

Gravitational instantons and fluxes from M/F-theory on Calabi-Yau fourfoldsThomas W. Grimm^{*} and Raffaele Savelli[†]*Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 Munich, Germany*

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We compactify four-dimensional $\mathcal{N} = 1$ gauged supergravity theories on a circle including fluxes for shift-symmetric scalars. Four-dimensional Taub-NUT gravitational instantons universally correct the three-dimensional superpotential in the absence of fluxes. In the presence of fluxes these Taub-NUT instanton contributions are no longer gauge-invariant. Invariance can be restored by gauge-instantons on top of Taub-NUT instantons. We establish the embedding of this scenario into M-theory. Circle fluxes and gaugings arise from a restricted class of M-theory four-form fluxes on a resolved Calabi-Yau fourfold. The M5-brane on the base of the elliptic fourfold dualizes into the universal Taub-NUT instanton. In the presence of fluxes this M5-brane is anomalous. We argue that anomaly-free contributions arise from involved M5-brane geometries dual to gauge-instantons on top of Taub-NUT instantons. Adding a four-dimensional superpotential to the gravitational instanton corrections leads to three-dimensional Anti-de Sitter vacua at stabilized compactification radius. We comment on the possibility to uplift these M-theory vacua, and to tunnel to four-dimensional F-theory vacua.

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I. INTRODUCTION

The study of four-dimensional supergravity theories within string theory has been approached in various compactification schemes. A rather general goal of these approaches is to understand the set of supergravity theories arising in a consistent quantum theory of gravity. Ideally one would thus be able to identify either a restrictive class of consistency conditions, or, even more ambitious, discover a dynamics which prefers certain string reductions. A more modest aim is to identify the concepts and tools required to study such four-dimensional (4d) effective theories. In this work we investigate 4d, $\mathcal{N} = 1$ supergravity theories from the perspective of a quantum corrected three-dimensional (3d) effective theory obtained by a flux compactification on a circle.

We start our investigation with a class of 4d, $\mathcal{N} = 1$ supergravity actions describing a set of complex scalars together with a non-Abelian gauge theory. Of particular interest will be cases where the imaginary parts of the scalars possess shift symmetries. We will include terms quadratic in the curvature in our investigation. Such effective theories naturally arise by compactifying F-theory on singular Calabi-Yau fourfolds. The singularities correspond to space-time filling 7-branes supporting the non-Abelian gauge groups. Switching on two-form fluxes on the world volume of the 7-branes can induce a minimal gauging of part of the shift symmetries. This generically breaks the gauge group to the Abelian Cartan torus. In such theories we study two types of 4d background configurations in detail: (1) a 3d Minkowski space times a circle with nontrivial field-strength of the shift-symmetric scalars,

(2) a 4d gravitational instanton solution which might admit a 4d gauge instanton bundle. Let us comment on these two configurations in turn.

A circle compactification with nontrivial vacuum expectation value for the field strength of the shift-symmetric scalars can be viewed as the simplest flux compactification [1,2]. Using a Kaluza-Klein reduction we determine the 3d gauged supergravity theory and show that it contains gaugings inherited from 4d, as well as new genuinely 3d gaugings induced by the fluxes around the circle. In three dimensions all degrees of freedom can be encoded by dynamical scalars and nondynamical vectors encoding the gauging. The 3d, $\mathcal{N} = 2$ supergravity theory formulated in this language will be our starting point to study the scalar potential, determine flux vacua, and discuss quantum corrections. These effective phenomena are then argued to have an M-theory interpretation which is inferred by the duality between M- and F-theory [3–5]. We link the above introduced 3d supergravities to an M-theory reduction on an elliptically fibered Calabi-Yau fourfold with a nontrivial flux for the M-theory four-form field strength. In our compactification schemes we will always neglect the effects of a nontrivial warp factor.

The gravitational instantons under consideration are solutions to the Euclidean Einstein equations with anti-self-dual two-form curvature tensor. More precisely, we will focus on a special class of solutions, known as Taub-NUT spaces [6]. The Taub-NUT spaces have finite action and are believed to contribute new saddle-points of the 4d path integral summing over gravity backgrounds with different topologies [7]. However, instead of computing directly in the four dimensions we will include the Taub-NUT instanton corrections in the effective three-dimensional supergravity theory obtained by the fluxed circle reduction. While Taub-NUT instantons generically

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break 4d supersymmetry in the background, they can supersymmetrically correct the 3d effective supergravity theory. This has a natural analog in the M-theory picture where the Taub-NUT instantons are identified with M5-brane instantons on the base of the elliptic fibration. In fact, such M5-branes naturally contribute to the 3d effective superpotential in the absence of fluxes [8].

To be more precise, we will find that in the absence of 3d gaugings the Taub-NUT gravitational instantons yield a universal contribution to the 3d superpotential proportional to $e^{-2\pi T_0}$. Here, the complex scalar T_0 parameterizes the square of the radion, i.e. the radius of the circle, complexified with the scalar obtained by dualizing the Kaluza-Klein vector in the 4d metric. This induces a potential for the radion which can lead to new supersymmetric AdS₃ vacua if already a 4d superpotential W_0 is present. The supersymmetric stabilization in an AdS vacuum is the lower-dimensional analog to the proposal of KKLT to stabilize Kähler moduli [9]. In this new vacuum the theory can be effectively three-dimensional and stable. A natural conjecture is to up-lift this setup to positive vacuum energies. This generates a meta-stable 3d de Sitter vacuum which can tunnel to a 4d de Sitter vacuum. Such transdimensional tunnelings have been discussed, for example, in [10–14]. In this work, we will rather focus on the first steps in implementing such a scenario. It turns out that the combination of the fluxed reduction and Taub-NUT instanton correction already imposes interesting compatibility conditions.

If we consider the complete flux compactification of the 4d gauged supergravity theory, the corrections due to Taub-NUT gravitational instantons will be altered. We show that the fluxes along the compactification circle enforce that the shift symmetry of the imaginary part of the radion modulus T_0 is gauged. This forbids a superpotential $\mathcal{A}e^{-2\pi T_0}$, when the prefactor $\mathcal{A}$ does not also transform under the $U(1)$ gauge symmetry. However, we find that one can indeed include a field-dependent $\mathcal{A}$ to cancel the shifts by using the gaugings already present in the 4d theory, together with fluxes on the Taub-NUT instanton. This yields a superpotential of the form recently discussed in [15], in which two scalars with gauged shift symmetries are appropriately combined using fluxes on the instantons. In the microscopic M-theory picture the absence of a superpotential with constant $\mathcal{A}$ has an anomaly interpretation. In different setups involving D-branes the possible conflicts of fluxes and nonperturbative superpotentials have been discussed in [16–19]. In the presence of fluxes the M5-brane on the base of the elliptic fibration becomes anomalous [20], in a way which is familiar from the Freed-Witten anomaly for D-branes [21]. Only for a more complicated M5-brane worldvolume, on which the four-form fluxes encoding the 4d gaugings vanish, this anomaly can be absent. On the one hand, this is precisely the string dual to a Type IIB compactification with a Taub-NUT instanton supporting gauge bundles. This is a realization of our

original 4d configuration. On the other hand, in the Type IIA limit of M-theory this setup can be interpreted in terms of D4-D6-NS5 branes, as we discuss briefly.

The paper is organized as follows: In Sec. II, we perform the dimensional reduction from 4d to 3d on a circle by allowing for a flux background for the shift-symmetric scalars, and discuss the inclusion of the Taub-NUT instantons from a 3d perspective. In Sec. II A, the Kaluza-Klein reduction is discussed in the absence of 4d vectors. We also introduce the 3d $\mathcal{N} = 2$ supergravities with dynamical scalars and nondynamical 3d gauge vectors. In Sec. II B, we generalize this reduction by introducing 4d vectors and gaugings of 4d shift symmetries. We determine the combined scalar potential. The stabilization of the radion modulus by Taub-NUT gravitational instantons is discussed in Sec. II C. We motivate the form of the Taub-NUT corrections by studying the instanton action. Furthermore, we comment on a KKLT-like scenario to stabilize the size of the compactification circle, and briefly speculate on the up-lift and tunneling from 3d to 4d. The form of the superpotential in the presence of circle fluxes, vectors and 4d gaugings is discussed in Sec. II D. This requires the introduction of gauge-instantons on top of the gravitational instanton.

In Sec. III, we provide the M-theory construction of the 3d effective theory and combine fluxes and instanton corrections. We discuss M-theory on elliptically fibered Calabi-Yau fourfolds in Sec. III A. We also introduce its F-theory limit in which the M-theory elliptic fiber shrinks to zero size which yields in the T-dual picture to a growing large extra dimension. The resulting setup can be matched with an F-theory compactification. Four-form fluxes in M-theory are introduced in Sec. III B, where we also comment on their four-dimensional interpretation. The resulting 3d D-term potential is introduced in Sec. III C. The identification of the Taub-NUT gravitational instanton as M5-brane is discussed in Sec. III D. We also study the interplay of four-form fluxes and the M5-brane instantons.

II. CIRCLE FLUX REDUCTIONS AND GRAVITATIONAL INSTANTONS

In this section, we analyze the compactification of 4d, $\mathcal{N} = 1$ supergravity theories to three dimensions on a circle by including fluxes for the field strengths of shift-symmetric scalars in Secs. II A and II B. The gravitational instantons, and their contribution to the 3d superpotential are discussed in Secs. II C and II D. Section II will be entirely formulated in the language of the supergravity theories, and hence can be understood independently of the M-theory realization discussed in Sec. III. However, the structure of certain 4d couplings will be familiar from a F-theory compactifications to four space-time dimensions [22–24], which will later make the embedding into M-theory more immediate.

A. Fluxed reduction from 4d to 3d

In this subsection, we will discuss how nontrivial fluxes can arise in the reduction on a circle and generate a landscape of vacua in the effective three-dimensional theory. In our discussion we will generalize the following simple observation. Consider a real scalar ϕ which only appears with a derivative $F_1 = d\phi$ in the four-dimensional action. In the S^1 reduction from four to three dimensions F_1 can acquire a background flux $M = \int_{S^1} \langle F_1 \rangle$. The constants M will induce a scalar potential in the three-dimensional effective theory. Note that this is the simple analog of the more complicated flux compactifications from ten to four dimensions reviewed in [1,2].

Let us include such fluxes when dimensionally reducing four-dimensional $\mathcal{N} = 1$ theories. In this section, we will assume that there are no 4d vectors in the spectrum, and postpone a more general discussion to the next subsection. The purely bosonic part of the 4d supergravity action reads

$$S^{(4)} = - \int \frac{1}{2} R_4 *_4 \mathbf{1} + K_{T_\alpha \bar{T}_\beta}^{4d} dT_\alpha \wedge *_4 d\bar{T}_\beta + V_F *_4 \mathbf{1} + \mathcal{L}_{\mathcal{R}^2}, \quad (2.1)$$

with

$$\mathcal{L}_{\mathcal{R}^2} = \frac{1}{2} \text{Re} \sigma \text{Tr}(\mathcal{R}_4 \wedge *_4 \mathcal{R}_4) + \frac{1}{2} \text{Im} \sigma \text{Tr}(\mathcal{R}_4 \wedge \mathcal{R}_4), \quad (2.2)$$

where $*_4$ is the 4d Hodge star, and $\mathcal{R}_4 = \frac{1}{2} R_{\mu\nu} dx^\mu \wedge dx^\nu$ is the $SO(3,1)$ -valued curvature two-form. Note that we have included the higher curvature terms, which are quadratic in the curvature. The inclusion of such corrections has been argued to be crucial in linking the effective theory and the stingy compactification geometries [24].

In order to perform the reduction we will assume that some of the real four-dimensional scalar fields $\text{Im} T_\alpha$ have global shift symmetries. These will later be gauged in Sec. II B. We denote the number of independent shift symmetries by n_s . This implies that n_s combinations of the fields $\text{Im} T_\alpha$ only appear with derivatives in the 4d action. To specify the fields $\text{Im} T_\alpha$, which have a shift symmetry, we introduce the constant “symmetry tensor” Q_α^Σ such that

$$\text{Im} T_\alpha \rightarrow \text{Im} T_\alpha + Q_\alpha^\Sigma \Lambda_\Sigma, \quad \Sigma = 1 \dots n_s \quad (2.3)$$

for constants Λ_Σ , is a symmetry of the effective action (2.1). To specify the shift symmetries we introduced a general Q_α^Σ with fixed rank n_s . Clearly, upon choosing an appropriate basis of the T_α one can set $Q_\alpha^\Sigma = (\delta_\alpha^\Sigma, 0)$ such that the first n_s T_α 's have shift symmetries. There are several reasons for a nonmaximal rank of Q_α^Σ . The Kähler metric $K_{T_\alpha \bar{T}_\beta}^{4d}$, which integrates to a Kähler potential K^{4d} , can explicitly depend on $\text{Im} T_\alpha$ breaking the shift symmetry. However, note that K^{4d} is nonholomorphic

and hence can consistently depend on $\text{Re} T_\alpha$ only. The latter fact should be contrasted with the holomorphic superpotential W , which specifies the scalar potential V_F in the action (2.1) via

$$V_F = e^{K^{4d}} (K_{4d}^{T_\alpha \bar{T}_\beta} D_{T_\alpha} W_{4d} \overline{D_{\bar{T}_\beta} W_{4d}} - 3|W_{4d}|^2). \quad (2.4)$$

The superpotential breaks the shift symmetry as soon as it depends on the corresponding T_α . A well-known example for a nontrivial W_{4d} are superpotentials of the schematic form $W_{4d} = W_0 + A e^{-n^\alpha T_\alpha}$. The T_α -dependent term arises in string theory compactifications from nonperturbative effects, and clearly breaks the shift-symmetry of the corresponding $n^\alpha \text{Im} T_\alpha$. Such nonperturbative contributions are essential in the study of moduli stabilization in the 4d effective theory as argued in [1,2,9].

In addition to K^{4d} and W_{4d} also a field-dependent coupling $\sigma(T)$ in front of the higher curvature terms can, in principle, break the shift symmetry. In string and M-theory compactifications, as we discuss below, a simple form of σ for large $\text{Re} T_\alpha$ is [24]

$$\sigma = \frac{1}{8} k^\alpha T_\alpha, \quad (2.5)$$

for a constant vector k^α . Since Q_α^Σ parameterizes the shift invariant $\text{Im} T_\alpha$ appearing in the action, a coupling (2.5) in the effective action suggests the natural restriction of Q_α^Σ :

$$k^\alpha Q_\alpha^\Sigma = 0. \quad (2.6)$$

Strictly speaking, for a gravitational coupling with (2.5) the action can be made invariant under the shift symmetry of $\text{Im} \sigma$ if boundary terms are properly taken into account, and one uses the gravitational Chern-Simons form. However, it turns out that assuming the absence of a shift symmetry for $\text{Im} \sigma$ simplifies the analysis significantly, and we will make the assumption (2.6) in the following. It will be interesting to see in the setups of Sec. III how this condition is realized in M-theory.

