

Stability analysis and area spectrum of three-dimensional Lifshitz black holes

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In this work, we probe the stability of a $z = 3$ three-dimensional Lifshitz black hole by using scalar and spinorial perturbations. We found an analytical expression for the quasinormal frequencies of the scalar probe field, which perfectly agree with the behavior of the quasinormal modes obtained numerically. The results for the numerical analysis of the spinorial perturbations reinforce the conclusion of the scalar analysis, i.e., the model is stable under scalar and spinor perturbations. As an application we found the area spectrum of the Lifshitz black hole, which turns out to be equally spaced.

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I. INTRODUCTION

Some decades ago Regge and Wheeler began a pioneering study of a small perturbation in the background of a black hole in order to get information of the stability of this object [1], a problem that was continued by Zerilli [2]. The oscillations found in these studies are not normal modes due to the emission of gravitational waves; thus, their frequencies are complex and, as a result, the oscillations are damped.

The terminology, quasinormal mode (QNM) and quasinormal frequency (QNF), aiming to name these new modes and their frequencies, was first pointed out by Vishveshwara [3] and Press [4]. Although initially studied in black hole backgrounds, the concept of QNM applies to a much broader class of systems. The QNMs of black holes were first numerically calculated by Chandrasekhar and Detweiler [5] showing that the amplitude is dominated by a ringing characteristic signal at intermediate times. The QNMs are independent of the particular initial perturbation that excited them. The frequencies and damping times of the oscillations depend only on the parameters of the black hole and are, therefore, the “footprints” of this structure. Soon, the connection of QNMs to astrophysics was established by noting that their existence can lead to the detection of black holes through the observation of the gravitational wave spectrum. The interest in QNMs has motivated the development of numerical and analytical techniques for their computation (see [6–8] for a review). Also, the study of the quasinormal spectrum gives information about the stability aspects of black hole solutions using probe classical matter fields (scalar, electromagnetic, spinorial) evolving in the geometry without backreacting

on the spacetime background. Much has been done in that direction, not only in four dimensions [9,10], but also in two [11], and in more than four [12].

Aside from the study of the stability of the solutions, the QNFs are important in the context of the gauge-gravity correspondence, whose most celebrated example is the duality between the type IIA-B string theory in $\text{AdS}_5 \times S^5$ spacetime and the four-dimensional supersymmetric Yang-Mills theory [13]. Such a correspondence can be generalized for those cases in which there is an event horizon in the gravity side. In this case, the Hawking temperature of the black hole is related to the temperature of a thermal field theory defined at the anti-de Sitter (AdS) boundary. Also, as a consequence of the correspondence, the quasinormal spectrum corresponds to the poles of thermal Green functions [14], more precisely, the inverse of the imaginary part of the fundamental quasinormal frequency can be interpreted as the dual field theory relaxation time [15].

Another interesting application of QNMs appears in the context of black hole thermodynamics. Some decades ago Bekenstein [16] suggested that the horizon area of a black hole must be quantized, so that the area spectrum has the form $A_n = \gamma n \hbar$, with γ a dimensionless constant to be determined. The first proposal to calculate this constant through QNMs was made by Hod [17]. Accordingly, the real part of the asymptotic quasinormal mode can be seen as a transition frequency in the semiclassical limit, and its quantum emission causes a change in the mass of the black hole, which is related to the area. In this way, the constant γ for a Schwarzschild black hole was determined as $\gamma = 4 \ln 3$. Later, Kunstatter [18] repeated the calculation, quantizing the adiabatic invariant $I = \int dE/\omega(E)$ via the Bohr-Sommerfeld quantization and using the real part of the QNF as the vibrational frequency. The result was exactly the same as Hod’s. However, recently Maggiore [19] pointed out that QNMs should be described as damped harmonic

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oscillators, thus, the imaginary part of the QNF should not be neglected, and the proper physical frequency is the module of the entire QNF. Moreover, when considering the quantization of the adiabatic invariant, the frequency to be considered is that corresponding to a transition between two neighboring quantum levels. With this identification, the constant γ for a Schwarzschild black hole becomes $\gamma = 8\pi$, a result that coincides with the value calculated by other methods [20]. The consequences of Hod's and Maggiore's proposals were promptly studied in several spacetimes [21,22].

In this paper, we are interested in the study of the stability of the $z = 3$, three-dimensional Lifshitz black hole found in the context of the so-called new massive gravity (NMG) [23]. Moreover, as an application of our QNM results we aim to calculate the area spectrum of this black hole.

NMG is a novel parity-preserving, unitary [24], power-counting super-renormalizable [25], three-dimensional model describing the propagation of a massive positive-energy graviton with two polarization states of helicity ± 2 in a Minkowski vacuum, whose linearized version is equivalent to the Pauli-Fierz theory for a massive spin-2 field in three dimensions. The action of NMG consists of a “wrong sign” Einstein-Hilbert term plus a quadratic curvature term given by a precise combination of the Ricci tensor and the curvature scalar, which introduces a mass parameter [26]. As with other massive gravity theories, NMG also admits black hole-type solutions with several asymptotics and additional parameters [27,28]. Even though this last feature could challenge the usual Einstein-Hilbert gravity, it is seen that the definition of mass in this new type of black holes is a conserved charge computed from a combination of the black hole parameters, which satisfies the first law of thermodynamics. A study of QNMs in these static new type of black holes has been performed in [29].

