed both for scalar and Dirac fields in the null mass case. The study of wave packets in this geometry could improve our understanding of scattering processes and could help us to understand the physical significance of the cross section.

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APPENDIX

The main steps leading to our amplitude (14) will be detailed in this section. First, we will use the well-known formula for Hankel functions in terms of Bessel functions J [24,25]:

$$H^{(1)}_\mu(z) = \frac{J_{-\mu}(z) - e^{-i\pi\mu}J_\mu(z)}{i\sin(\pi\mu)},\qquad H^{(2)}_\mu(z) = \frac{e^{i\pi\mu}J_\mu(z) - J_{-\mu}(z)}{i\sin(\pi\mu)}. \tag{A1}$$

This transformation finally leads to integrals of the type Weber-Schafheitlin [24,26]. These integrals can be solved in two distinct cases:

$$\int_0^\infty dz z^{-s} J_\mu(\alpha z)J_\nu(\beta z) = \frac{\alpha^\mu \Gamma(\frac{\mu+\nu-s+1}{2})}{2^s\beta^{\mu-s+1}\Gamma(\mu+1)\Gamma(\frac{\nu-\mu+s+1}{2})} \times {}_2F_1\left(\frac{\mu+\nu-s+1}{2}, \frac{\mu-\nu-s+1}{2}; \mu+1; \frac{\alpha^2}{\beta^2}\right), \tag{A2}$$

$$\beta > \alpha > 0, \qquad \mathrm{Re}(s) > -1, \qquad \mathrm{Re}(\mu+\nu-s+1) > 0,$$

and

$$\int_0^\infty dz z^{-s} J_\mu(\alpha z)J_\nu(\beta z) = \frac{\beta^\nu \Gamma(\frac{\mu+\nu-s+1}{2})}{2^s\alpha^{\nu-s+1}\Gamma(\nu+1)\Gamma(\frac{\mu-\nu+s+1}{2})} \times {}_2F_1\left(\frac{\mu+\nu-s+1}{2}, \frac{\nu-\mu-s+1}{2}; \nu+1; \frac{\beta^2}{\alpha^2}\right), \tag{A3}$$

$$\alpha > \beta > 0, \qquad \mathrm{Re}(s) > -1, \qquad \mathrm{Re}(\mu+\nu-s+1) > 0,$$

both cases being fulfilled in our analysis.

Now, one sees that our integrals in (13) demand $s = -1$. This is problematic because the integral becomes oscillatory for $z \to \infty$. Then, a simple way to solve this problem is to consider a parameter s of the form

$$s = -1 + \epsilon \tag{A4}$$

and let, in the end, $\epsilon \to 0$. No notable differences appear when evaluating the integrals in (13) directly with $s = -1$ and with $\mu, \nu = 1/2 \pm ik$.

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