We now want to reduce such a theory to a three-dimensional $\mathcal{N} = 2$ supergravity theory on a background geometry of topology $\mathbb{R}^{1,2} \times S^1$ with metric

$$\langle g^{(4)} \rangle = \begin{pmatrix} \eta_{\mu\nu}^{(3)} & 0 \\ 0 & \langle r^2 \rangle \end{pmatrix}, \quad (2.7)$$

$\langle r \rangle$ being the vev of the radion field which parameterizes the circumference of S^1 . In this background we also allow for a nontrivial profile of the $\text{Im} T_\alpha$ along the S^1 which possess a shift-symmetry (2.3). This allows us to include n_s independent background fluxes M_Σ as

$$M_\alpha \equiv Q_\alpha^\Sigma M_\Sigma = \int_{S^1} d \langle \text{Im} T_\alpha \rangle. \quad (2.8)$$

Note that this form of M_α implies together with (2.6) that

$$k^\alpha M_\alpha = 0. \quad (2.9)$$

Clearly such a flux should be proportional to the unique harmonic one-form on S^1 , which with a slight abuse of notation may be written in the form dy .¹ Hence (2.8) implies that the local expression for $\langle \text{Im} T_\alpha \rangle$ is, up to an additive constant, $\langle \text{Im} T_\alpha \rangle = M_\alpha y$. So far we did not give any reason for the M_α or M_Σ to obey a quantization rule, and thus, for now, they are real numbers. However, we will see in Sec. III C that their M-theory origin forces them to satisfy some quantization condition dictated by global consistency constraints.

The three-dimensional effective theory is computed by including the fluctuations around this background. The fluctuation of the metric takes the following generic form

$$g^{(4)}(x^\mu) = \begin{pmatrix} g_{\mu\nu}^{(3)} + r^2 A_\mu^0 A_\nu^0 & r^2 A_\mu^0 \\ r^2 A_\nu^0 & r^2 \end{pmatrix} (x^\mu), \quad (2.10)$$

where $g^{(3)}$ is the fluctuation of the 3d metric in the Einstein frame and A^0 is the Kaluza-Klein vector of the reduction. Restricting to the lowest modes r , A^0 , and $g^{(3)}$ are only functions of the 3d coordinates x^μ . Note that for the reduction defined by (2.10), the real scalar accompanying A^0 in a 3d, $\mathcal{N} = 2$ vector multiplet will be r^{-2} , and we define

$$R = r^{-2}. \quad (2.11)$$

We also include the fluctuations of the four-dimensional fields T_α which correspond to three-dimensional complex scalars. By abuse of notation we also denote them by T_α , keeping in mind that they only depend on the 3d coordinates x^μ when used in a three-dimensional context.

Performing the reduction, the kinetic term in (2.1) simply reduces to the usual 3d kinetic term for the fields T_α promoted to 3d scalar fields plus a scalar potential. After integration over S^1 and performing the Weyl rescaling $g^{(3)} \rightarrow r^{-2} g^{(3)} = R g^{(3)}$ in order to bring the 3d Einstein-Hilbert term to the canonical form, we find

$$\begin{aligned} S^{(3)} = & - \int \frac{1}{2} R_3 *_3 \mathbf{1} - \frac{1}{4} \tilde{K}_{RR} (F^0 \wedge *_3 F^0 + dR \wedge *_3 dR) \\ & + \tilde{K}_{T_\alpha \bar{T}_\beta} \nabla T_\alpha \wedge *_3 \overline{\nabla T}_\beta + (R^2 \tilde{K}_{T_\alpha \bar{T}_\beta} M_\alpha M_\beta \\ & + R^2 V_F) *_3 \mathbf{1} + \tilde{\mathcal{L}}_{\mathcal{R}^2}, \end{aligned} \quad (2.12)$$

where now $*_3$, R_3 are, respectively, the Hodge star and Ricci scalar with respect to the 3d Einstein frame metric. $\tilde{\mathcal{L}}_{\mathcal{R}^2}$ is a complicated expression for the reduction of $\mathcal{L}_{\mathcal{R}^2}$ given in (2.2), and we will not need this form in the following. The negative definite metric $\tilde{K}_{RR}$ and the positive Kähler metric $\tilde{K}_{T_\alpha \bar{T}_\beta}$ can be determined from the function $\tilde{K}$, the kinetic potential, of the form

$$\tilde{K}(T_\alpha + \bar{T}_\alpha | R) = \log R + K^{4d}(T_\alpha + \bar{T}_\alpha) + 1. \quad (2.13)$$

¹Note that one would have to introduce two patches on S^1 to give a proper global definition of the harmonic one-form.

Note that due to the nontrivial background fluxes (2.8) the ordinary derivatives in (2.1) are reduced to the three-dimensional covariant (or rather invariant) derivatives

$$\nabla T_\alpha = dT_\alpha + i M_\alpha A^0. \quad (2.14)$$

This implies that even in the absence of a gauging in four dimensions, the three-dimensional theory will be a gauged supergravity theory. In this derivation we have used the inverse of the metric fluctuation (2.10), which looks like

$$g^{(4)-1}(x^\mu) = \begin{pmatrix} g^{(3)\mu\nu} & -g^{(3)\mu\nu} A_\nu^0 \\ -A_\mu^0 g^{(3)\mu\nu} & \frac{1}{r^2} + g^{(3)\mu\nu} A_\mu^0 A_\nu^0 \end{pmatrix} (x^\mu). \quad (2.15)$$

It is easy to see that the scalar potential in (2.12) arises from the $1/r^2$ part of the last component of (2.15). The A^0 -dependent part of (2.15), instead, is responsible for the appearance of the covariant derivative acting on the T_α .

The gaugings generated by the flux M_α have a direct geometrical interpretation in terms of the 1-dimensional diffeomorphisms of S^1 , which, after reduction, turns into a gauge symmetry of the 3d theory with gauge boson A^0 . Indeed, by the Kaluza-Klein ansatz (2.10), the coordinate transformation $y \rightarrow y + \lambda^0(x^\mu)$ induces the gauge transformation $A_\mu^0 \rightarrow A_\mu^0 - \partial_\mu \lambda^0$. Now, since we required an explicit dependence on y of the 4-dimensional field $\text{Im} T_\alpha$, the corresponding 3d fluctuations $\text{Im} T_\alpha(x^\mu)$ acquire a charge with respect to such a symmetry. In fact, one has for the 4d scalars:

$$\text{Im} T_\alpha(x, y + \lambda^0(x)) = \text{Im} T_\alpha(x, y) + \partial_y \text{Im} T_\alpha(x, y) \lambda^0(x). \quad (2.16)$$

This clearly induces the following local Peccei-Quinn symmetry for the imaginary parts of the 3d complex scalars:

$$\text{Im} T_\alpha(x) \rightarrow \text{Im} T_\alpha(x) + M_\alpha \lambda^0(x), \quad (2.17)$$

where it is manifest that the M_α 's play the role of the charges of the corresponding scalars under this symmetry. The latter is spontaneously broken by $\langle \text{Im} T_\alpha \rangle$ and therefore an equivalent, gauge-fixed description of the same theory is possible, in terms of a massive vector field: The gauge boson $A^0(x)$ acquires mass by the Higgs mechanism, i.e. by eating up the Goldstone boson $\text{Im} T_\alpha(x)$.

Since we have supersymmetrically reduced a four-dimensional $\mathcal{N} = 1$ theory, the resulting action will be a $\mathcal{N} = 2$ supergravity theory in three-dimensions. Therefore our aim is now to recast the action (2.12) in the canonical way for such a theory. Let us start by recalling the general form of an 3d, $\mathcal{N} = 2$ action in terms of chiral multiplets only, which we call T_Σ . The purely bosonic part of the action is given by [25–27]

$$S_{\mathcal{N}=2}^{(3)} = - \int \frac{1}{2} R_3 *_3 \mathbf{1} + K_{T_\Lambda \bar{T}_\Sigma} \nabla T_\Lambda \wedge *_3 \overline{\nabla T_\Sigma} + \frac{1}{2} \Theta_{IJ} A^I \wedge F^J + V *_3 \mathbf{1}, \quad (2.18)$$

where Θ_{IJ} is a symmetric, constant “embedding tensor”, giving rise to a Chern-Simons term for some nondynamical vector fields A^I , such that $F^I = dA^I$, without kinetic term. The covariant derivatives in this action are taking the form

$$\nabla T_\Sigma = dT_\Sigma + \Theta_{IJ} X_\Sigma^I A^J, \quad (2.19)$$

where $X^I = X_\Lambda^I \partial_{T_\Lambda} + \bar{X}_\Lambda^I \partial_{\bar{T}_\Lambda}$ are the Killing vectors generating a subgroup of isometries of the Kähler manifold with coordinates T_Λ which are gauged by A^I . Such gaugings generate nontrivial D-terms given by momentum maps D^I , which fulfill the following relation:

$$i \partial_{T_\Lambda} D^I = K_{T_\Lambda \bar{T}_\Sigma} \bar{X}_\Sigma^I, \quad (2.20)$$

a condition known for the four-dimensional D-terms. The scalar potential in (2.18) is:

$$V = K^{T_\Lambda \bar{T}_\Sigma} \partial_{T_\Lambda} \mathcal{T} \partial_{\bar{T}_\Sigma} \mathcal{T} - \mathcal{T}^2 + e^K (K^{T_\Lambda \bar{T}_\Sigma} \mathcal{D}_{T_\Lambda} W \overline{\mathcal{D}_{\bar{T}_\Sigma} W} - 4|W|^2), \quad (2.21)$$

where $\mathcal{D}_{T_\Lambda} = \partial_{T_\Lambda} + K_{T_\Lambda}$ is the Kähler covariant derivative. The scalar potential is given in terms of two functions which, together with the Kähler potential K , specify completely the 3d, $\mathcal{N} = 2$ action: The holomorphic superpotential $W(T)$, and the D-term potential $\mathcal{T}(T, \bar{T})$, which depends on the momentum maps as

$$\mathcal{T} = -\frac{1}{2} D^I \Theta_{IJ} D^J, \quad (2.22)$$

and requires the specification of the embedding tensor Θ_{IJ} .

Let us now specify the characteristic data of the 3d, $\mathcal{N} = 2$ gauged supergravity obtained by dimensionally reducing on S^1 a 4d, $\mathcal{N} = 1$ theory [Eq. (2.1)] with chiral fields T_α with shift symmetries and no vectors. In this situation, the indices Λ and I appearing in (2.18) have the same nature and take value in the set $\{0, \alpha\}$, where 0 corresponds to the new chiral field arising after the reduction, the radion. The superpotential $W = W_{4d}$ can only depend on the scalars which do not have shift symmetry. Moreover, the shift symmetries for $\text{Im} T_\alpha$ give rise to isometries of the corresponding moduli space, which are generated by the subset of Killing vectors

$$X_\Sigma^\Lambda = -2i \delta_\Sigma^\Lambda. \quad (2.23)$$

Therefore, by inserting (2.23) in (2.19) and comparing it with (2.14), we are led to identify circle fluxes with off-diagonal components of the constant embedding tensor, i.e.

$$\Theta_{\alpha 0} \equiv -\frac{M_\alpha}{2}. \quad (2.24)$$

In this particular case these are the only nontrivial components of the constant embedding tensor. Thus, being the latter of rank one, the shift symmetry of only one of the T_α 's is effectively gauged. The Kähler potential in (2.18) is related to the kinetic potential (2.13) via a Legendre transformation with R and $\text{Re} T_0$ conjugate variables. Thus, we find

$$K(T_\Sigma + \bar{T}_\Sigma) = -\log(\text{Re} T_0) + K^{4d}(T_\alpha + \bar{T}_\alpha), \quad \text{Re} T_0 = \frac{1}{R}. \quad (2.25)$$

Note that $K_{T_0 \bar{T}_\alpha} = 0$ and $K_{T_\alpha \bar{T}_\beta} = \tilde{K}_{T_\alpha \bar{T}_\beta}$. Thus, the kinetic terms for the scalars T_α in (2.18) correctly reproduce the ones in the action (2.12). Moreover, we observe that for such a simple Kähler potential, a no-scale property holds for the field T_0 alone, namely:

$$K^{T_0 \bar{T}_0} K_{T_0} K_{\bar{T}_0} = 1. \quad (2.26)$$

Inserting (2.26) in (2.21), and using that W is independent of T_0 at this point, we recover the standard scalar potential of 4d, $\mathcal{N} = 1$ supergravity,² with the factor of 3 in front of $|W|^2$.

It is easy to see that the D-term potential, which in the present case becomes

$$\mathcal{T} = \frac{1}{2} D^0 M_\alpha D^\alpha, \quad (2.27)$$

gives rise exactly to the scalar potential in (2.12). Indeed, by inserting (2.23) in (2.20), one finds

$$D^\Lambda = 2K_{T_\Lambda}. \quad (2.28)$$

Therefore, by inserting (2.27) in (2.21) and using (2.28) and (2.25), one gets

$$K^{T_\alpha \bar{T}_\beta} \partial_{T_\alpha} \mathcal{T} \partial_{\bar{T}_\beta} \mathcal{T} = R^2 \tilde{K}_{T_\alpha \bar{T}_\beta} M_\alpha M_\beta. \quad (2.29)$$

Moreover, the contribution to the 3d scalar potential of the other two $\mathcal{T}$ -dependent terms in (2.21) exactly cancel, i.e.

$$K^{T_0 \bar{T}_0} \partial_{T_0} \mathcal{T} \partial_{\bar{T}_0} \mathcal{T} - \mathcal{T}^2 = 0. \quad (2.30)$$

In order to recover the other terms in the action (2.12), we have to systematically perform the dualization, which brings (2.18) to the dual action written in terms of the chiral multiplets T_α and the vector multiplet (R, A^0) . As we have just seen, at the level of the Kähler potential, such a procedure simply amounts to a Legendre transformation. The relevant terms of the action (2.18) are:

$$-K_{T_0 \bar{T}_0} d\text{Re} T_0 \wedge *_3 d\text{Re} T_0 - K_{T_0 \bar{T}_0} (d\text{Im} T_0 - 2\Theta_{0\alpha} A^\alpha) \wedge *_3 (d\text{Im} T_0 - 2\Theta_{0\beta} A^\beta) - \Theta_{0\alpha} A^\alpha \wedge F^0. \quad (2.31)$$

²In 4 noncompact dimensions, the fluxes (2.8) must vanish in order not to break Poincaré invariance of the vacuum.

Note that the action (2.31) is invariant under the gauge transformation:

$$\text{Im } T_0 \rightarrow \text{Im } T_0 + 2\Theta_{0\alpha}\lambda^\alpha(x), \quad A^\alpha \rightarrow A^\alpha + d\lambda^\alpha, \quad (2.32)$$

which, in contrast to the one in (2.17) for the 0 index, has no geometrical origin. Let us now use the following dualization algorithm:

- (1) Fix the gauge (2.32) by setting $\text{Im } T_0 = 0$: Hence $\Theta_{0\alpha}A^\alpha$ becomes a massive vector by the Higgs mechanism.
- (2) Treat $\Theta_{0\alpha}A^\alpha$ as a Lagrange multiplier and eliminate it by setting it to its on-shell value.