The black holes we take into account for our study are asymptotically Lifshitz, i.e., they exhibit the anisotropic scale invariance, $t \rightarrow \lambda^z t$, $\vec{x} \rightarrow \lambda \vec{x}$, where z is the dynamical critical exponent. Specifically, we deal with the solutions found for the particular case of $z = 3$ and a precise value of the mass parameter [23]. The general class of these solutions are important in the context of gauge-gravity duality [30,31] and were also investigated in other background theories [32–35]. No stability study of black holes with Lifshitz asymptotics in three dimensions in the scenario of NMG has been performed yet. We aim to give some contribution to this issue by considering the QNFs of scalar and spinorial matter fields in the probe limit, i.e., there are no backreaction effects upon the asymptotically Lifshitz black hole metric. Spinor fields have been extensively studied in general relativity [36,37], and their quasinormal frequencies have also been considered [38–40].

The paper is organized as follows. In Sec. II we introduce the Lifshitz black holes and discuss their causal

structure. Sections III and IV are dedicated to the study of stability under scalar and spinorial perturbations with special emphasis on the massless spinor for the latter. In Sec. V we present the numerical analysis for both kinds of perturbations showing the QNMs and the corresponding QNFs computed in each case. Section VI is devoted to the calculation of the area spectrum of these black holes as an application of our quasinormal spectrum. Finally, we discuss our results and conclude in Sec. VII.

II. LIFSHITZ BLACK HOLES IN THREE DIMENSIONS

In this section we review the black hole solutions we will consider within this paper, and we comment some of their features.

The NMG theory [26] is defined by the $(2 + 1)$ -dimensional action,

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[R - 2\lambda - \frac{1}{m^2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right) \right], \quad (1)$$

where m is the so-called “relative” mass parameter, and λ is the three-dimensional cosmological constant. Defining the dimensionless parameters, $y = m^2 l^2$ and $w = \lambda l^2$, it is found that the theory exhibits special properties at the points $y = \pm 1/2$. When looking for black hole solutions with Lifshitz asymptotics, it is precisely at the point $y = -1/2$, $w = -13/2$, with Lifshitz scaling $z = 3$, where the field equations turn out to be solved by [23]

$$ds^2 = -a(r) \frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\phi^2, \quad (2)$$

where

$$a(r) = \frac{r^4}{l^4}, \quad (3)$$

and

$$\Delta = -Mr^2 + \frac{r^4}{l^2}, \quad (4)$$

with M an integration constant and $l^2 = -\frac{13}{2\lambda}$. Also, the NMG admits as a solution, the well-known Bañados-Teitelboim-Zanelli black hole with the dynamical critical exponent $z = 1$. As we shall see below in more detail, this metric (2) exhibits a regular single event horizon located at $r = r_+ = l\sqrt{M}$ and a spacetime singularity at $r = 0$. Besides, the surface $r = r_+$ acts as a one-way membrane for physical objects as we can see from the norm of a vector χ normal to a given surface s . Since s has to be null in order to be a one-way membrane, the norm of χ must be null as well, i.e., $g^{rr} = 0$, which occurs at $r = r_+$.

From the behavior of the Kretschmann invariant for the metric (2),

$$R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} = -\frac{4}{l^4 r^4} [8r_+^4 - 48r_+^2 r^2 + 91r^4], \quad (5)$$

we see that for $r \rightarrow r_+$,

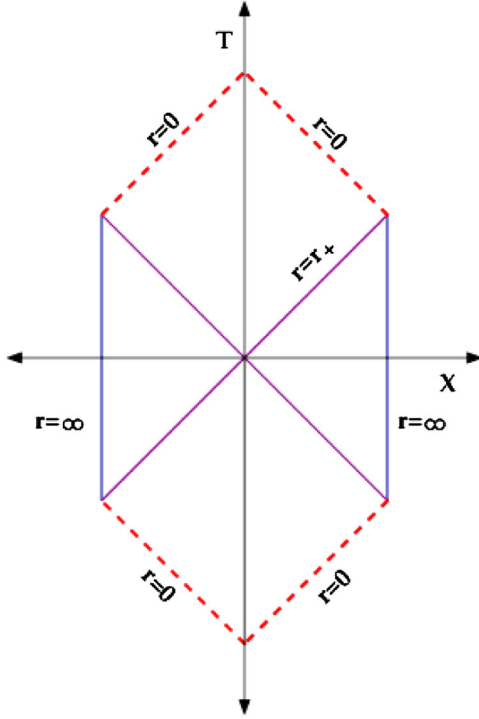


FIG. 1 (color online). Penrose-Carter diagram for the Lifshitz black hole. The singularity at $r = 0$ is light-like and covered by a regular event horizon r_+ .

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \rightarrow -\frac{204}{l^4}, \quad (6)$$

and for $r \rightarrow 0$,

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \rightarrow \infty. \quad (7)$$

Thus, the black hole solution has a genuine spacetime singularity at the origin $r = 0$ and an event horizon at $r = r_+$. Nevertheless, to see if the singularity is timelike, spacelike, or null we have to construct the Penrose-Carter diagram. First of all, we must remove the coordinate singularity at $r = r_+$. Rewriting the metric in terms of null coordinates (U, V) ,

$$U = e^{r_+^3(t+r_*)}, \quad V = -e^{-r_+^3(t-r_*)}, \quad (8)$$

where r_* is the tortoise coordinate shown in the next section, we get

$$ds^2 = -\frac{1}{4}\left(\frac{r}{r_+}\right)^6\left(1 + \frac{r_+}{r}\right)^2 e^{-(2r_+/r)} dU dV, \quad (9)$$

which is manifestly regular at $r = r_+$.