The equation of motion for $\Theta_{0\alpha}A^\alpha$ reads:

$$\Theta_{0\alpha}A^\alpha = \frac{1}{8}K^{T_0\bar{T}_0} *_3 F^0 \quad (2.33)$$

where we have used that $*_3^2 = -1$ with Minkowski signature $(-, +, +)$. Inserting (2.33) back into (2.31) we get exactly the kinetic term for the vector multiplet (A^0, R) in (2.12), which now makes A^0 a dynamical vector field. In the derivation we have used (2.13), which in turn implies $\text{Re } T_0 = R^{-1}$ and the relation $K^{T_0\bar{T}_0} = -4\tilde{K}_{RR}$. Note that no Chern-Simons term survives after the dualization, in the action (2.12).

To summarize, once we turn on $d(\text{Im } T_\alpha)$ fluxes (allowing a dependence of the imaginary parts of the 4d scalars on the compact coordinate), the corresponding 3d scalars will transform under a local Peccei-Quinn symmetry with charges equal to the flux quanta. From the 4d perspective such gauging is a consequence of the 1d diffeomorphisms along the S^1 , while from the 3d perspective it is induced by nontrivial Chern-Simons couplings. As remarked above, such symmetry is spontaneously broken and its gauge boson, which is the Kaluza-Klein vector A^0 , can acquire mass by an “affine” Higgs mechanism. Nevertheless, the 3d vector multiplet (A^0, R) can still be dualized to the 3d chiral multiplet T_0 since, thanks to the nontrivial embedding tensor Θ , the field A^0 will still appear in the dual action as a nondynamical vector, i.e. without kinetic term.

One can actually think of yet another, much more subtle contribution to the 3d landscape, namely, the one related to the Θ_{00} term of the constant embedding tensor. In analogy to the fluxes above, this is induced by a nontrivial dependence of the purely 3d field $\text{Im } T_0$ on the parameter y . This clearly implies an additional gauging for T_0 , which will now be charged also under the S^1 -diffeomorphisms. One therefore expects new terms in the scalar potential, dependent on this new charge. These are expected to arise from the reduction of the 4d Ricci scalar R_4 in the presence of this more involved geometry. We will not attempt to perform such a reduction here, but rather we will tackle this problem in the M-theory context in Sec. III, where the structure of the internal elliptic fibration allows to intro-

duce these new fluxes, and at the same time suggests a way to reabsorb them by a suitable field redefinition.

B. Inclusion of 4d vectors and gaugings

Let us now introduce vectors in 4d and perform again the fluxed compactification of the previous section. We will need this extension in order to generalize nonperturbative contributions to the superpotential in the case there are circle fluxes which gauge the shift symmetry of T_0 . Since the reduction is now much more involved, we give here only the result in terms of the 3d, $\mathcal{N} = 2$ action (2.18), with all the fields dualized to complex scalars, leaving to appendix A the details of the dualization to the vector multiplet formalism. For the discussion of such reductions without circle fluxes, see also [22,23,27,28]

For simplicity we will focus on an $SU(N)$ gauge theory. In string constructions such a theory can arise from stacks of D-branes leading to a $U(N) = SU(N) \times U(1)$ gauge theory. The overall $U(1)$ will often become massive at the Kaluza-Klein scale by geometric Stückelberg couplings even in the absence of fluxes (see [23] for a recent discussion). We will therefore neglect it in the following, and only make some additional comments on this $U(1)$ in the discussion of instanton corrections. The action for the non-Abelian field strength $F = dA + A \wedge A$ of $SU(N)$ is given by

$$S_F^{(4)} = - \int \frac{1}{2} \text{Re} \tau \text{Tr}(F \wedge *_4 F) + \frac{1}{2} \text{Im} \tau \text{Tr}(F \wedge F), \quad (2.34)$$

where τ is the gauge coupling function which depends holomorphically on the complex scalar fields. As before we will only consider effective theories including the fields T_α , and do not allow for charged fundamental matter.

In the string compactifications the $SU(N)$ can appear in a broken phase due to Abelian gaugings of the T_α arising from background fluxes. More precisely, the ordinary derivatives in (2.1) of the T_α are replaced by covariant derivatives

$$\nabla T_\alpha = dT_\alpha + \text{Tr}(f_\alpha A), \quad (2.35)$$

where f_α is a constant matrix in the adjoint of $SU(N)$.

For generic gaugings, provided $n_s \geq N - 1$, the $SU(N)$ is completely broken to the Cartan subgroup $U(1)^{N-1}$ with Abelian vectors A^i . The $\mathcal{N} = 1$, 4d action (2.1) then becomes

$$\begin{aligned} S^{(4)} = & - \int \frac{1}{2} R_4 *_4 \mathbf{1} + K_{T_\alpha \bar{T}_\beta}^{4d} \nabla T_\alpha \wedge *_4 \overline{\nabla T_\beta} \\ & + \frac{1}{2} \text{Re} \tau_{ij} F^i \wedge *_4 F^j + \frac{1}{2} \text{Im} \tau_{ij} F^i \wedge F^j \\ & + (V_F + V_D) *_4 \mathbf{1} + \mathcal{L}_{\mathcal{R}^2}, \end{aligned} \quad (2.36)$$

where $\mathcal{L}_{\mathcal{R}^2}$ is the higher curvature term given in (2.2). Since this theory arises from a broken non-Abelian group

the classical gauge coupling functions τ_{ij} are determined from the τ in (2.34) by

$$\tau_{ij} = C_{ij}\tau, \quad (2.37)$$

with C_{ij} obtained from the trace of two Cartan generators in a chosen basis. The non-Abelian gauge symmetry is broken by gaugings of the form

$$\nabla T_\alpha = dT_\alpha + X_{i\alpha}A^i, \quad X_{i\alpha} = \text{Tr}(f_\alpha \mathcal{G}_i), \quad (2.38)$$

where $\mathcal{G}_i$ are the chosen Cartan generators for $SU(N)$ and $X_i = X_{i\alpha}\partial_{T_\alpha} + \bar{X}_{i\bar{\alpha}}\partial_{\bar{T}_\alpha}$ become the set of Killing vectors generating a subgroup of isometries of the Kähler manifold with coordinates T_α , which are gauged by A^i . Such gaugings induce a nontrivial D-term potential

$$V_D = \frac{1}{2} (\text{Re}\tau)^{ij} D_i D_j, \quad (2.39)$$

where $\text{Re}\tau^{ij}$ is the inverse of $\text{Re}\tau_{ij}$, and the D-terms satisfy the relation

$$i\partial_{T_\alpha} D_i = K_{T_\alpha \bar{T}_\beta} \bar{X}_{i\bar{\beta}}. \quad (2.40)$$

Note that we will restrict to the case of gauged shift symmetries (2.3) with constant $X_{i\alpha}$ such that (2.40) is readily integrated. The integration constant is believed to be zero in string compactifications.

In the next step we will restrict to a leading gauge coupling function τ linear in the complex scalars T_α . This is analogous to (2.5), and in summary we have

$$\tau = \frac{1}{2} C^\alpha T_\alpha, \quad \sigma = \frac{1}{8} k^\alpha T_\alpha, \quad (2.41)$$

with C^α and k^α being constant vectors. We will assume that these two couplings are well-defined along S^1 , which implies that the field strength of their imaginary parts cannot have any flux. In other words, when imposed on the shift symmetries, we have the restriction

$$C^\alpha Q_\alpha^\Sigma = 0, \quad (2.42)$$

where Q_α^Σ is parameterizing the shift symmetries as in (2.3). This is in complete analogy to (2.6) for the $\mathcal{R} \wedge \mathcal{R}$ -term. Using the definition of the fluxes M_α in (2.8) we find that

$$C^\alpha M_\alpha = 0. \quad (2.43)$$

Note that gauging the shift symmetries parameterized by Q_α^Σ amounts to specifying constant Killing vectors $X_{i\alpha}$ picked among the set of Q 's. From the conditions (2.6) and (2.42) one thus infers

$$C^\alpha X_{i\alpha} = 0, \quad k^\alpha X_{i\alpha} = 0 \quad \forall i. \quad (2.44)$$

These conditions assure that in the 4d theory τ and σ are not gauged by any of the A^i , so that the action (2.36) is classically gauge invariant.

We now aim to reduce this more general 4d theory down to 3 dimensions on S^1 with the background metric given in (2.7) and fluxes (2.8). We take again the metric fluctuation of the form (2.10), while the 4d vector fields split as follows:

$$A_p^i = (A_\mu^i - A_\mu^0 \zeta^i, \zeta^i), \quad (2.45)$$

$$\langle A^i \rangle = (0, \langle \zeta^i \rangle)(y), \quad (2.46)$$

where by abuse of notation we denote also the 3d vectors by A^i , and ζ^i are 3d real scalars. Therefore, in addition to the vector multiplet of the graviphoton, we will now have new 3d vector multiplets, with bosonic content given by:

$$(A^i, \xi^i) \quad \xi^i \equiv R\zeta^i. \quad (2.47)$$

Note that nontrivial background values (2.46) for the 4d vectors are in general allowed along the S^1 . Because of the large gauge transformations of the 4d vectors, these vevs undergo integral shifts

$$\langle \zeta^i \rangle \rightarrow \langle \zeta^i \rangle + \ell^i, \quad \ell^i \in \mathbb{Z}. \quad (2.48)$$

In addition, the definition (2.8) for the fluxes slightly changes here due to the presence of 4d gaugings which make (2.8) no longer invariant under large gauge transformations. This can be cured by replacing the simple derivative with the covariant one as

$$M_\alpha = \int_{S^1} \langle \nabla \text{Im} T_\alpha \rangle, \quad (2.49)$$

which is now a consistent, gauge-invariant definition. Let us observe that, if we want to keep Poincaré invariance in three-dimensions, no fluxes can be introduced which arise from the 4d vectors.³

As we noted above, in three-dimensions we can eliminate all dynamical vectors in favor of scalars to bring the action into the form (2.18). In order to do that for the vectors in (A^i, ξ^i) and (A^0, R) we introduce the dual complex coordinates

$$T_i = -\frac{2}{R} \text{Re}\tau_{ij} \xi^j + i\tilde{\zeta}_i, \quad (2.50)$$

$$T_0 = \frac{1}{R} + \frac{1}{R^2} \text{Re}\tau_{ij} \xi^i \xi^j + i\tilde{\zeta}_0. \quad (2.51)$$

Here $(\tilde{\zeta}_i, \tilde{\zeta}_0)$ are the dual real scalars which carry the dynamical degrees of freedom of the vectors (A^i, A^0) . Note that $\text{Re}T_i$ and $\text{Re}T_0$ are connected with R , ξ^i via a Legendre transform as we discuss in appendix A. The resulting Kähler potential is given by

³One could only think of integrating their field strengths on the compact direction, but this would clearly generate a Lorentz-breaking flux in 3d.

$$K(T_{\Sigma} + \bar{T}_{\Sigma}) = -\log(\text{Re}T_0 - \frac{1}{4} \text{Re}\tau^{ij}\text{Re}T_i\text{Re}T_j) + K^{4d}(T_{\alpha} + \bar{T}_{\bar{\alpha}}), \quad (2.52)$$

where $T_{\Sigma} = (T_0, T_i, T_{\alpha})$. As a simple check, one notes that the expression for K and T_0 reduces to (2.25) in the absence of vectors, i.e. when $T_i = 0$. Furthermore, let us observe that this Kähler potential no longer fulfills the no-scale property (2.26). The latter is indeed violated by the presence of vectors in the following way:

$$K^{T_0\bar{T}_0}K_{T_0}K_{\bar{T}_0} = 1 + \frac{2\text{Re}\tau_{ij}\xi^i\xi^j}{R}. \quad (2.53)$$

It is then easy to see that the Kähler potential (2.52) satisfies the new no-scale property:

$$K^{T_I\bar{T}_I}K_{T_I}K_{\bar{T}_I} = 1, \quad (2.54)$$

where $I = \{0, i\}$.

After integrating over S^1 , performing the Weyl rescaling, and dualizing all the vector multiplets to chiral multiplets, the 3d action can be brought into the form (2.18), with an additional term given by the reduction of the higher derivative correction $\mathcal{L}_{\mathcal{R}^2}$. In the following we will not need the explicit form of this higher curvature terms. However, as a crucial result of the reduction including the fluxes M_{α} and gaugings $X_{\alpha i}$ we find a modified covariant derivative of the form:

$$\nabla T_{\Lambda} = dT_{\Lambda} - 2i\Theta_{\Lambda\Sigma}A^{\Sigma}, \quad (2.55)$$

where generally all vectors $A^{\Lambda} = (A^0, A^i, A^{\alpha})$ can appear in the gauging. Explicitly the gaugings are determined by

$$\Theta_{0\alpha} = -\frac{M_{\alpha}}{2}, \quad \Theta_{i\alpha} = \frac{i}{2}X_{i\alpha}, \quad (2.56)$$

$$\Theta_{\alpha\beta} = \Theta_{00} = \Theta_{0i} = \Theta_{ij} = 0,$$

which generalizes (2.24). The scalar potential is given by Eq. (2.21) with $W = W_{4d}$ and a D-term potential

$$\mathcal{T} = -\frac{1}{2}D^{\Lambda}\Theta_{\Lambda\Sigma}D^{\Sigma} = -\frac{K_{T_{\alpha}}(M_{\alpha} + \frac{i}{2}X_{i\alpha}\text{Re}\tau^{ij}\text{Re}T_j)}{\text{Re}T_0 - \frac{1}{4}\text{Re}\tau^{ij}\text{Re}T_i\text{Re}T_j}, \quad (2.57)$$

with 3d D-terms D^{Λ} still given by Eq. (2.28). In the last equality we used (2.52) and (2.56). The explicit computation of the scalar potential can be found in the Appendix, where it is shown to coincide with the one obtained by dimensionally reducing the 4d action (2.36).

C. Taub-NUT instantons and the stabilization of the radion

In this section we will include a particular set of supersymmetric corrections to the three-dimensional effective theory. In particular we will argue that four-dimensional gravitational instantons, more precisely Taub-NUT geome-

tries, will appear as corrections in the three-dimensional superpotential W . This will also provide us with a mechanism to stabilize the radion field r , or rather the combination T_0 introduced in (2.50). In the presence of a four-dimensional superpotential $W_0 = W_{4d}$, and the absence of fluxes M_{α} along the S^1 this stabilization mechanism will be completely universal. The generalization including fluxes will be significantly more complicated and discussed separately in Sec. IID.

Let us first assume that the three-dimensional superpotential is independent of the scalar T_0 parameterizing the size of the fourth dimension. In this case the superpotential is of the form [29,30]

$$W = W_{4d}(T_{\alpha}) + \sum_i A_i e^{-2\pi T_i} + \tilde{A} e^{-2\pi\tau} e^{2\pi\sum_i a_i T_i} \quad (2.58)$$

where a_i are the Dynkin labels of the gauge group in (2.34), e.g. $a_i = 1$ for $SU(N)$. Inserting this into the three-dimensional scalar potential (2.21) using the Kähler potential (2.52), one finds that the radion r has no local minimum at finite r . If in the moduli stabilization the negative term in (2.21) is canceled, r tends to minimize the potential in the limit $r \rightarrow \infty$. This is due to the fact that the scalar potential (2.21) admits an overall factor $e^K = \frac{e^{K^{4d}}}{r^2}$, where K^{4d} is the four-dimensional Kähler potential. Clearly, this limit corresponds to the decompactification limit to four space-time dimensions.

Let us now ask if one can obtain a local minimum at finite r by including further corrections to the 3d superpotential. Firstly, note that there cannot be any perturbative corrections in T_0 to W . This is due to the fact that $\text{Im}T_0$ has a perturbative shift-symmetry, since it arises from dualizing a 3-dimensional vector A^0 . The immediate possibility is a new nonperturbative correction of the form

$$W(T_0) = \mathcal{A} e^{-2\pi T_0}, \quad (2.59)$$

where $\mathcal{A}$ is some T_0 -independent prefactor, which will be discussed in more detail below. The first check on such a superpotential is provided by gauge invariance. However, since we focus on the fluxless reduction $M_{\alpha} = 0$ this is trivially the case since T_0 is not charged. In addition, one has to check the zero mode condition, and hence identify the source of the correction (2.59). Since T_0 parameterizes degrees of freedom of the 4d metric, the superpotential is expected to be induced by a four-dimensional gravitational instanton. Let us note that the gravitational instanton of interest is a Euclidean Taub-NUT space and its AdS generalizations.⁴

To motivate that (2.59) arises from a Taub-NUT gravitational instanton let us evaluate the four-dimensional action. We first summarize the key properties of the

⁴We Wick-rotate the time direction because we consider Taub-NUT instantons which will later admit gauge instantons.

Taub-NUT gravitational solution. The Taub-NUT is a special case of an ALF space (asymptotically locally flat space). It is a noncompact manifold, which is everywhere a fibration of S^1 over $\mathbb{R}^3$, apart of a single point, the nut, at which the circle shrinks to zero size. Its asymptotic boundary ∂TN is topologically a three-sphere S^3 realized as an Hopf fibration of S^1 over S^2 with first Chern number 1. The metric of Taub-NUT is well-known [6,7] and can be written as follows:

$$ds^2 = \left(\frac{1}{\lambda^2} + \frac{1}{\rho^2}\right)(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) + \frac{1}{\frac{1}{\lambda^2} + \frac{1}{\rho^2}}(d\psi + \cos\theta d\phi)^2, \quad (2.60)$$

where $\rho \in [0, \infty)$ parameterizes the radial distance from the nut, $\theta \in [0, \pi)$, and $\phi \in [0, 2\pi)$ are the angular coordinates of the S^2 at infinity of the base ($\rho \rightarrow \infty$), and $\psi \in [0, 4\pi)$ parameterizes the fiber S^1 . The real number λ represents the asymptotic radius of the fiber and therefore it is proportional to the expectation value of the radion field on this Tub-NUT background, i.e. $\lambda = \langle r \rangle / 4\pi$. In the limit $\lambda \rightarrow \infty$, the metric (2.60) reduces to the flat metric on $\mathbb{R}^4$. Using (2.60) one finds that the curvature form on this space is anti-self-dual, i.e. $*\mathcal{R}_4 = -\mathcal{R}_4$. The topology of TN is that of $\mathbb{R}^4$ and yields a first Pontrjagin number

$$p = -\frac{1}{2} \int_{\text{TN}} \text{Tr}(\mathcal{R}_4 \wedge \mathcal{R}_4) = 2. \quad (2.61)$$

Because of the noncompactness of the Taub-NUT space the evaluation of the action is nontrivial, since it has to be regularized. This was first done in Ref. [31] by subtracting the action of Euclidian flat space, despite the fact that the boundaries of these two noncompact spaces do not match. The resulting action was found to be

$$S_{\text{TN}}^{(4)} = -\frac{4\pi n^2}{G} = -\frac{\langle r \rangle^2 M_p^2}{2} = -2\pi \langle r \rangle^2, \quad (2.62)$$

where we have expressed the so-called nut-charge $n = \langle r \rangle / 8\pi$ in terms of the circumference $\langle r \rangle$ of the periodic Euclidian direction at infinity. In order to evaluate (2.62) we have used that our action (2.1) is negative, and the reduced Planck mass square M_p^2 is 4π in the conventions of [23]. The action (2.62) was later confirmed using the AdS-CFT prescription of [32] to regularize the gravitational action of Taub-NUT-AdS₄ in [33]. For Taub-NUT-AdS₄ the action is given by

$$S_{\text{TN-AdS}}^{(4)} = -\frac{4\pi n^2}{G} \left(1 - \frac{2n^2}{l^2}\right) \quad (2.63)$$

where l is the AdS-radius. This indeed yields (2.62) when sending l to infinity. One realizes that in the absence of vectors the action $S_{\text{TN}}^{(4)}$ in (2.62) agrees with the real part of T_0 , $\text{Re}T_0 = r^2$. It would be interesting to perform the

reduction around an AdS-background and match the action (2.63).

Let us include the higher curvature terms (2.2) and denote this corrected action by $\tilde{S}_{\text{TN}}^{(4)}$. One then obtains by using the anti-self-duality of $\mathcal{R}$ and (2.61) that the Taub-NUT gravitational instanton contributes in the 3d effective theory with an exponential $e^{\tilde{S}_{\text{TN}}^{(4)}}$. Hence, one finds that the correction to the superpotential is of the form

$$W(T_0) = \mathcal{A} e^{-8\pi\sigma} e^{-2\pi T_0}. \quad (2.64)$$

A by now well-known proposal to stabilize moduli using instanton corrections was given by KKLT [9]. Here we suggest that the correction (2.64) precisely allows us to generate a new vacuum at finite r . In order to do that one can proceed in various ways, depending on the scale at which the fields T_α are stabilized. We discuss two scenarios in turn.

Let us first consider the case that all fields T_α have been supersymmetrically fixed in 4d solving $D_{T_\alpha} W_{4d} = 0$ with the T_α appearing in the superpotential. If the masses of such stabilized T_α are sufficiently large they can be integrated out from the 4d effective theory. In the vacuum this can lead to four-dimensional $W_{4d} = W_0$ in (2.58), where W_0 is a small constant. For example, in a Type IIB compactification W_0 can be determined by a 4d flux superpotential as in [9,34,35]. Clearly, this implies that the 4d effective theory has a scalar potential

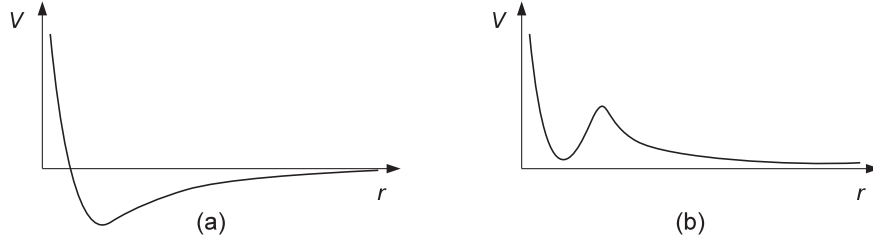
$$V_{4d} = -3e^{K_{4d}} |W_0|^2, \quad (2.65)$$

with negative cosmological constant. The supersymmetric solutions are AdS₄ vacua. Considering this effective theory compactified on a circle, the three-dimensional superpotential is expected to be of the form

$$W = W_0 + \mathcal{A} e^{-2\pi T_0}, \quad (2.66)$$

where we note that the nonperturbative contribution should now arise from Taub-NUT-AdS₄ gravitational instantons. Together with the Kähler potential (2.25) one obtains a supersymmetric AdS₃-vacuum at a value $\langle T_0 \rangle$ obtained by solving $D_{T_0} W = 0$. This is depicted in Fig. 1(a). For sufficiently small $\langle \text{Re}T_0 \rangle$ the theory is effectively three-dimensional.

A second possibility is to consider the case where instanton corrections in T_α and T_0 are included altogether in the 3d effective superpotential. This can be the case if the masses of the T_α are of the same order as the mass of T_0 . In the evaluation of the Taub-NUT instanton action one then neglects all instanton corrections to the 4d superpotential. If the 4d Kähler potential for the T_α is of no-scale type [36], the scalar potential depending on W_{4d} will vanish identically even in the presence of a nonvanishing $W_{4d} = W_0$. Nonperturbative corrections in the T_α , together with the Taub-NUT instanton corrections (2.64) then induce a 3d superpotential, $W = W_0 + \sum_\alpha \mathcal{B}_\alpha e^{-2\pi n_\alpha T_\alpha} + \mathcal{A} e^{-2\pi T_0}$,

FIG. 1. Scalar potential for the radion r before (a) and after (b) the uplift.

which generically admits AdS_3 vacua at values T_0, T_α obtained by solving $D_{T_0} W = D_{T_\alpha} W = 0$. As we will see in Sec. III this possibility is most directly realized in an M-theory compactification on Calabi-Yau fourfolds to three dimensions. In this case, the T_0, T_α are on the same footing, since they measure sizes of divisors. Also, the nonperturbative corrections uniformly arise from wrapping M5-branes on these divisors, as we discuss in more detail below.

Let us end this section by commenting on a version of this scenario which admits a 3d vacuum with positive cosmological constant. Vacua with positive cosmological constant, in particular, dS_4 vacua, are hard to realize within a string compactification. A prominent proposal was given by KKLT [9], where anti-D3-branes in a warped throat uplift an AdS_4 to a dS_4 vacuum. While this suggestion has been disputed over the last years, see e.g. [37–42], let us assume for a moment that such an uplift is possible. We can imagine two possibilities to link it with the 4d-3d story, which match the two given ways to stabilize the T_0, T_α introduced in the last two paragraphs. Firstly, one could consider a 4d theory in which the T_α are sufficiently massive and can be stabilized at a nonsupersymmetric dS_4 vacuum. Both in deriving the effective 3d theory, and the gravitational instanton solution, which is expected to be Taub-NUT- dS_4 in this case, the fact that the fields T_α settle in a nonsupersymmetric vacuum should be taken into account. The potential is expected to be of the form as given in Fig. 1(b). The second possibility is the analog of KKLT up-lift in 3d, in which the positive cosmological constant is treated as a perturbation of a supersymmetric vacuum. The string realization would be an M-theory compactification with anti-M2-branes localized in a warped throat. Clearly it would be interesting to make these proposals more concrete. In particular, this would open the possibility to study tunnelings from an effectively 3d theory with a dS_3 vacuum to an effectively 4d theory with a cosmologically relevant vacuum. The cosmological implications of such transdimensional tunnelings have been discussed in [10–14].

D. Superpotentials in the presence of fluxes

In this subsection, we want to generalize the discussion of the superpotential to the case where vector fields and circle fluxes M_α are included in the reduction as in Sec. IIB. Recall that in this more general situation new

gaugings are present. Using (2.56) one finds that under the 3d $U(1)$ gauge transformations for A^α the scalars have to shift as

$$T_0 \rightarrow T_0 - iM_\alpha \lambda^\alpha, \quad (2.67)$$

$$T_i \rightarrow T_i - X_{i\alpha} \lambda^\alpha, \quad (2.68)$$

where $\lambda^\alpha(x)$ are the local gauge shifts. These transformations ensure the invariance of the covariant derivatives (2.55). One realizes that the first transformation (2.67) implies that a superpotential $W = \mathcal{A} e^{-2\pi T_0}$ is no longer gauge invariant for a constant prefactor $\mathcal{A}$ due to the nonvanishing fluxes M_α . In the following, we want to discuss a simple way to restore gauge invariance by using a T_i dependent prefactor $\mathcal{A}$ and the transformation (2.68). This is analog to the situation for fluxed D3-brane instantons in Type IIB and F-theory [15]. In [15] the nontrivial cancellation of the gauge transformations arises from the possibility to switch on two-form fluxes on the world-volume of the D3-brane instanton wrapped on a four-cycle with nontrivial topology. Here we also suggest an interpretation of the “fluxes” which have to be included on top of the Taub-NUT geometry.

To begin with let us first note that any correction to the holomorphic superpotential now has to be holomorphic in the complex coordinates T_0, T_i given in (2.50) and (2.51). In particular we realize that the vectors also correct the definition of T_0 and a simple matching of the Taub-NUT action alone with the T_0 is not possible. This is not surprising, since one now has to include a nontrivial field configuration for the 4d vector fields A^i into the discussion. We are looking for instanton solutions for the gauge fields which have an anti-self-dual field strength

$$\mathcal{F} = - * \mathcal{F}. \quad (2.69)$$

This relation should be equally true for the case of $SU(N)$ and $U(1)$ field strengths. In the absence of gaugings it is not hard to show that in the case of an anti-self-dual gauge bundle the Einstein equations are not modified. This implies that the Taub-NUT geometries are still solutions to the equations of motion of the more general action.

We now want to evaluate the 4d Euclidean action including the instanton gauge field configuration. As in the previous subsection we will focus on the real part of the Euclidean action, and later use holomorphicity of the

superpotential to find the complete coordinate dependence. Of crucial importance will be the kinetic term of the gauge fields⁵

$$\begin{aligned} S_{\mathcal{F}} &= -2\pi \int_{\text{TN}} \text{Re} \tau \text{Tr}(\mathcal{F} \wedge * \mathcal{F}) \rightarrow S_{\mathcal{F}} \\ &= -2\pi \int_{\text{TN}} \text{Re} \tau C_{ij} \mathcal{F}^i \wedge * \mathcal{F}^j, \end{aligned} \quad (2.70)$$

where τ is constant in the instanton configuration. The gauge bundle configuration on Taub-NUT can be constructed for non-Abelian gauge groups $U(N)$ as discussed in [43–46]. Since we aim to compute corrections to the 3d action in the Coulomb phase $U(1)^N$ we will focus on the $U(1)$ case with vectors $\mathcal{A}^i$ here. One first realizes that due to the triviality of the second cohomology $H^2(\text{TN}, \mathbb{Z})$ of TN one finds that the $U(1)$ field-strength configuration is globally exact $\mathcal{F}^i = d\mathcal{A}^i$. As boundary conditions for the gauge configurations $\mathcal{A}^i$ we will impose a matching with the reduction of Sec. II B and expand

$$\mathcal{A}^i = \langle \zeta^i \rangle \Lambda + n^i \Lambda, \quad (2.71)$$

where Λ is defined as

$$\Lambda = \frac{\rho}{4\pi(\rho + \lambda^2)} (d\psi + \cos\theta d\varphi), \quad (2.72)$$

with coordinates as in the metric (2.60). Hence the background gauge field strengths will be

$$\mathcal{F}^i = (\langle \zeta^i \rangle + n^i) d\Lambda, \quad (2.73)$$

where $d\Lambda$ is the unique normalizable harmonic two-form on TN. Using the metric (2.60) one checks that $*d\Lambda = -d\Lambda$, such that $\mathcal{F}^i$ is anti-self-dual. Let us comment on the split of (2.71) into $\langle \zeta^i \rangle$ and n^i . The $\langle \zeta^i \rangle$ parameterize the monodromies (Wilson lines) of the gauge fields $\mathcal{A}^i$ on the S^1 fiber at infinity ($\rho \rightarrow \infty$), and are all chosen to be in the range $[0, 1)$. The n^i are integers and determine the topological sector which the instanton configurations $\mathcal{A}^i$ belong to. They have an intrinsic meaning because they make the field strengths in (2.73) invariant under the large gauge transformations (2.48).⁶ In other words, n^i determines a canonical representative in the holonomy class of the configuration $\mathcal{A}^i$, and can be thought of as the first Chern class of the i -th Abelian instanton on TN. Since, as already pointed out, TN has trivial topology, such a class takes values in a “geometrical” version of $H^2(\text{TN}, \mathbb{Z})$ [44]. All this has a natural, topological formulation on the manifold which compactifies TN, namely $\mathbb{CP}^2$ (see e.g. [47]). Indeed, we have a natural surjective map

$$f: \text{TN} \rightarrow \mathbb{CP}^2, \quad f|_{S_\infty^3} \equiv \pi^H: S_\infty^3 \rightarrow S^2, \quad (2.74)$$

⁵In this expression we used the convention $M_p^2 = 4\pi$, as in (2.62).