Finally, to construct the Penrose-Carter diagram (Fig. 1) we use the following set of null coordinates:

$$T = \tilde{U} + \tilde{V}, \quad X = \tilde{U} - \tilde{V}, \quad (10)$$

with $\tilde{U} = \arctan(U)$ and $\tilde{V} = \arctan(V)$.

From this diagram we see that the spacetime singularity is located at $r = 0$, as previously observed from the behavior of the Kretschmann invariant. Moreover, it is light-like and covered by a regular event horizon at $r = r_+$.

III. SCALAR PERTURBATION

In this section, we analyze the behavior of a scalar field perturbation in the background of a three-dimensional Lifshitz black hole.

The scalar field obeys the Klein-Gordon equation,

$$\square\Phi = \frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}g^{MN}\partial_N)\Phi = m^2\Phi, \quad (11)$$

where m is the mass of the field Φ . Performing the decomposition,

$$\Phi(t, r, \phi) = \Psi(t, r)e^{i\kappa\phi}. \quad (12)$$

The Klein-Gordon equation takes the form,

$$\begin{aligned} -\partial_t^2\Psi + \frac{r^4}{l^6}\left(1 - \frac{Ml^2}{r^2}\right)\left(\frac{5r^3}{l^2} - 3Mr\right)\partial_r\Psi + \frac{r^8}{l^8}\left(1 - \frac{Ml^2}{r^2}\right)^2\partial_r^2\Psi \\ - \frac{r^4}{l^6}(m^2r^2 + \kappa^2)\left(1 - \frac{Ml^2}{r^2}\right)\Psi = 0. \end{aligned} \quad (13)$$

Even though this equation has an analytical solution, as we will see in what follows, it is also useful to check the numerical results. With this goal we further decompose $\Psi = X(t, r_*)/\sqrt{r}$, where the tortoise coordinate r_* is given by

$$r_* = l^4\left[-\frac{1}{M^{3/2}l^3}\operatorname{arccoth}\left(\frac{r}{l\sqrt{M}}\right) + \frac{1}{Ml^2r}\right]. \quad (14)$$

In this way the Klein-Gordon equation reduces to

$$-\partial_t^2X + \partial_{r_*}^2X = V(r)X, \quad (15)$$

where $V(r)$ is the scalar effective potential given by

$$\begin{aligned} V(r) = \left(\frac{7}{4l^8} + \frac{m^2}{l^6}\right)r^6 - \left(\frac{5M}{2l^6} + \frac{Mm^2}{l^4} - \frac{\kappa^2}{l^6}\right)r^4 \\ + \left(\frac{3M^2}{4l^4} - \frac{M\kappa^2}{l^4}\right)r^2. \end{aligned} \quad (16)$$

Now let us come back to the issue of finding an exact solution for Eq. (13). We set the time dependence of the field $\Psi(t, r_*)$ as $R(r)e^{-i\omega t}$ and redefine the radial coordinate as $r = l\sqrt{M}/y$. Thus, Eq. (13) turns out to be

$$\begin{aligned} \partial_y^2R + \frac{y^2 - 3}{y(1 - y^2)}\partial_yR - \frac{l^2}{(1 - y^2)}\left[-\frac{\omega^2y^4}{M^3(1 - y^2)}\right. \\ \left. + \frac{m^2}{y^2} + \frac{\kappa^2}{Ml^2}\right]R = 0, \end{aligned} \quad (17)$$

whose solution is given in terms of Heun confluent functions,

$$\begin{aligned} R(y) = C_1y^{2+\alpha}(1 - y^2)^{\beta/2}\operatorname{HeunC}\left(0, \alpha, \beta, -\frac{\beta^2}{4}, \frac{\alpha^2}{4} + \frac{\kappa^2}{4M}, y^2\right) \\ + C_2y^{2-\alpha}(1 - y^2)^{\beta/2} \\ \times \operatorname{HeunC}\left(0, -\alpha, \beta, -\frac{\beta^2}{4}, \frac{\alpha^2}{4} + \frac{\kappa^2}{4M}, y^2\right), \end{aligned} \quad (18)$$

where C_1 and C_2 are integration constants, while $\alpha = \sqrt{4 + m^2 l^2}$ and $\beta = -i l \omega / M^{3/2}$.

Imposing the Dirichlet condition at infinity, we set $C_1 = 0$. In order to apply the boundary condition of incoming waves at the horizon we use the following connection formula [41]:

$$\begin{aligned} & \text{HeunC}(0, b, c, d, e, z) \\ &= \frac{c_1 \Gamma(1-b) \Gamma(c)}{\Gamma(1+c+k) \Gamma(-b-k)} \text{HeunC}(0, c, b, -d, e \\ &+ d, 1-z) + \frac{c_2 \Gamma(1-b) \Gamma(-c)}{\Gamma(1-c+s) \Gamma(-b-s)} (1-z)^{-c} \\ &\times \text{HeunC}(0, -c, b, -d, e+d, 1-z). \end{aligned} \quad (19)$$

This formula connects a solution around the singular regular point $z = 0$ to the corresponding solution around the singular regular point $z = 1$ of the confluent Heun equation given by

$$z(z-1)H'' + [(b+1)(z-1) + (c+1)z]H' + (dz - \epsilon)H = 0. \quad (20)$$

$$\omega = 2i \frac{M^{3/2}}{l} \left[1 + 2N + \sqrt{4 + m^2 l^2} - \sqrt{7 + \frac{3}{2} m^2 l^2 + \frac{\kappa^2}{2M} + (3 + 6N)\sqrt{4 + m^2 l^2} + 6N(N+1)} \right], \quad (25)$$

where N is a positive integer. The imaginary part of the fundamental frequency ($N = 0$) is negative provided that

$$\sqrt{7 + \frac{3}{2} m^2 l^2 + \frac{\kappa^2}{2M} + 3\sqrt{4 + m^2 l^2}} > 1 + \sqrt{4 + m^2 l^2}. \quad (26)$$

While the asymptotic frequency ($N \rightarrow \infty$) is given by

$$\omega_\infty = -2(\sqrt{6} - 2)i \frac{M^{3/2}}{l} N < 0. \quad (27)$$

Thus, since the imaginary part of the quasinormal frequencies is negative provided that the parameters respect the relation (26), we can conclude that the model is stable under scalar perturbations.

IV. SPINORIAL PERTURBATION

In this section, we are going to consider a spinorial field as a perturbation in the spacetime given by the three-dimensional Lifshitz black hole. We analyze the covariant Dirac equation for a two component spinor field Ψ with mass μ_s . This equation is given by

$$i \gamma^{(a)} e_{(a)}^\mu \nabla_\mu \Psi - \mu_s \Psi = 0, \quad (28)$$

where Greek indices refer to spacetime coordinates (t, r, ϕ) , and the Latin indices enclosed in parentheses describe the flat tangent space in which the triad basis

The parameters k and s are obtained from

$$k^2 + (b + c + 1)k - \epsilon + d/2 = 0, \quad (21)$$

$$s^2 + (b - c + 1)s - \epsilon + d/2 = 0, \quad (22)$$

and ϵ is related to e as

$$\epsilon = -\frac{bc}{2} - \frac{c}{2} - \frac{b}{2} - e. \quad (23)$$

Thus, near $y = 1$, Eq. (18) can be written as

$$\begin{aligned} R(y \rightarrow 1) &\approx \xi_1 (1 - y^2)^{\beta/2} \frac{\Gamma(1 + \alpha) \Gamma(\beta)}{\Gamma(\alpha - k) \Gamma(1 + \beta + k)} \\ &+ \xi_2 (1 - y^2)^{-\beta/2} \frac{\Gamma(1 + \alpha) \Gamma(-\beta)}{\Gamma(\alpha - s) \Gamma(1 - \beta + s)}, \end{aligned} \quad (24)$$

with ξ_i as constants. As we are looking for quasinormal frequencies with negative imaginary parts, which give stable solutions, we find that for $\beta < 0$ we need $\Gamma(1 + \beta + k) \rightarrow \infty$. Thus, the quasinormal frequencies are

$e_{(a)}^\mu$ is defined. The spinor covariant derivative ∇_μ is given by

$$\nabla_\mu = \partial_\mu + \frac{1}{8} \omega_\mu^{(a)(b)} [\gamma_{(a)}, \gamma_{(b)}], \quad (29)$$

where $\omega_\mu^{(a)(b)}$ is the spin connection, which can be written in terms of the triad $e_{(a)}^\mu$ as

$$\omega_\mu^{(a)(b)} = e_\nu^{(a)} \partial_\mu e^{(b)\nu} + e_\nu^{(a)} \Gamma_{\mu\sigma}^\nu e^{\sigma(b)}, \quad (30)$$

where $\Gamma_{\mu\sigma}^\nu$ are the metric connections. The $\gamma^{(a)}$ denotes the usual flat gamma matrices, which can be taken in terms of the Pauli ones. In this work, we take $\gamma^{(0)} = i\sigma_2$, $\gamma^{(1)} = \sigma_1$, and $\gamma^{(2)} = \sigma_3$.

We can write the triad basis $e_{(a)}^\mu$ for the metric (2) as follows:

$$e_0^{(a)} = \frac{\sqrt{a(r)\Delta}}{r} \delta_0^{(a)}, \quad e_1^{(a)} = \frac{r}{\sqrt{\Delta}} \delta_1^{(a)}, \quad e_2^{(a)} = r \delta_2^{(a)}, \quad (31)$$

and the metric connections,

$$\begin{aligned} \Gamma^0_{01} &= \frac{d}{dr} \left[\ln \left(\frac{a(r)\Delta}{r^2} \right)^{1/2} \right], \quad \Gamma^1_{11} = \frac{d}{dr} \left[\ln \left(\frac{r^2}{\Delta} \right)^{1/2} \right], \\ \Gamma^1_{00} &= \frac{\Delta}{2r^2} \frac{d}{dr} \left[\frac{a(r)\Delta}{r^2} \right], \quad \Gamma^1_{22} = -\frac{\Delta}{r}, \quad \Gamma^2_{12} = \frac{1}{r}. \end{aligned} \quad (32)$$

Using these quantities it is straightforward to write down the expressions for spin connection components. In the present case, we have only two nonvanishing components,

$$\omega_0^{(0)(1)} = \frac{1}{2\sqrt{a(r)}} \frac{d}{dr} \left(\frac{a(r)\Delta}{r^2} \right), \quad \omega_2^{(1)(2)} = -\frac{\sqrt{\Delta}}{r}. \quad (33)$$