⁶This can be viewed as an analog to the situation in D-brane configurations where the NS-NS B-field appears with the brane fluxes in the gauge-invariant combination $B + F$.

which fixes TN at finite ρ , while sending its infinity to an S^2 via the Hopf projection. This two-sphere describing the infinity of TN in its compactification represents (the Poincaré dual of) the hyperplane class ω of $\mathbb{CP}^2$, which generates $H^2(\mathbb{CP}^2, \mathbb{Z}) \simeq \mathbb{Z}$. Moreover, one has

$$d\Lambda = f^*(\omega). \quad (2.75)$$

Using (2.69), (2.73), and (2.75) we can easily evaluate (2.70) and we obtain⁷ [44]:

$$\begin{aligned} S_{\text{TN}+\mathcal{F}}^{(4)} &= S_{\text{TN}}^{(4)} - 2\pi \int_{\text{TN}} \text{Re} \tau C_{ij} \mathcal{F}^i \wedge * \mathcal{F}^j \\ &= -2\pi r^2 - 2\pi \text{Re} \tau (n^i n^j C_{ij} + 2n^i C_{ij} \zeta^j \\ &\quad + C_{ij} \zeta^i \zeta^j) \\ &= -2\pi (\text{Re} T_0 - n^i \text{Re} T_i + n^i n^j C_{ij} \text{Re} \tau), \end{aligned} \quad (2.76)$$

where we have used $\xi^i = R\zeta^i$, and the definitions (2.50) and (2.51) of the coordinates $\text{Re} T_0$, $\text{Re} T_i$. Note that (2.76) is only a part of the full 4d, $\mathcal{N} = 1$ supergravity action (2.36) evaluated on this instanton background. In fact, one uses that in the background the kinetic terms of the scalars $\text{Re} T_\Lambda$ vanish for their constant vacuum expectation values. The higher curvature terms can be included as in the previous subsection and yield an additional term linear in the complex field σ . More subtle are the proper inclusion of the imaginary parts of T_Λ , and the scalar potential. Firstly, $\text{Im} T_0$, $\text{Im} T_i$ only arise after dualizing components of the 4d metric and 4d vectors, respectively. Secondly, already in 4d the fields $\text{Im} T_\alpha$ appear in gauge covariant derivatives. The scalar potential can have a background value (2.65) and hence can also contribute deforming the TN into an AdS-Taub-NUT geometry with action (2.63). Clearly, it would be desirable to include these modification in the analysis. In particular, it might be that the gaugings actually modify the metric solution itself. We will leave the study of the backreaction to future work.

In the following we will take a more basic approach to analyze the superpotential. We simply complexify (2.76) such that it can describe a correction to the holomorphic superpotential W , by replacing the real parts of the complex Kähler coordinates with their complete expressions. This leads to the following form of the T_0 -dependent contribution:

$$W(T_0) = \mathcal{C} e^{-8\pi\sigma} \left(\sum_{\vec{n}, n^i} e^{-2\pi \vec{n} \tau} e^{-2\pi C_{ij} n^i n^j \tau + 2\pi n^i T_i} \right) e^{-2\pi T_0}, \quad (2.77)$$

provided the following additional condition holds

$$M_\alpha = -in^i X_{i\alpha} \quad \forall \alpha, \quad (2.78)$$

⁷We omit the brackets $\langle \rangle$, but we remind that all fields are set to their expectation values on the instanton background under consideration.

which assures gauge invariance of (2.77). The form of $W(T_0)$ is reminiscent of a constraint Jacobi form, as will be also apparent from the discussion of the M-theory realization using M5-branes [48].⁸

If $N - 1$, which is the rank of $SU(N)$, is bigger than n_s , i.e. the number of 4d scalars which possess shift symmetry, then⁹ (2.78) could in principle be satisfied by suitably choosing the n^i s. However, in general, Eq. (2.78) is rather a constraint on the fluxes M_α , which therefore cannot be completely arbitrary if we want gravitational instantons to contribute at all to the 3d superpotential. They must be linear combinations of the gaugings appearing in 4d, with coefficients equal to the weights n^i appearing in the non-perturbative superpotential. Notice that the σ , τ -dependent factors in (2.77) are separately gauge invariant. Their charges vanish due to conditions (2.9), (2.43), and (2.44).

Let us conclude this section with an observation. The term in the superpotential (2.77) with coefficient $\bar{n}$ depends on the holomorphic gauge coupling function τ , which in our case is simply given by $\tau = \frac{1}{2}C^\alpha T_\alpha$. The coefficient $\bar{n}$ has an interpretation analogous to the n^i s for the central $U(1)$ gauge field in $U(N)$. If we restore the non-Abelian phase of the gauge theory by switching off the gaugings, we can have a fully non-Abelian $U(N)$ instanton. More precisely, requiring that the Wilson line background of the instanton configuration is trivial, $\bar{n}$ is an integer instanton number given by the second Chern character

$$\bar{n} = -\frac{1}{2} \int_{\text{TN}} \text{Tr} \mathcal{F} \wedge \mathcal{F}, \quad (2.79)$$

where $\mathcal{F}$ is the $U(N)$ -valued curvature of the instanton bundle in units of 2π .

In section III we will provide a description of the Abelian and non-Abelian instanton bundles which appeared here by means of a M5-brane construction. In the Type IIA weak coupling limit we interpret $\bar{n}$ and n^i as wrapping numbers of extended D-brane and NS-brane instantonic sources.

III. M-THEORY REALIZATION AND THE F-THEORY LIMIT

In this section we will show how the supergravity scenario outlined in Sec. II can be embedded into an M-theory compactification. We first briefly review in Sec. III A the dimensional reduction of M-theory on an elliptically fibered Calabi-Yau fourfold, and summarize how the resulting 3d effective theory can be connected to a 4d F-theory compactification. The inclusion of four-form fluxes and the discussion of their induced D-term potential is given in

⁸See also Refs. [49,50] for related discussions.

⁹If we suppose the gauge group $SU(N - n_s)$ left unbroken by the gaugings to be further broken, say to its Cartan torus, by other fluxes, we can allow $X_{i\alpha}$ to be an $(N - 1) \times n_s$ matrix, as for $n_s > N - 1$.

Secs. III B and III C. This allows us to make contact with the 3d gauged supergravity theories obtained in Sec. II by fluxed reduction on a circle. Finally, in Sec. III D we present a microscopic realization of the gravitational and gauge instanton corrections introduced in the 4d context in Secs. II C and II D. This requires a discussion of M5-brane instantons and their anomalies.

A. M-theory on Calabi-Yau fourfolds and the F-theory limit

We first recall the basic steps to link M-theory and Type IIB string theory [4,5]. Consider M-theory compactified on a two-torus T^2 , naming one of the 1-cycles the A-cycle, and the other 1-cycle the B-cycle. The metric background is of the form

$$ds_{11}^2 = \frac{v^0}{\text{Im}\tau} ((dx + \text{Re}\tau dy')^2 + (\text{Im}\tau)^2 dy'^2) + ds_9^2, \quad (3.1)$$

where τ is the complex structure modulus of the T^2 , and v^0 describes its volume. If the volume v^0 of the two-torus is small, one can pick one of the 1-cycles, say the A-cycle, to obtain Type IIA string theory. T-duality along the B-cycle leads to the corresponding Type IIB set-up, and identifies τ as the Type IIB dilaton-axion $\tau = C_0 + ie^{-\phi}$.

This construction can be now applied fiberwise for the elliptically fibered Calabi-Yau fourfold X_4 . Consider M-theory on X_4 , which leads to a three-dimensional theory with $\mathcal{N} = 2$ supersymmetry. The reduction and T-duality on the elliptic fiber leads¹⁰ to Type IIB string theory on $B_3 \times S^1$, where B_3 is the base. Indeed, it is possible to see that the nontriviality of the original elliptic fibration is now encoded by the IIB dilaton-axion profile and thus the Type IIB Einstein frame metric gives simply a direct product geometry. Let us have a closer look at this set-up from a four-dimensional point of view. As in (2.10) we label the S^1 dimension by y and note that the four-dimensional metric is of the form

$$ds^4 = r^{-2} g_{\mu\nu}^{(3)} dx^\mu dx^\nu + r^2 (dy + A_\mu^0 dx^\mu)^2, \quad (3.2)$$

$\mu, \nu = 0, 1, 2,$

where r is the circumference of the fourth dimension, $g_{\mu\nu}^{(3)}$ is the three-dimensional Einstein frame metric, and A_μ^0 is a three-dimensional vector. In the following we will identify (r, A^0) with their analogs arising in the M-theory reduction on X_4 .

Let us consider for the moment M-theory on a non-singular Calabi-Yau fourfold X_4 with an elliptic fibration (with 0-section), so that there is no non-Abelian gauge symmetry in three dimensions. Here and in the rest of the paper we will restrict to geometries with full $SU(4)$ -holonomy, and demand $h^{2,1}(X_4) = 0$. On smooth elliptic fibrations there is a natural set of divisors which

¹⁰At least in the absence of metric fluxes; see Sec. III B.

span $H_6(X_4, \mathbb{R})$. Firstly, one has the section of the fibration which is homologous to the base B_3 . Secondly, there is the set of vertical divisors D_α which are obtained as $D_\alpha = \pi^{-1}(D_\alpha^b)$, where D_α^b is a divisor of B_3 and π is the projection to the base $\pi: X_4 \rightarrow B_3$. For these smooth elliptic fibrations one has $h^{1,1}(B_3) = h^{1,1}(X_4) - 1$ such divisors. Classically Euclidean M5-branes wrapped on such divisors will couple to the complex coordinates

$$\begin{aligned} T_0 &= \frac{1}{6} \int_{B_3} J \wedge J \wedge J + i \int_{B_3} C_6, \\ T_\alpha &= \frac{1}{6} \int_{D_\alpha} J \wedge J \wedge J + i \int_{D_\alpha} C_6, \end{aligned} \quad (3.3)$$

where C_6 is the dual of the M-theory threeform C_3 . In order to make T_0, T_α dimensionless one would need to multiply the volume terms by ℓ_M^{-6} . The three-dimensional kinetic terms of these chiral multiplets are obtained from a Kähler potential $K^M(T + \bar{T})$. To determine K^M one analyses the Weyl rescaling to the three-dimensional Einstein frame. In a large volume compactification, only the classical volume $\mathcal{V}$ arises as prefactor of the Einstein-Hilbert term. Comparing this with the e^{K^M} prefactor in the scalar potential, one infers [28]

$$\begin{aligned} K^M &= -3 \log \mathcal{V} - \log \int \Omega_4 \wedge \bar{\Omega}_4, \\ \mathcal{V} &= \frac{1}{4!} \int_{X_4} J \wedge J \wedge J \wedge J. \end{aligned} \quad (3.4)$$

To evaluate K^M as a function of $T + \bar{T}$ one first expands the Kähler form J as

$$J = v^0 \omega_0 + v^\alpha \omega_\alpha \quad (3.5)$$

where ω_0, ω_α are the two-forms Poincaré dual to B_3, D_α . Using this expansion one has to solve (3.3) for the modes v^0, v^α of J and insert the result into (3.4). This evaluation is more conveniently performed in a dual picture. Simply performing the dimensional reduction, one notes that the v^0, v^α arise as scalar components of three-dimensional vector multiplets (v^α, A^α) and (v^0, A^0) , where (A^0, A^α) appear in the expansion of the M-theory threeform

$$C_3 = A^0 \wedge \omega_0 + A^\alpha \wedge \omega_\alpha. \quad (3.6)$$

Let us now discuss the map of the M-theory data to the 4d/3d supergravity data of Sec. II A following [22]. The complex coordinate (3.3) are related to the three-dimensional chiral multiplets $T_\Sigma = (T_0, T_\alpha)$ of Sec. II A via a Legendre transform. One now notes that A^0 is precisely the vector which appears in the metric (3.2). Indeed, C_3 with two legs along the elliptic fiber becomes the B-field after reduction to Type IIA and the Kaluza-Klein vector after T-duality to Type IIB. Hence, the T-duality operation in going from an M-theory compactification to a Type IIB compactification is the Legendre transform in the direction T_0 with respect to the Kähler potential K^M . The dual coordinate is given by

$$R = -2\partial_{T_0} K^M = \frac{v^0}{\mathcal{V}}. \quad (3.7)$$

One thus identifies the circumference r in (3.2) with R in the way we have already seen in Sec. II A, i.e.

$$R = r^{-2}. \quad (3.8)$$

One can now evaluate T_0 more explicitly by again using the fact that X_4 is elliptically fibered. For an elliptic fibration the intersection numbers satisfy

$$\mathcal{K}_{\alpha\beta\gamma\delta} = D_\alpha \cap D_\beta \cap D_\gamma \cap D_\delta = 0, \quad (3.9)$$

for the vertical divisors. This allows us to split T_0 in a small R expansion as

$$\text{Re } T_0 = \frac{1}{6} \int_{B_3} J \wedge J \wedge J = \frac{1}{R} + p(R) = r^2 + p(r^{-2}), \quad (3.10)$$

where $p(R)$ is a power series in R with no further poles and we have used (3.3), (3.7), and (3.8).

In the presence of non-Abelian seven-brane stacks the Calabi-Yau manifold itself will become singular due to the singularities in the elliptic fibration. To nevertheless perform the reduction one can resolve these singularities and introduce additional two-form classes for the resolution divisors. Such fourfold resolutions have recently been studied in Refs. [51–55]. Let us assume for simplicity to have just one such stack and let us call ω_i the Poincaré duals of the exceptional divisors D_i , with $i = 1 \dots \text{rk } G$, where G is the non-Abelian gauge group corresponding to the singularity. Let the 7-brane stack wrap a generic divisor of the base given by the Poincaré dual in B_3 of $\omega_{(\tau)} \equiv \frac{1}{2} C^\alpha \omega_\alpha$, C^α being integral coefficients. Then, after blow-up, the singular fibration is replaced by the following resolved one:

$$\hat{\omega}_{(\tau)} = \omega_{(\tau)} + a^i \omega_i, \quad (3.11)$$

where a^i are the Dynkin numbers characterizing the Dynkin diagram of G and $\omega_{(\tau)}$ now corresponds to the fibration of the extended node over the 7-brane stack. Each blow-up divisor is instead a fibration of the corresponding Cartan node over the 7-brane stack. In addition to (3.9), one has the following useful relations for some intersection numbers which involve the blow-up divisors:

$$\begin{aligned} D_0 \cap D_i &= 0, \\ \mathcal{K}_{i\alpha\beta\gamma} &= D_i \cap D_\alpha \cap D_\beta \cap D_\gamma = 0, \\ \mathcal{K}_{ij\alpha\beta} &= D_i \cap D_j \cap D_\alpha \cap D_\beta = -C_{ij}^\gamma \mathcal{K}_{\alpha\beta\gamma}, \end{aligned} \quad (3.12)$$

where $C_{ij}^\alpha \equiv C^\alpha C_{ij}$, the latter being the Cartan matrix of G and $\mathcal{K}_{\alpha\beta\gamma} \equiv \mathcal{K}_{0\alpha\beta\gamma}$. Because of the additional harmonic 2-forms, we will now have further 3d vector multiplets after compactification, whose bosonic content is given by

$$(A^i, \xi^i), \quad \xi^i \equiv \frac{v^i}{V}, \quad (3.13)$$

where, as usual, A^i and v^i come from the expansion along ω_i of C_3 and J respectively.

As opposed to the 4d/3d context, in the M-theory picture it is natural to express all the relevant quantities of the effective theory in a formulation where the dynamical vector multiplets (R, A^0) , (L^α, A^α) and (ξ^i, A^i) are kept in the spectrum [22,27], where we have set $L^\alpha = v^\alpha/V$. So instead of working with a Kähler potential as in the general form of the 3d action (3.4), one is using the Legendre transform

$$\tilde{K}^M(R, L, \xi) = K^M + \frac{1}{2}(T_\Lambda + \bar{T}_\Lambda)L^\Lambda, \quad (3.14)$$

where $L^\Lambda = (R, L^\alpha, \xi^i)$. This expression can be evaluated for $\text{Re}T_\Lambda = \frac{1}{3!} \int_{D_\Lambda} J^3$ and K^M as given in (3.4). Using (3.9) and (3.12) one finds

$$\begin{aligned} \tilde{K}^M = & \log R + \log \left[\left(\frac{L^\alpha L^\beta L^\gamma}{6} - \frac{RL^\alpha L^\beta k^\gamma}{4} \right. \right. \\ & + \frac{R^2 L^\alpha k^\beta k^\gamma}{6} - \frac{R^3 k^\alpha k^\beta k^\gamma}{24} - \frac{L^\alpha L^\beta \xi^i \xi^j C_{ij}^\gamma}{4R} \Big) \mathcal{K}_{\alpha\beta\gamma} \\ & \left. \left. + \frac{\xi^i \xi^j \xi^k L^\alpha}{6R} \mathcal{K}_{ijk\alpha} + \frac{\xi^i \xi^j \xi^k \xi^l}{24R} \mathcal{K}_{ijkl} \right] + 4. \end{aligned} \quad (3.15)$$

In this expression we have introduced the expansion coefficients k^α , and used the fact that $\hat{X}$ is a resolved elliptic fibration. More precisely, one has

$$c_1(B_3) = k^\alpha \omega_\alpha|_{B_3}, \quad \omega_0 \wedge \omega_0 = -\hat{\pi}^* c_1(B_3) \wedge \omega_0 \quad (3.16)$$

with $\hat{\pi}: \hat{X}_4 \rightarrow B_3$.¹¹ All the volumes here are expressed in units of the 11d Planck length. For later purposes, it is useful to write down the precise behavior of these quantities in the small R expansion (F-theory limit). Using the fact that V is quartic in the v 's with leading term $v^0 \cdot (v^\alpha)^3$, and that the L^α remain finite in this limit, it is easy to find the rules [22],

$$\begin{aligned} v^0 &\sim \epsilon, & R &\sim \epsilon^{3/2}, & v^\alpha &\sim \epsilon^{-1/2}, \\ L^\alpha &\sim L_b^\alpha \equiv \frac{v_b^\alpha}{V_b}, & v^i &\sim \epsilon^{3/2}, & \xi^i &\sim \epsilon^2, \end{aligned} \quad (3.17)$$

where the subscript b refers to the corresponding quantities of the base B_3 . Notice that the behavior for the ξ^i s follows from the fact that the ξ^i s of (2.45) should be also vanishing in this limit, in order not to generate a Poincaré-breaking background in four dimensions.

¹¹Notice that the expansion coefficients of $\hat{\pi}^* c_1(B_3)$ along ω_i , which in general are nontrivial, do not contribute to ω_0^2 because of the first relation in (3.12).

B. M-theory fluxes

In M-theory compactifications on Calabi-Yau fourfolds one has to distinguish two classes of four-form flux G_4 . One notes that $H^4(X_4, \mathbb{C})$ splits into a horizontal and a vertical part, orthogonal to each other with respect to the nondegenerate scalar product given by wedge and integration:

$$H^4(X_4, \mathbb{C}) = H_H^4(X_4, \mathbb{C}) \oplus H_V^4(X_4, \mathbb{C}). \quad (3.18)$$

The M-theory fluxes G_4 split accordingly. The elements in $H_H^4(X_4, \mathbb{C})$ are of varying Hodge-type and the whole space $H_H^4(X_4, \mathbb{C})$ is spanned by the complex structure variations of the holomorphic (4, 0)-form Ω on X_4 , such that

$$\begin{aligned} H_H^4(X_4, \mathbb{C}) &= H^{4,0} \oplus H^{3,1} \oplus H_H^{2,2} \oplus H^{1,3} \oplus H^{0,4}(X_4, \mathbb{C}) \\ H_H^{2,2} &\subset H^{2,2}|_{\text{primitive}}(X_4, \mathbb{C}). \end{aligned} \quad (3.19)$$

For elliptically fibered Calabi-Yau fourfolds, in this group there are fluxes with one and only one leg along the elliptic fiber and as such do not break 4d Poincaré invariance after the F-theory limit $R \rightarrow 0$. Such fluxes have been studied in the F-theory context recently in Refs. [56–58]. In contrast, the elements in $H_V^4(X_4, \mathbb{C})$ are necessarily of Hodge-type (2, 2), and can be represented as a wedge product of two elements of $H^{1,1}(X_4, \mathbb{C})$. Given the basis $(\omega_0, \omega_\alpha)$ of $H^{1,1}(X_4)$ considered in Sec. III A for a smooth elliptic fourfold, we see that here there are fluxes with either 2 or no legs along T^2 and thus they break 4d Poincaré invariance after the F-theory limit. It is worthwhile to remark here that if the elliptic Calabi-Yau has full $SU(4)$ -holonomy, its base B_3 is not any Kähler manifold, but it has $h^{0,1} = h^{0,2} = h^{0,3}(B_3) = 0$. Hence, the fourth cohomology of B_3 splits as $H^4(B_3) = H^{2,2}(B_3) \simeq H^{1,1} \wedge H^{1,1}(B_3)$. This implies that all fluxes with 0 or 2 legs along T^2 (4d Poincaré breaking) lie in the second summand of (3.18).

Let us further analyze fluxes when X_4 is nonsingular. Since we have given an explicit basis of $H^{1,1}(X_4)$ in Sec. III A, we can classify the parts of G_4^V as

$$G_4^V = N^\alpha \omega_0 \wedge \omega_\alpha + \tilde{N}_\alpha \tilde{\omega}^\alpha, \quad (3.20)$$

where $\tilde{\omega}^\alpha$ are the pull-backs of four-forms on B_3 such that $\int_{B_3} \omega_\alpha \wedge \tilde{\omega}^\beta = \delta_\alpha^\beta$. The coefficients $N^\alpha, \tilde{N}_\alpha$ are appropriately quantized to satisfy (3.27). Note that the coefficient of the term $\omega_0 \wedge \omega_0$ has been absorbed in the N^α . One also has in cohomology

$$\omega_\alpha \wedge \omega_\beta = \mathcal{K}_{\alpha\beta\gamma} \tilde{\omega}^\gamma, \quad \mathcal{K}_{\alpha\beta\gamma} = \int_{B_3} \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma, \quad (3.21)$$

such that the $\tilde{N}_\alpha$ parameterize independent degrees of freedom where the relations among the $\omega_\alpha \wedge \omega_\beta$ have been modded out.

The integers N^α correspond to the Chern class of an S^1 -fibration on B_3 integrated over the 2-cycles $PD_{B_3}(\tilde{\omega}^\alpha)$ in Type IIB. Note that this is consistent with performing the Type IIA limit of M-theory and performing a T-duality around the second circle in the elliptic fiber. In the IIA limit N^α labels H_3 NS-NS fluxes which, as is well-known, are translated to metric fluxes under T-duality along one of their legs. The S^1 fiber is the T-dual of the B -cycle, which we call B_T in the following. Such fluxes make the mere existence of the 4d effective theory questionable, as this circle constitutes the third spatial direction of the 4d space-time. We will later demand that these components of G_4 vanish.

In contrast, the $\tilde{N}_\alpha$ correspond to the flux quanta of the axions obtained by expanding C_4 in four-forms of B_3 in the Type IIB picture as

$$\tilde{N}_\alpha = \int_{B_T} d\text{Im}t_\alpha, \quad C_4 = \text{Im}t_\alpha \tilde{\omega}^\alpha|_{B_3}, \quad (3.22)$$

where $\text{Im}t_\alpha$ are real 4d scalars. Such fluxes do not induce any nontrivial fibration of B_T on B_3 and are S-duality invariant. Note that the fields t_α are related to the fields T_α of Sec. II and III A, whose imaginary parts are 3d dual to the A^α , by the relation

$$T_\alpha = \frac{t_\alpha}{2}, \quad (3.23)$$

as can be checked by comparing the $\mathcal{N} = 1$ gauge coupling functions [23]. The $1/2$ factor is compatible with the gauge transformation of C_6 in 11d supergravity

$$C_6 \rightarrow C_6 + \frac{1}{2} \Lambda_2 \wedge G_4, \quad (3.24)$$

where Λ_2 defines the gauge transformation for C_3 , i.e. $C_3 \rightarrow C_3 + d\Lambda_2$. Equation (3.24) induces a $1/2$ in the gaugings of the shift symmetry of $\text{Im}T_\alpha$, because these gaugings arise from the term $\frac{1}{2} C_3 \wedge G_4$ integrated over the vertical divisors of X_4 . Therefore, one has

$$M_\alpha = \frac{\tilde{N}_\alpha}{2}, \quad (3.25)$$

where M_α define the gaugings we considered in Sec. II A in the context of the 4d/3d reduction.¹²

In the presence of a non-Abelian singularity one has additional vertical fluxes. The complete expansion of G_4^V reads

$$G_4^V = N^\alpha \omega_0 \wedge \omega_\alpha + \tilde{N}_\alpha \tilde{\omega}^\alpha + f^{i\alpha} \omega_i \wedge \omega_\alpha + \tilde{f}^{ij} \omega_i \wedge \omega_j. \quad (3.26)$$

The two new terms will be crucial for the M-theory interpretation of the 4d/3d reduction with vectors performed in

¹²When the Calabi-Yau fourfold is singular there is an additional contribution to the 4d fluxes M_α , as it will be shown in Sec. III C.

Sec. II B. Following the duality recalled in Sec. III A to link M-theory on X_4 with Type IIB on $S^1 \times B_3$ one expects that the $f^{i\alpha}$ map to the periods of a seven-brane two-form flux F_2^i along the i -th Cartan direction of the non-Abelian gauge group G . They exist in the four-dimensional gauge theory in the form of the gaugings $X_{i\alpha}$ introduced in Sec. II B and are not special to three-dimensions. As we will show below, this matching will be modified by the fluxes $\tilde{f}^{ij}$.

Let us close this subsection by commenting on the quantization of the fluxes in (3.26). Notice that $c_2(X_4)$ lies in the second summand of (3.18), and the G_4 fluxes in this part of the cohomology have to satisfy a shifted quantization condition [59]. Vertical fluxes are in fact quantized in the following way

$$G_4^V + \frac{c_2(X_4)}{2} \in H_V^4(X_4, \mathbb{C}) \cap H^4(X_4, \mathbb{Z}). \quad (3.27)$$

This condition has been recently studied explicitly for elliptic, possibly singular, fourfolds [60]. In contrast, horizontal fluxes are never affected by nontrivial quantization conditions.

C. The D-term potential

We are now ready to present an M-theory realization of the terms of the 3d effective action derived in Secs. II A and II B by flux compactification of a 4d, shift symmetric, $\mathcal{N} = 1$ theory.

While the fluxes G_4^H in $H_H^4(X_4, \mathbb{C}) \cap H^4(X_4, \mathbb{Z})$ appear in the Gukov-Vafa-Witten superpotential [61]

$$W_{\text{GVW}} = \int_{X_4} \Omega_4 \wedge G_4^H, \quad (3.28)$$

the vertical fluxes appear in three dimensions in the function [28]

$$\mathcal{T} = \frac{1}{8\mathcal{V}^2} \int_{X_4} J \wedge J \wedge G_4^V. \quad (3.29)$$

They determine the 3d scalar potential as in Eq. (2.21).

Let us focus on the Kähler moduli of the M-theory compactification, which correspond to the fields T_Σ of Sec. II. Using (3.26), the D-term potential (3.29) reads:

$$\mathcal{T} = \frac{1}{8} \int_{\hat{X}_4} (R\omega_0 + L^\alpha \omega_\alpha + \xi^i \omega_i)^2 \wedge (N^\beta \omega_0 \wedge \omega_\beta + \tilde{N}_\gamma \tilde{\omega}^\gamma + f^{j\delta} \omega_j \wedge \omega_\delta + \tilde{f}^{kl} \omega_k \wedge \omega_l) \quad (3.30)$$

where the integral is performed on the blown-up Calabi-Yau fourfold. After using (3.9) and (3.12) to simplify the expression (3.30), one can easily read off the embedding tensor $\Theta_{\Lambda\Sigma}$ in front of the term $L^\Lambda L^\Sigma$, i.e. we have

$$\Theta_{\Lambda\Sigma} = -\frac{1}{4} \int_{\hat{X}_4} G_4 \wedge \omega_\Sigma \wedge \omega_\Lambda. \quad (3.31)$$

It is then clear that in general $\Theta_{0\alpha}$ and $\Theta_{i\alpha}$ will not be the only nonzero components. This is due to the fact that the

flux content and the geometric structure of the full M-theory compactification are much richer than the ones we had in the 4d theory discussed in Sec. II. Therefore, in order that the present M-theory compactification leads to a three-dimensional theory which can also be obtained by a 4d/3d flux compactification, as performed in Sec. II B, we have to impose the following two restrictions:

- (1) We constrain some of the fluxes in order to ensure the vanishing of the components of the embedding tensor which have no analog in the reduction from 4d to 3d presented in Sec. II B. Namely, we impose the conditions (see Eq. (2.56))

$$\Theta_{\alpha\beta} = \Theta_{ij} = \Theta_{00} = \Theta_{0i} = 0. \quad (3.32)$$

- (2) We discard the terms in the kinetic potential (3.15) which lead in the 3d scalar potential to terms $\mathcal{O}(\epsilon^{x>4})$ in the F-theory limit (3.17).