At this point we are able to write the Dirac equation for the two component spinor,

$$\Psi = \begin{pmatrix} \Psi_1(t, r, \phi) \\ \Psi_2(t, r, \phi) \end{pmatrix}, \quad (34)$$

which turns to be the set of coupled differential equations,

$$\begin{aligned} \frac{ir}{\sqrt{a(r)\Delta}} \partial_t \Psi_2 + \frac{i\sqrt{\Delta}}{r} \partial_r \Psi_2 + \frac{i}{r} \partial_\phi \Psi_1 + \frac{i}{4} \left[\frac{a(r)'\Delta}{a(r)r} + \frac{\Delta'}{r\sqrt{\Delta}} \right] \Psi_2 \\ - \mu_s \Psi_1 = 0, \end{aligned} \quad (35)$$

$$\begin{aligned} -\frac{ir}{\sqrt{a(r)\Delta}} \partial_t \Psi_1 + \frac{i\sqrt{\Delta}}{r} \partial_r \Psi_1 - \frac{i}{r} \partial_\phi \Psi_2 \\ + \frac{i}{4} \left[\frac{a(r)'\Delta}{a(r)r} + \frac{\Delta'}{r\sqrt{\Delta}} \right] \Psi_1 - \mu_s \Psi_2 = 0. \end{aligned} \quad (36)$$

In order to simplify our problem, we redefine Ψ_1 and Ψ_2 as

$$\begin{aligned} \Psi_1 &= i[a(r)\Delta]^{1/4} e^{-i\omega t + im\phi} \Phi_+(r), \\ \Psi_2 &= [a(r)\Delta]^{1/4} e^{-i\omega t + im\phi} \Phi_-(r), \end{aligned} \quad (37)$$

and the tortoise coordinate as in the scalar case (14),

$$\frac{d}{dr_*} = \frac{\Delta\sqrt{a(r)}}{r^2} \frac{d}{dr}. \quad (38)$$

Thus, we can rewrite Eqs. (35) and (36) as

$$\partial_{r_*} \Phi_- - i\omega \Phi_- = i \frac{\sqrt{a(r)\Delta}}{r^2} (\hat{m} - i\mu_s r) \Phi_+, \quad (39)$$

$$\partial_{r_*} \Phi_+ + i\omega \Phi_+ = i \frac{\sqrt{a(r)\Delta}}{r^2} (\hat{m} + i\mu_s r) \Phi_-, \quad (40)$$

where $m = im$.

Furthermore, we define a new function θ , a new rescaling for the spinorial components $R_\pm$, and a new tortoise coordinate $\hat{r}_*$ through the expressions,

$$\begin{aligned} \theta &= \arctan\left(\frac{\mu_s r}{\hat{m}}\right), \quad \Phi_\pm = e^{\pm i\theta/2} R_\pm(r), \\ \hat{r}_* &= r_* + \frac{1}{2\omega} \arctan\left(\frac{\mu_s r}{\hat{m}}\right). \end{aligned}$$

In this way Eqs. (39) and (40) become

$$(\partial_{\hat{r}_*} \pm i\omega) R_\pm = W R_\mp, \quad (41)$$

where W is the so-called superpotential,

$$W = \frac{i\sqrt{a(r)\Delta}(\hat{m}^2 + \mu_s^2 r^2)^{3/2}}{r^2(\hat{m}^2 + \mu_s^2 r^2) + \frac{\mu_s \hat{m} \sqrt{a(r)\Delta}}{2\omega}}. \quad (42)$$

Notice that when $a(r) = 1$, Eq. (42) reduces to the Bañados-Teitelboim-Zanelli superpotential [42].

Finally, letting $X_\pm = R_+ \pm R_-$ we have

$$(\partial_{\hat{r}_*}^2 + \omega^2) X_\pm = V_\pm X_\pm, \quad (43)$$

where $V_\pm$ are the superpartner potentials,

$$V_\pm = W^2 \pm \frac{dW}{d\hat{r}_*}, \quad (44)$$

which, in the case of a massless spinor ($\mu_s = 0$), reduces to

$$\begin{aligned} V_\pm &= \left(-\frac{m^2 M}{l^4} \mp \frac{mM}{l^5} \sqrt{r^2 - Ml^2} \right) r^2 \\ &+ \left(\frac{m^2}{l^6} \pm \frac{2m}{l^7} \sqrt{r^2 - Ml^2} \right) r^4. \end{aligned} \quad (45)$$

V. NUMERICAL ANALYSIS

In this section, we numerically solve Eqs. (15) and (43), which correspond to the scalar and massless spinorial perturbations, respectively. Although in the scalar case we found an analytical solution and the corresponding QNF, our motivation to perform the numerical analysis is to verify the applicability of certain numerical methods in asymptotically Lifshitz spacetimes. In particular, it would be interesting to check if the Horowitz-Hubeny method [15] works well when finding the QNF.

Using the finite difference method, we define $\psi(r_*, t) = \psi(-j\Delta r_*, l\Delta t) = \psi_{j,l}$, $V(r_*) = V(-j\Delta r_*) = V_j$, and rewrite Eqs. (15) and (43) as

$$\begin{aligned} -\frac{\psi_{j,l+1} - 2\psi_{j,l} + \psi_{j,l-1}}{\Delta t^2} + \frac{\psi_{j+1,l} - 2\psi_{j,l} + \psi_{j-1,l}}{\Delta r_*^2} \\ - V_j \psi_{j,l} + O(\Delta t^2) + O(\Delta r_*^2) = 0, \end{aligned} \quad (46)$$

which can be rearranged as

$$\begin{aligned} \psi_{j,l+1} &= -\psi_{j,l-1} + \frac{\Delta t^2}{\Delta r_*^2} (\psi_{j+1,l} + \psi_{j-1,l}) \\ &+ \left(2 - 2\frac{\Delta t^2}{\Delta r_*^2} - \Delta t^2 V_j \right) \psi_{j,l}. \end{aligned} \quad (47)$$

The initial conditions $\psi(r_*, 0) = f_0(r_*)$ and $\dot{\psi}(r_*, 0) = f_1(r_*)$ define the values of $\psi_{j,l}$ for $l = 0$ and $l = 1$, and we use Eq. (47) to obtain the values of $\psi_{j,l}$ for $l > 1$. At $j = 0$ we impose Dirichlet boundary conditions since $V(r_*)$ tends to infinity as r_* tends to zero. The numerical solution is stable if

$$\frac{\Delta t^2}{\Delta r_*^2} + \frac{\Delta t^2}{4} V_{\max} < 1, \quad (48)$$

where $V_{\max} = V_1$ is the largest value of V_j in our domain. This condition is verified in all cases studied here.

Now we are going to analyze the potential for the scalar case. By rewriting Eq. (16) in terms of a new variable $z = r^2$, we obtain

$$V(r) = \frac{z}{l^8} \left[\left(\frac{7}{4} + m^2 l^2 \right) z^2 - \left(\frac{5}{2} + m^2 l^2 - \frac{\kappa^2 l^2}{z_h} \right) z_h z + \left(\frac{3}{4} - \frac{\kappa^2 l^2}{z_h} \right) z_h^2 \right], \quad (49)$$

where $z_h = r_h^2$. The parable in brackets tends to infinity as long as $(\frac{7}{4} + m^2 l^2) > 0$, which is consistent with the Breitenlohner-Freedman-type bound for the present case. The roots of this polynomial potential are given by

$$z_0 = 0, \quad z_+ = z_h, \quad (50)$$

$$z_- = z_h \left[\frac{\frac{3}{4} - \frac{\kappa^2 l^2}{z_h}}{\frac{7}{4} + m^2 l^2} \right]. \quad (51)$$

If $m^2 l^2 > -1$, we see that $z_- < z_+$. Thus, going back to the original variable r , the roots of the potential are $r = 0$ with double multiplicity, $r = \sqrt{z_-}$ and $r = r_h$ ($r = -\sqrt{z_-}$ and $r = -r_h$ are excluded as $r > 0$). Then, since r_h is the biggest root and $V(r)$ tends to ∞ when r tends to ∞ , the potential is positive-definite in the region (r_h, ∞) . Therefore, the quasinormal modes for $m^2 l^2 > -1$ are necessarily stable [15].

The numerical results regarding the decay of the scalar field are shown in Figs. 2 and 3, and the comparison between the numerical and analytical results is displayed in Fig. 4. Our results reinforce the conclusion already found analytically; the $z = 3$ Lifshitz black hole is stable under scalar perturbations. Moreover, according to Fig. 4,

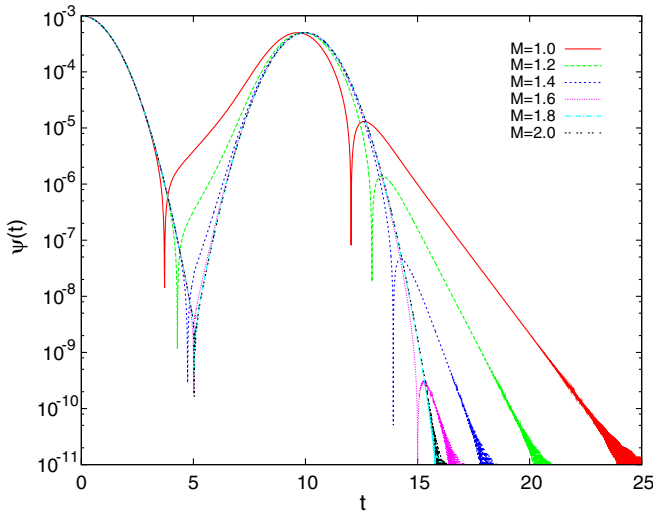


FIG. 2 (color online). Decay of scalar field with mass $m = 1$ and $l = 1$ for different values of black hole mass M .

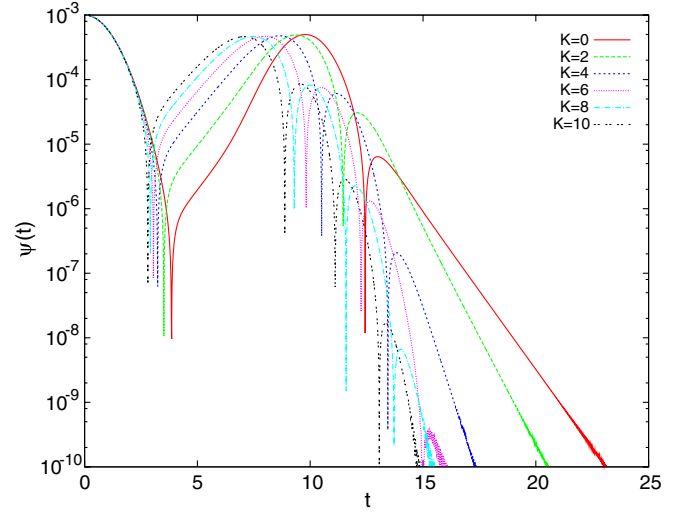


FIG. 3 (color online). Decay of scalar field with mass $m = 1$ and $l = 1$ varying the azimuthal parameter κ .

the numerical results have a very good agreement with the analytical calculation.