The second restriction comes from the fact that, after reduction of the 4d theory with vectors and the Weyl rescaling $g^{(3)} \rightarrow Rg^{(3)}$, the most suppressed term we get in the large S^1 expansion of the 3d scalar potential looks schematically like $R^2 \zeta^2 \sim \mathcal{O}(\epsilon^4)$, where the first factor is due to the Weyl rescaling. From the direct computation of the scalar potential given in appendix A, one easily infers that, to satisfy such a condition, we need to keep those terms in (3.15) which are at most of order $\mathcal{O}(\epsilon^{5/2})$. The term linear in the first Chern class, which by the rules (3.17) still survives, can be easily eliminated by the following field redefinition¹³

$$L^\alpha \rightarrow L^\alpha + \frac{k^\alpha}{2} R, \quad A^\alpha \rightarrow A^\alpha + \frac{k^\alpha}{2} A^0, \quad (3.33)$$

with all the other fields unchanged. We can then regard the new fields as the right 3d Kähler coordinates in terms of which the 4d/3d reduction goes exactly as described in Sec. II A. Expanding (3.15) according to (3.17) and applying restriction 2 above, the kinetic potential, in the new basis of fields, reads

$$\tilde{K} = \log R - \frac{1}{R} \text{Re} \tau_{ij} \xi^i \xi^j + \log \left(\frac{1}{6} L_b^\alpha L_b^\beta L_b^\gamma \mathcal{K}_{\alpha\beta\gamma} \right) + 4, \quad (3.34)$$

where we have used that the leading order term for τ is given by

$$\tau \sim \frac{L_b^\alpha L_b^\beta C^\gamma}{4 \mathcal{V}_b^{-2}} \mathcal{K}_{\alpha\beta\gamma} \quad \text{with} \quad \mathcal{V}_b^{-2} = \frac{1}{6} L_b^\alpha L_b^\beta L_b^\gamma c K_{\alpha\beta\gamma}. \quad (3.35)$$

¹³Remarkably, such field redefinition eliminates also the term of order three in the first Chern class, which in our context is irrelevant.

This is compatible with the fact that $\tau = \frac{1}{2} C^\alpha T_\alpha$. It is easy to verify that the kinetic potential (3.35) leads via Legendre transformation exactly to the complex coordinates defined in (2.50) and (2.51), as well as to the Kähler potential (2.52).¹⁴ Further details on this theory, including its dualization to the vector multiplet formalism, are given in the Appendix.

Let us now come back to restriction 1 above, and deduce the right constraints on fluxes. After the field redefinition (3.33), the expressions of the various components of the embedding tensor in terms of the flux quanta of G_4 read

$$\begin{aligned} \Theta_{00} &= -\frac{1}{16} (N^\alpha - \tilde{f}^{ij} C_{ij}^\alpha) k^\beta k^\gamma \mathcal{K}_{\alpha\beta\gamma}, \\ \Theta_{0\alpha} &= -\frac{1}{4} \tilde{N}_\alpha + \frac{1}{8} (N^\beta + \tilde{f}^{ij} C_{ij}^\beta) k^\gamma \mathcal{K}_{\alpha\beta\gamma}, \\ \Theta_{\alpha i} &= -\frac{1}{4} (\tilde{f}^{jk} \mathcal{K}_{\alpha ijk} - f^{j\beta} C_{ij}^\gamma \mathcal{K}_{\alpha\beta\gamma}), \\ \Theta_{0i} &= \frac{1}{2} k^\alpha \Theta_{\alpha i}, \\ \Theta_{\alpha\beta} &= -\frac{1}{4} (N^\alpha - \tilde{f}^{ij} C_{ij}^\alpha) \mathcal{K}_{\alpha\beta\gamma}, \\ \Theta_{ij} &= -\frac{1}{4} (f^{k\alpha} \mathcal{K}_{\alpha k i j} + \tilde{f}^{kl} \mathcal{K}_{i j k l} - \tilde{N}_\alpha C_{ij}^\alpha), \end{aligned} \quad (3.36)$$

where $\mathcal{K}_{\alpha ijk}$ and $\mathcal{K}_{ijkl}$ are new intersection numbers whose explicit forms in terms of invariants of the group G are not relevant for our purposes. Imposing restriction 1 amounts to two further constraints on the fluxes given by

$$N^\alpha - \tilde{f}^{ij} C_{ij}^\alpha = 0, \quad (3.37)$$

$$\tilde{f}^{kl} \mathcal{K}_{ijkl} = \tilde{N}_\alpha C_{ij}^\alpha - f^{k\alpha} \mathcal{K}_{\alpha k i j}. \quad (3.38)$$

The conditions (3.37) and (3.38) respectively specify the fluxes N^α (explicitly) and $\tilde{f}^{ij}$ (implicitly) in terms of $\tilde{N}_\alpha$ and $f^{k\alpha}$. However, the latter remain completely unfixed. In this way we achieved $\Theta_{\alpha\beta} = \Theta_{ij} = \Theta_{00} = 0$.

There is one further constraint which we still need to impose, $\Theta_{0i} = 0$. However, as already given in (3.36), this component obeys $\Theta_{0i} = \frac{1}{2} k^\alpha \Theta_{\alpha i}$. The latter identity can be inferred by realizing that the k^α are related to the first Chern class of B_3 and appear in the geometrical constraint (3.16). The vanishing constraint then reads

$$k^\alpha \Theta_{\alpha i} = 0. \quad (3.39)$$

It is interesting to relate this condition to a 4d constraint. In order to do that it is necessary to read off the k^α in a purely 4d context. It was argued in [24] that this can be done using the higher curvature terms (2.2). In fact, the coefficients k^α appearing in the expansion of the 4d scalar σ , Eq. (2.5), coincide with the ones of the expansion of the first Chern

¹⁴One also neglects derivatives of τ with respect to the L 's, because they lead to higher order corrections in ϵ .

class of the M-theory fourfold. This implies that the constraint (3.39) has to be identified with the second condition in (2.44) by using $\Theta_{ai} = \frac{i}{2} X_{i\alpha}$ as in (2.56).

The nonvanishing components $\Theta_{0\alpha}$ and $\Theta_{i\alpha}$ are given in terms of the unconstrained fluxes $\tilde{N}_\alpha, f^{i\alpha}$ together with $\tilde{f}^{ij}$ solving Eq. (3.38). To summarize, except for the conditions (2.44), which already exist in 4 dimensions, $\Theta_{i\alpha}$ is completely arbitrary, exactly as it is in the 4d theory for $X_{i\alpha}$. Analogously, recalling Eqs. (2.9) and (2.43) for the M_α in the 4d/3d reduction, $\Theta_{0\alpha}$ must be such that $C^\alpha \Theta_{0\alpha} = k^\alpha \Theta_{0\alpha} = 0$.

Moreover, let us stress that, when non-Abelian singularities are present, the condition of absence of “metric” fluxes, which in the smooth situation is simply $N^\alpha = 0$, becomes Eq. (3.37). Similarly, the 7-brane gauge flux, which is related to Θ_{ai} gets contributions also from $\tilde{f}^{ij}$ fluxes and not only from $f^{i\alpha}$. As a final comment, notice that in the case $h^{1,1}(B_3) = 1$, the 4d conditions (2.44) are equivalent to absence of 4d gaugings.

D. M5-brane instantons, Taub-NUTs and Fluxes

In this final subsection we provide the M-theory realization of the gravitational and gauge instanton corrections discussed in Secs. II C and II D. M-theory unifies all these corrections as coming from a single source: M5-branes wrapped on divisors of the blown-up Calabi-Yau fourfold $\hat{X}_4$.

Let us first concentrate on the simplest situation of a Taub-NUT instanton without fluxes correcting the 3d superpotential as in (2.59). In Ref. [8] it has been studied when M5-brane instantons have the right number of zero-modes to contribute to the superpotential W of the 3d effective theory. For M-theory on an elliptically fibered Calabi-Yau fourfold with 0-section, and in the absence of G_4 -fluxes, there is always an M5-brane wrapped on the base B_3 which contributes to the superpotential. Using the fact that $h^{i,0}(X_4) = 0$, $i = 1, 2, 3$ one concludes that $h^{i,0}(B_3) = 0$ for the base B_3 of an elliptic fibration. In particular, this implies that the holomorphic Euler characteristic satisfies $\chi_h(B_3) = \sum_i (-1)^i h^{i,0} = 1$. Accordingly, the M5-brane with action T_0 given in (3.3) and (3.10) contributes a superpotential of the form $W = \mathcal{A} e^{-2\pi T_0}$. Note that upon reduction and T-duality this M5-brane instanton turns into a Taub-NUT gravitational instanton. Indeed, reducing M-theory to Type IIA on the A -cycle the M5-brane reduces to an NS5-brane. After T-duality along the B -cycle the NS5-brane is dualized into a Taub-NUT geometry. More precisely, the 10d Type IIB dual to the M5-brane instanton background is $TN \times B_3$. This duality explains the precise match of the 3d M-theory superpotential and the superpotential (2.59) discussed in Sec. II C.

It is important to remark that in the decompactification limit induced by shrinking the torus fiber of X_4 , the F-theory limit $R \rightarrow 0$, the action (3.10) of the M5-brane on B_3 becomes infinite and the corresponding correction to

the superpotential is absent. The dual 4d analog is the fact that the Taub-NUT action is infinite in the limit $r \rightarrow \infty$, in which the metric (2.60) becomes the flat metric on $\mathbb{R}^4$.

Let us now include a nontrivial G_4 -flux into the discussion of the instanton corrections. This will be the M-theory realization of the set-up described in Sec. II D, and provide a microscopic point of view of the compatibility between fluxes and the gravitational instantons. In Sec. II we have seen that circle fluxes induce nontrivial gaugings for the shift symmetries of the scalars T_0, T_α . This in turn forbids the appearance of a superpotential of the form $\mathcal{A} e^{-2\pi T_0}$ with constant $\mathcal{A}$ since this would break the gauge symmetries. Therefore a natural question to ask is, what the lack of gauge invariance is corresponding to for the M5-brane sources. On general grounds, the background fluxes responsible for this phenomenon should restrict nontrivially to the world-volume of the brane sources. For instance, in a string background, D-branes with a nontrivial H-flux on their world-volume can suffer from a Freed-Witten anomaly [21], and thus cannot modify the effective action as instanton corrections. This relation between gauge invariance of the superpotential and absence of Freed-Witten anomalies has already been pointed out in [16,19] for D-brane instantons and in [17,18] in the context of space-time filling D-branes.

In our case, we need the generalization of the Freed-Witten anomaly to the M5-branes of M-theory. This was conjectured by Witten [20], and takes an analogous formulation to the string case. An M5-brane is nonanomalous if and only if the G_4 flux, when restricted to the M5-brane world-volume, satisfies the following condition:

$$G_4|_{M5} = \theta_4(M5), \quad (3.40)$$

where $\theta_4(M5) \in \text{Tor } H^4(M5, \mathbb{Z})$ is a 2-torsion class of the M5-brane. The four-form θ_4 generalizes the third Stiefel-Whitney class W_3 of the D-brane case. We expect that θ_4 vanishes for complex manifolds, as W_3 does. Therefore, the M5 brane wrapping the base B_3 does suffer from this anomaly, because of the nonvanishing G_4 flux (3.26) on its world-volume. In particular, the G_4 fluxes dual to the circle fluxes pull back nontrivially to B_3 .

However, in Sec. II D, we have seen a way to cure the lack of gauge invariance of the term $\mathcal{A} e^{-2\pi T_0}$, by allowing the prefactor $\mathcal{A}$ to depend on the moduli T_i . As the form of superpotential (2.77) suggests the modified correction should no longer be simply generated by an M5-brane instanton wrapping the B_3 , but by an M5-brane wrapping a more complicated divisor $\mathcal{D}$ of the resolved fourfold $\hat{X}_4$. Cohomologically the Poincaré dual two-form to $\mathcal{D}$ is

$$\text{PD}_{\hat{X}_4}(\mathcal{D}) = \omega_0 - n^i \omega_i. \quad (3.41)$$

The M5-branes wrapping B_3 and the blow-up divisors D_i dual to ω_i are separately anomalous, due to the fluxes $\Theta_{0\alpha}$ and $\Theta_{i\alpha}$ respectively. In contrast the M5-brane wrapping $\mathcal{D}$ is anomaly free if the constraint

$$G_4|_{\mathcal{D}} = 0: \Theta_{0\alpha} - n^i \Theta_{i\alpha} = 0, \quad (3.42)$$

is satisfied. Using the duality to the 4d setup of Sec. II, we recognize (3.42) to be the constraint (2.78) ensuring a gauge-invariant superpotential. Concerning the first two terms appearing in the 3d superpotential (2.77), they are gauge-invariant by themselves, thanks to the conditions $C^\alpha \Theta_{i\alpha} = k^\alpha \Theta_{i\alpha} = C^\alpha \Theta_{0\alpha} = k^\alpha \Theta_{0\alpha} = 0$, which are the analogs of Eqs. (2.9), (2.43), and (2.44). Hence, in the present language, the corresponding M5-brane sources are anomaly free. Let us finally remark that in case there exists only 1 Kähler modulus, namely $h^{1,1}(B_3) = 1$, in order the M5-brane on B_3 to be nonanomalous, or equivalently the Taub-NUT instanton contribution to be gauge invariant, one has to turn off also the flux for $\text{Im}T$ due to Eq. (2.78), ending up with a fluxless 4d/3d reduction.

It is illuminating to also give the Type IIA interpretation of the M5-brane wrapping $\mathcal{D}$. We do that by reducing M-theory along a small A -cycle of the torus fiber. The result is a system of NS5-D6-D4-branes wrapping various cycles in B_3 . The B -cycle of the torus is still fibered over B_3 , and the various branes are points or segments on this circle as depicted in Fig. 2. More precisely, the D6-branes and NS5-branes are points in the B -circle, with the NS5-brane sitting on the marked point of the circle obtained by the 0-section of the M-theory elliptic fibration. The support of any D4-brane is a fibration of a line segment in the B -cycle over the divisor $S = \mathcal{D} \cap B_3$. These line segments either connect two neighboring D6-branes or a D6-brane with the NS5-brane. The singularity of the intersection between the D4-brane and the NS5-brane is only an artifact of the weak coupling limit [62], as it is smoothed by growing the A -cycle in M-theory, where the two branes both become M5-branes, intersecting on a smooth holomorphic surface (diffeomorphic to S). Upon T-duality along the B -cycle, the D6-branes map to D7-branes on $\text{TN} \times S$ and the D4-branes to D3-branes on S .

Let us now use this brane setting to reproduce the configuration of gauge instantons on Taub-NUT presented in Sec. IID. We consider N D6-branes on $\mathbb{R}^3 \times S$. If they are coincident on the B -cycle they give rise to an $U(N) = SU(N) \times U(1)$ gauge theory on their world-volume. In contrast, if their positions along the circle are generic, the gauge theory is in its Coulomb phase and the group is

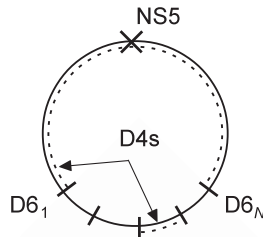


FIG. 2. Brane configuration along the Type IIA compactification circle.

broken to the Cartan torus $U(1)^N$. Notice that this is a different breaking with respect to the one seen in Sec. IID, which is due to gaugings, arising due to brane fluxes. Here, instead, the breaking arises from the resolution of the M-theory singular fiber or, in the dual Type IIB, from a nontrivial Wilson line background of the instanton connection.

Let us first focus on the non-Abelian phase of the gauge theory. T-duality along the B -cycle dualizes the NS5-brane into a Taub-NUT space with a gauge instanton bundle E on it. Adapting the construction of [43,44] to this situation, we can specify the topology of this bundle in terms of the brane construction just described.