Figure 4 shows that the Horowitz-Hubeny method gives unreliable results. In [43], it is argued that the frequencies do not converge as required by the method, and that may be explained by ill-conditioned polynomials. However, in this work, the frequencies converge, but they do not agree with the analytic expression and with the results from finite difference method. In [44], the authors find cases where this method does not work either, and they do so by comparing the results with other methods. For instance, they find that the method is unreliable for dimensions bigger than 6. Even in the original work [15], the method is unreliable for small black holes, and there is no clear explanation for this limitation. In our case, the asymptotic behavior of the black hole under study might play an

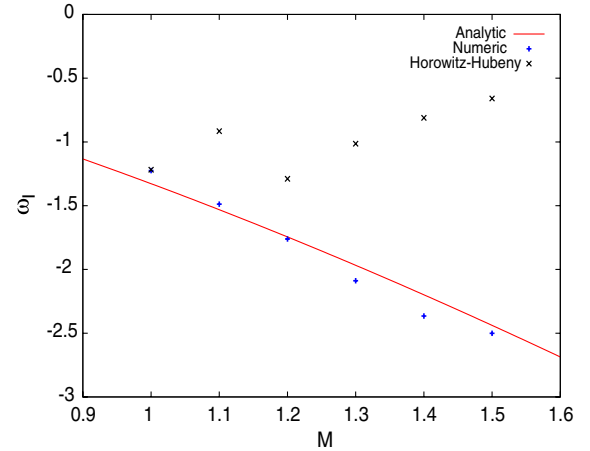


FIG. 4 (color online). Imaginary part of scalar quasinormal frequencies. We display the results obtained using different approaches.

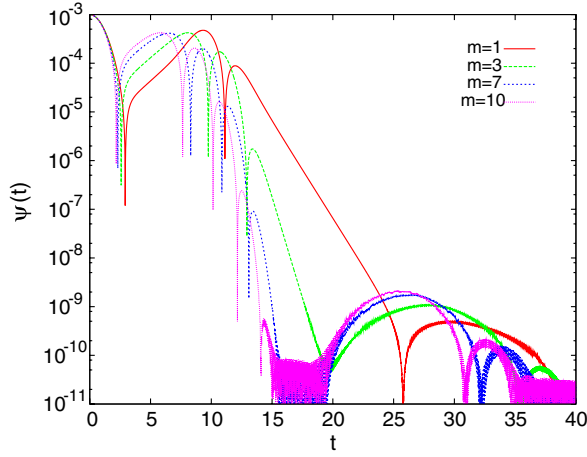


FIG. 5 (color online). Decay of massless spinor with $l = 1$ and black hole mass $M = 1.0$ for different values of the azimuthal parameter m .

important role in the convergence of the method. Nevertheless, a general criteria for the convergence of the Horowitz-Hubeny method remains an open question.

In the case of the massless spinorial perturbation, the superpartner potentials (45) can be written as

$$V_{\pm} = \frac{1}{l^8} \left[(ml)^2 r^2 (r^2 - r_+^2) \pm (ml) r^2 (2r^2 - r_+^2) \sqrt{r^2 - r_+^2} \right], \quad (52)$$

and their derivative turns to be

$$V'_{\pm} = \frac{1}{l^8} \left\{ (ml)^2 r (2r^2 - r_+^2) \pm (ml) \left[2r(r^2 - r_+^2) \sqrt{r^2 - r_+^2} + r^3 \frac{2r^2 - r_+^2}{\sqrt{r^2 - r_+^2}} \right] \right\}. \quad (53)$$

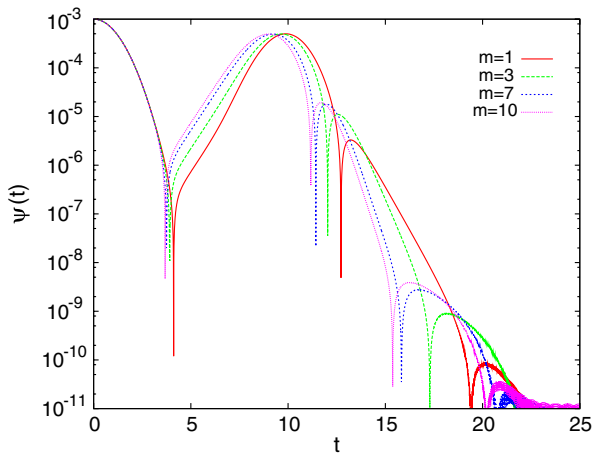


FIG. 6 (color online). Decay of massless spinor with $l = 1$ and black hole mass $M = 1.5$ for different values of the azimuthal parameter m .

We can see that, outside the event horizon, V_+ is positive-definite if $ml > 0$, and $\lim_{r \rightarrow \infty} V_+(r) = -\infty$ if $ml < 0$. Whereas V_- is positive-definite if $ml < 0$, and $\lim_{r \rightarrow \infty} V_-(r) = -\infty$ if $ml > 0$. Moreover, we notice that if $ml = 0$, we have a free-particle case. The decaying behavior of the massless spinor is given in Figs. 5 and 6. Thus, we conclude that the $z = 3$ Lifshitz black hole is stable under massless spinorial perturbations.

VI. AREA SPECTRUM

One of the applications of our results for the quasinormal frequencies is the relation they have with the area spectrum of a black hole. According to Maggiore [19], the proper physical frequency of the damped harmonic oscillator equivalent to the black hole quasinormal mode is given by

$$\omega_p = \sqrt{\omega_R^2 + \omega_I^2}, \quad (54)$$

where ω_R and ω_I stand for the real and imaginary part of the asymptotic QNF, respectively. Thus, using (27) we have

$$\omega_p = 2(\sqrt{6} - 2) \frac{M^{3/2}}{l} N. \quad (55)$$

According to Myung *et al.* [45], the Arnowitt-Deser-Misner mass of the Lifshitz black hole we are studying is given by

$$\mathcal{M} = \frac{M^2}{2}. \quad (56)$$

Applying Maggiore's method, we calculate the adiabatic invariant I as

$$I = \int \frac{d\mathcal{M}}{\Delta\omega} = \int \frac{M}{\Delta\omega} dM, \quad (57)$$

where $\Delta\omega$ is the change of proper frequency between two neighboring modes, i.e.,

$$\Delta\omega = 2(\sqrt{6} - 2) \frac{M^{3/2}}{l}. \quad (58)$$

Thus,

$$I = \frac{lM^{1/2}}{(\sqrt{6} - 2)}, \quad (59)$$

which is quantized via Bohr-Sommerfeld quantization in the semiclassical limit. Recalling that the horizon area of the black hole is given by $A = 2\pi r_+$, with $r_+ = l\sqrt{M}$, and using (59), we arrive at the result,

$$A = 2\pi(\sqrt{6} - 2)n\hbar, \quad (60)$$

with n an integer number. Therefore, we see that the horizon area for the $z = 3$ Lifshitz black hole is quantized and equally spaced.

This result would not be expected for a theory containing higher order curvature corrections since, in general, black hole solutions in such theories do not have a proportional relation between their entropy and area, and consequently, both of them (if any) should not be quantized with an equally spaced spectrum for large quantum numbers [21,46,47]. However, it was already demonstrated that the $z = 3$, three-dimensional Lifshitz black hole has an entropy proportional to its horizon area [45,48]. Thus, our result (60) also states that the entropy should be quantized with a spectrum evenly spaced. Nevertheless, we should stress that a generalization of this result for Lifshitz black holes should wait for the calculation of the area spectra of other black holes of such a type. Solely, these studies can give a definite answer on this subject.

VII. CONCLUDING REMARKS

We have studied the stability of the three-dimensional Lifshitz black hole under scalar and spinorial perturbations in the probe limit through the computation of quasinormal modes. In addition, we have found the event horizon area quantization as an application of the results for quasinormal modes using Maggiore's prescription.

Regarding the stability, we have not found unstable quasinormal modes in the range of parameters that we have considered; all the frequencies have a negative imaginary part indicating that the modes are damped and thus, the perturbations decay, leaving the system stable against this particular sort of probe fields.

In the case of a scalar probe field, such results totally agree with the analytical expressions for the quasinormal frequencies; they show a very large imaginary part and a very small real part. These modes are almost purely imaginary. We have implemented two different numerical methods in order to obtain the quasinormal frequencies and modes: the finite difference and the Horowitz-Hubeny methods. The former allows us to obtain the temporal behavior of the fields showing all the stages of the decay, while the latter gives only the frequencies values. As explained in Sec. V, the Horowitz-Hubeny method failed in the calculation of the scalar frequencies as it can be observed in Fig. 4. On the contrary, the finite difference method has a very good agreement with the analytical

expression (25). Apart from the numerical factor, the asymptotic scalar frequency found in the present work is the same as the one calculated in the hydrodynamic limit of the scalar perturbations in the context of gauge-gravity duality [49].

Regarding the spinorial perturbation, our numerical results show that the probe massless spinor decays and thus, the $z = 3$ Lifshitz black hole is stable also under spinorial perturbations.

As a by-product we also obtained the area spectrum of this black hole by means of the application of Maggiore's method using our results for the scalar asymptotic quasinormal frequencies. Equation (60) shows that the horizon area is quantized and equally spaced. Furthermore, in light of the conclusions shown in [45,48], the corresponding entropy should also have an evenly spaced spectrum.

Finally, although we have demonstrated the stability of the $z = 3$, three-dimensional Lifshitz black hole under scalar and spinor perturbations, we should stress that the definite answer on stability should come from the gravitational perturbations, in particular, from the tensor part of the metric perturbation. It is well known that Einstein gravity in three dimensions has no propagating degrees of freedom, however, the massive versions of the theory, e.g., NMG, allow the propagation of gravitational waves. Albeit this subject deserves further study, the calculation of metric perturbations is a formidable task that is out of the scope of the present paper. The analysis is not dead easy because the perturbation equation is a fourth order differential equation. Thus, some other techniques need to be used together with the traditional QNM analysis [50]. This study will be discussed elsewhere.

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