- (i) The first Chern class of the instanton bundle vanishes:

$$\text{ch}_1(E) = 0. \quad (3.43)$$

This is due to the fact that, when we have only one NS5-brane localized on a circle, there is no invariant information behind its relative position (or its linking number) with respect to a localized D6-brane, as we can move the D6-brane all along the S^1 always avoiding Hanany-Witten transitions. This is clearly consistent with the fact that $H^2(\text{TN}, \mathbb{Z}) = 0$.

- (ii) The second Chern character of the instanton bundle is given by the number of “complete” D4-branes, namely, those which wrap the entire B -cycle (i.e. from the NS5 back to itself). They lift to M5 instantons wrapping the entire singular fiber. In the singular limit for the blown-up fourfold $\hat{X}_4$, when the full non-Abelian gauge theory on the stack of D7-branes is restored, the class $\omega_{(\tau)}$ represents the full singular fiber over S . Consequently, $e^{-2\pi\bar{n}\tau}$ is the contribution of $\bar{n}$ complete D4-branes and hence we have Eq. (2.79) as the second Chern character of E .

Let us now introduce a Wilson line background for the instanton connection $\mathcal{A}$. As already stated, this will generically break $U(N)$ to $U(1)^N$. Therefore, T-duality will now yield a system of Abelian instantons on Taub-NUT. Their properties can be summarized as follows.

- (i) The monodromy of the instanton connection $\mathcal{A}$ along the circle B_T at infinity of TN is

$$e^{2\pi i \mathcal{A}} = \text{diag}(e^{2\pi i \langle s^1 \rangle}, \dots, e^{2\pi i \langle s^N \rangle}), \quad (3.44)$$

where $\langle s^I \rangle$ is the position of the I -th D6-brane along the asymptotic B -cycle and has been defined in Eq. (2.71). These N positions are related to the $N - 1$ $\langle \zeta^i \rangle$ used in Sec. IID via $\langle \zeta^i \rangle = \langle s^i \rangle - \langle s^{i+1} \rangle$.

- (ii) The second Chern character of the total instanton configuration gets now further contributions. In the Coulomb phase, the contribution $\bar{n}$ replacing Eq. (2.79) is only due to the Abelian field-strength corresponding to the central $U(1)$ inside $U(N)$. More precisely, it coincides with the number of

D4-branes wrapping the segment between the two outer D6-branes, i.e. $D6_1$ and $D6_N$, containing the NS5-brane (see Fig. 2). These lift in M-theory to M5-branes wrapping the extended node of the resolved fiber. In addition, we have all the terms in the action (2.70), which provide continuous contributions depending on the relative positions $\langle \zeta^i \rangle$ of the various D6-branes along the B -cycle. In total we have

$$\text{ch}_2(E) = \bar{n} + n^i n^j C_{ij} + 2n^i C_{ij} \langle \zeta^j \rangle + C_{ij} \langle \zeta^i \rangle \langle \zeta^j \rangle, \quad (3.45)$$

where, for simplicity, we have set to zero the position along the circle of the center of mass of the D6-brane stack, i.e. $\sum_i \langle s^i \rangle$. Here n^i is interpreted as the number of D4-branes stretching along the i -th segment joining two consecutive D6-branes. These lift in M-theory to M5-branes wrapping the i -th Cartan node of the Dynkin diagram of $SU(N)$.

Let us conclude this section with two remarks. One should not forget that the B -cycle is fibered over B_3 , while its T-dual B_T is not. Thus the whole discussion above should be applied fiberwise. However, topological quantities, like linking and winding numbers, do not change as we slide the S^1 along the base. This is confirmed by the fact that they get mapped to Chern characters of the instanton bundle by T-duality. In contrast, continuous information like brane locations along the circle will depend on the internal point and this is reflected after T-duality in a non-trivial dependence of the instanton configuration $\mathcal{A}$ on the internal coordinates.

As a second remark, we stress that in the more complicated M5-brane setups we have analyzed mainly the aspect of anomalies. However, another necessary condition for an M5-brane instanton to contribute to the superpotential is that the M5-brane instantons have the correct number of fermionic zero modes [8,63]. It would be interesting to address this question for the M5-brane configuration on $\mathcal{D}$. M5-brane instantons, relevant for F-theory compactifications to four dimensions, and their zero mode structure have recently been studied in [64–68].

IV. CONCLUSIONS

In the first part of the paper we performed a circle compactification with fluxes starting with a class of 4d, $\mathcal{N} = 1$ supergravity theories, and analyzed in detail the resulting 3d, $\mathcal{N} = 2$ effective actions obtained by Kaluza-Klein reduction. The original 4d theories typically arise from flux compactifications of Type IIB string theory with 7-branes, or their F-theory counterparts. The non-Abelian gauge symmetry can be broken already in 4d to the Cartan torus by gauging the shift symmetries of the scalars. The 4d/3d fluxed reduction includes nontrivial vacuum expectation values for the field strengths of the shift-symmetric 4d scalars. After reduction the 4d S^1 -diffeomorphisms

become 3d gauge symmetries with the graviphoton as gauge boson, and we have shown that the circle fluxes yield additional 3d gaugings of a subset of 3d scalars under this extra gauge symmetry.

The three-dimensional action can be written in a compact form upon dualizing all degrees of freedom to complex scalars, while keeping nondynamical vectors only in order to perform the gaugings. According to the general structure of a 3d, $\mathcal{N} = 2$ gauged supergravity theory, the action is defined by three functions of the scalars: The Kähler potential K , the D-term potential $\mathcal{T}$, and the holomorphic superpotential W . Because of the gaugings, a nontrivial constant embedding tensor appears, inducing Chern-Simons terms for the vector fields. We have evaluated the 3d characteristic data for our S^1 compactification as a function of the 4d characteristic data and the circle fluxes.

Elaborating on the 4d to 3d reduction, we have then discussed certain nonperturbative corrections to the 3d effective theory. More precisely, we have argued that 4d gravitational instantons with the geometry of a Taub-NUT space provide a nontrivial contribution to the 3d effective superpotential. Such a correction depends exponentially on the radion field and is of the form $\mathcal{A} e^{-2\pi T_0}$. In the absence of gaugings, such a term allows us to implement the usual KKLT procedure in order to fix the value of the radion at an AdS_3 minimum. We have discussed various aspects of such a stabilization scheme. In particular, the backreaction of a stepwise moduli stabilization will deform the geometry of the gravitational instanton to Taub-NUT-AdS or Taub-NUT-dS. In a second step one might hope that this vacuum can be uplifted to a metastable 3d de Sitter vacuum. If the existence of such a dS₃ state can be established, as it would be the case if one believes in the KKLT up-lift, one can study the tunneling from an effectively 3d theory to an 4d cosmological vacuum. It would be interesting to explore this possibility in more detail. The 4d/3d perspective will shed some new light on the backreaction of stepwise moduli stabilization and the generation of de Sitter vacua influences the compactification geometry.

Focusing on the inclusion of fluxes, one can study how the gravitational instanton corrections contribute in the presence of 3d gaugings induced by circle fluxes. We have shown that these gaugings forbid a contribution $\mathcal{A} e^{-2\pi T_0}$ to the 3d superpotential if $\mathcal{A}$ is field independent due to its lack of gauge invariance. This conclusion can be avoided by considering a field-dependent prefactor containing the terms $\mathcal{A} \propto e^{n^i T_i}$. For suitably chosen integers n^i 's gauge invariance is restored. This setup corresponds to a system of Abelian gauge instanton bundles on top of the Taub-NUT space. Within such a configuration we have discussed the structure of the superpotential and commented on the origin of contributions arising from higher 4d curvature terms, and the central $U(1)$ in a non-Abelian $U(N)$ gauge theory.

In the second part of the paper, we have provided an M-theory realization of the 4d/3d flux compactifications with gravitational instanton corrections. We studied the compactification of M-theory on a singular, elliptically fibered Calabi-Yau fourfold with G_4 fluxes. The resulting 3d effective theory can be matched with the gauged supergravity obtained by the 4d/3d flux compactification if appropriate restrictions are imposed on both setups. Firstly, the 4d theory used in the circle reduction has to arise from a Type IIB or F-theory compactification, such that the Kähler potentials and 4d gaugings can be matched with the M-theory gaugings. Secondly, also the M-theory reduction has to be restricted, since not all G_4 fluxes have a 4d/3d interpretation. We have found the precise constraints on the G_4 flux in order to match the 3d gaugings derived in the context of the 4d/3d flux reduction. Moreover, a specification of the scaling behavior of the various fields in the F-theory limit allowed us to consistently match the M-theory Kähler potential, with the Kähler potential arising in a 4d/3d reduction.

In the last part of the paper we have also given an M-theory dual of the Taub-NUT gravitational instanton corrections. By reducing M-theory to Type IIA at weak coupling and performing 1 T-duality, the M5-brane wrapped on the base B_3 of the elliptically fibered fourfold was mapped to the Taub-NUT geometry. In the absence of gaugings such an M5-brane universally corrects the 3d superpotential. The precise match of the 3d gaugings and G_4 fluxes allowed us then to provide a microscopic explanation for the lack of gauge invariance of the effective instanton contribution in the presence of a flux background. We have shown that the fluxes inducing genuine 3d gaugings restrict nontrivially to the world-volume of the M5-brane instanton on B_3 . This induces an anomaly on the M5-brane which is analogous to the Freed-Witten anomaly for D-branes. This anomaly can be avoided by replacing the M5-brane on B_3 with an M5-brane which has a world-volume $\mathcal{D}$ with a more involved geometry. Topologically $\mathcal{D}$ is specified by a number of integers n^i which parameterize the wrappings of $\mathcal{D}$ on the resolution divisors of a non-Abelian singularity of the fourfold X_4 . For appropriate choices of n^i , a given G_4 flux, determining the gaugings, can now vanish when restricted to $\mathcal{D}$.

We have also given a Type IIA string description of this M5-brane instanton obtained at weak coupling. The M5-brane splits in this case into a system of D6-branes, D4-branes and an NS5-brane. The D6-branes and the NS5-brane are localized at points on the 4d/3d circle, while the D4-branes stretch between them along intervals. It was possible to connect the geometrical features of the gauge instanton bundle on Taub-NUT to wrapping numbers and relative positions of the Type IIA branes. It will be very interesting to extend this analysis of the M5-brane instanton and corresponding Type IIA configuration. In particular, the structure of zero modes have to be explored further

to guarantee that the instantons actually correct the superpotential instead of other effective 3d couplings.

There are various further directions which can be explored to extend this work. As mentioned above, within the 4d/3d reduction one might hope to get a better understanding of the backreacted 4d geometry of the effectively 3d vacuum obtained by radion stabilization through Taub-NUT instantons. This also includes an extension to allow for a nontrivial warp factor both on the 4d/3d side, as well as in the M-theory reduction. In particular, one might hope that the various 4d geometries parameterizing the points in the effective 3d potential might be found explicitly. The transition between the effectively 3d vacuum and the asymptotic 4d vacuum could admit an interpretation as phase transition in the 4d gravity theory as in [69].

Independent of the precise interpretation one should be able to link the up-lifted 3d vacua with the 4d vacua by a transdimensional tunneling. In the stringy realization this corresponds to a tunneling from a 3d M-theory compactification to a 4d F-theory compactification. The 3d M-theory landscape will allow for more flux vacua than the 4d counterpart, and it deserves further study. It would be also interesting to use the tunneling process to generate the initial conditions for an inflationary model realized in F-theory.

On a more formal side, the analysis carried out in this paper reveals deep connections between consistency conditions of the string/M-theory compactifications and constraints on the fields found in the resulting 4d/3d supergravity theories [24]. In the study of consistent effective theories arising from string compactifications it will be important to also consider solutions of varying dimensionalities and topologies to constrain the couplings. A prominent role in our analysis took the couplings σ , τ of the F^2 and $\mathcal{R}_4^2$ terms in the 4d effective theory as in [24]. We have used circle fluxes to probe the connection of these couplings with the geometric data of the fourfold. It would be interesting to generalize this to include fluxes for the imaginary parts of σ and τ since this will lead to genuinely 3d Chern-Simons gauge and gravity theories.

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APPENDIX A: 4D/3D REDUCTION WITH VECTORS

In this appendix we add some more details about the 4d/3d reduction in the presence of 4d vectors, discussed in Sec. II B. A related analysis can be found with additional details in [22,23].

After integration of (2.36) over S^1 and performing the Weyl rescaling $g^{(3)} \rightarrow r^{-2}g^{(3)} = Rg^{(3)}$ in order to bring the 3d Einstein-Hilbert term to the canonical form, we find the 3d action in the vector multiplet formalism

$$S^{(3)} = \int -\frac{1}{2}R_3 *_3 \mathbf{1} + \frac{1}{4}\tilde{K}_{L'L'}F^I \wedge *_3 F^I + \frac{1}{4}\tilde{K}_{L'L'}dL^I \wedge *_3 dL^I \quad (A1)$$

$$+ F^I \wedge \text{Im}(\tilde{K}_{L'\tau}d\tau) - \tilde{K}_{T_\alpha\bar{T}_\beta} \nabla T_\alpha \wedge *_3 \overline{\nabla T}_\beta - (V_F^{3d} + V_D^{3d}) *_3 \mathbf{1}, \quad (A2)$$

where $I = \{0, i\}$ and the kinetic potential $\tilde{K}$ is given in (3.34). Note that due to the nontrivial background fluxes (2.49) the covariant derivatives in (2.38) are improved and

become the three-dimensional covariant (or rather invariant) derivatives

$$\nabla T_\alpha = dT_\alpha + X_{i\alpha}A^i + iM_\alpha A^0. \quad (A3)$$

The D-term scalar potential obtained from the reduction of (2.36) is given by

$$V_D^{3d} = 4K_{T_\alpha\bar{T}_\beta}(R\Theta_{0\alpha} + \Theta_{i\alpha}\xi^i)(R\Theta_{0\beta} + \Theta_{j\beta}\xi^j) + 2R(\text{Re}\tau)^{ij}\Theta_{i\alpha}\Theta_{j\beta}K_{T_\alpha}K_{T_\beta}. \quad (A4)$$

It is easy to see that it coincides with the D-term potential in the general form (2.21), with the $\mathcal{T}$ -potential given by Eq. (2.57) and the Kähler potential by Eq. (2.52). Indeed the first term in (A4) arises from $K^{T_\alpha\bar{T}_\beta}\partial_{T_\alpha}\mathcal{T}\partial_{\bar{T}_\beta}\mathcal{T}$. As for the second term, one has

$$K^{T_i\bar{T}_j}\partial_{T_i}\mathcal{T}\partial_{\bar{T}_j}\mathcal{T} = 16(K_{T_i\bar{T}_j} - K_{T_i}K_{\bar{T}_j})\Theta_{i\alpha}\Theta_{j\beta}K_{T_\alpha}K_{\bar{T}_\beta}, \quad (A5)$$

and for the Kähler potential (2.52) one finds

$$K_{T_i\bar{T}_j} - K_{T_i}K_{\bar{T}_j} = \frac{1}{8}R(\text{Re}\tau)^{ij}. \quad (A6)$$

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