PHYSICAL REVIEW D **85,** 024012 (2012)

Dynamics of perfect fluids in nonminimally coupled gravity

Orfeu Bertolami*

Departamento de Física e Astronomia, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007, Porto, Portugal

António Martins†

Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

(Received 11 October 2011; published 11 January 2012)

In this work we explore the consequences that a nonminimal coupling between geometry and matter can have on the dynamics of perfect fluids. It is argued that the presence of a static, axially symmetric pressureless fluid does not imply a Minkowski space-time like as is in general relativity. This feature can be attributed to a pressure mimicking mechanism related to the nonminimal coupling. The case of a spherically symmetric black hole surrounded by fluid matter is analyzed, and it is shown that under equilibrium conditions the total fluid mass is about twice that of the black hole. Finally, a generalization of the Newtonian potential for a fluid element is proposed and its implications are briefly discussed.

DOI: 10.1103/PhysRevD.85.024012 PACS numbers: 04.50.Kd, 04.70.Bw

I. INTRODUCTION

The recent revival of interest in alternative gravity models, motivated chiefly by the possibility of explaining the accelerated expansion of the Universe and the flattening of the rotation curves of galaxies without the need to introduce dark energy and matter, has led to a number of studies on the so-called "$f(R)$ theories," where the linear scalar curvature term of the Einstein-Hilbert action is replaced by a more general function of the same invariant [1]. Although these models have attracted a great deal of attention, it is interesting to take the generalization a step further and consider a nonminimal coupling between matter and geometry [2–4], which can be achieved by postulating an action of the form

$$S = \int \left[\frac{1}{2} f_1(R) + (1 + \lambda f_2(R)) \mathcal{L}_m \right] \sqrt{-g} dx^4, \quad (1)$$

where $f_i(R)$ are arbitrary functions of the scalar curvature R, $\mathcal{L}_m$ is the Lagrangian density of matter, g is the metric determinant, and λ is a coupling constant that can be used to gauge the contribution of $f_2(R)$. The standard Einstein-Hilbert action is recovered by taking $f_2 = 0$ and $f_1(R) = 2\kappa(R - 2\Lambda)$, where $\kappa = \frac{c^4}{16\pi G}$ and Λ is the cosmological constant.

Varying action (1) with respect to the metric coefficients yields the field equations

$$\begin{aligned}(F_1 + 2\lambda F_2 \mathcal{L}_m) R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} \\ = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box)(F_1 + 2\lambda F_2 \mathcal{L}_m) + (1 + \lambda f_2) T_{\mu\nu},\end{aligned} \quad (2)$$

where $F_i(R) = f_i'(R)$, and the energy-momentum tensor is defined, as usual, by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \quad (3)$$

Among the various interesting features of this model, perhaps the most striking is the nonconservation law

$$\nabla_\mu T^{\mu\nu} = \frac{\lambda F_2}{1 + \lambda f_2} (g^{\mu\nu} \mathcal{L}_m - T^{\mu\nu}) \nabla_\mu R. \quad (4)$$

This work aims to examine the implications of Eq. (4) for Kerr-like metrics and for the generalization of the Newtonian potential.

This paper is organized as follows: Section II explores the possibility of adapting static, axially symmetric metrics to regions of space-time permeated by a pressureless perfect fluid in or near hydrostatic equilibrium, and discusses the Schwarzschild-like case in some detail; a generalization of the Newtonian potential is examined in Sec. III, and its implications are briefly discussed. In Sec. IV we present our conclusions.

II. AXIAL SYMMETRY IN A STATIC PRESSURELESS FLUID WITH NONMINIMAL GRAVITATIONAL COUPLING

Consider a static, axially symmetric system described by a metric of the form

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi, \quad (5)$$

with $\partial_t g_{\mu\nu} = \partial_\phi g_{\mu\nu} = 0$. Since we want to study the effects of a nonminimal coupling between matter and geometry, it is pointless to consider a vacuum situation. As such, we will admit a matter distribution modeled by a pressureless perfect fluid in or close to hydrostatic equilibrium. This case is chosen as it exhibits two attractive

*Also at Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa. orfeu.bertolami@fc.up.pt

†antonio.fpn.martins@gmail.com

features: first, it is sufficiently simple to be treated analytically, while still displaying the consequences of a nonminimal coupling; second, it can be used to model a number of physically interesting systems.

One can then write

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu}, \qquad u^{\mu} \simeq (u^t, 0, 0, 0), \tag{6}$$

where ρ is the density, and $(u^t)^2 = -(g_{tt})^{-1}$. Notice that the last relation implies $g_{tt} < 0$, which immediately disables a Kerr metric identification. Using $\mathcal{L}_m = -\rho$ (see Ref. [5] for a discussion), one can readily show that Eq. (4) imposes the restrictions

$$\partial_i \ln(-g_{tt}) = \frac{\partial_i(-g_{tt})}{(-g_{tt})} = -\frac{2\lambda F_2(R)}{1 + \lambda f_2(R)} \partial_i R, \tag{7}$$

with $i = r, \theta$. Equation (7) introduces a rather surprising functional relation between g_{tt} and R. It is interesting to compare the above result with the equation of hydrostatic equilibrium (for a fluid with pressure p and density ρ) arising from the spherically symmetric case in general relativity, often used in studies of stellar structure [6]:

$$\frac{\partial_r(-g_{tt})}{(-g_{tt})} = -\frac{2}{\rho + p} \partial_r p. \tag{8}$$

The similarity of Eqs. (7) and (8) suggests the nonminimal coupling acts as an effective pressure. Indeed, by noting that in the spherically symmetric case $R \equiv R(r)$, one has $\frac{\partial}{\partial R} = \frac{\partial r}{\partial R} \frac{\partial}{\partial r}$ [7], so that the numerator of Eq. (7) reads $2(\frac{\partial R}{\partial r} \frac{\partial r}{\partial R})\partial_r(\lambda f_2) = 2\partial_r(\lambda f_2)$, where we have used $\frac{\partial R}{\partial r} \frac{\partial r}{\partial R} = 1$. If one now considers $\rho \simeq \rho_0 =$ constant [we shall argue later that matter in the vicinity of a black hole behaves asymptotically as $\rho \sim (\frac{r_s}{r})^5$, where $r_s = 2GM_{\mathrm{BH}}$ is the Schwarzschild radius, so this assumption is not too unrealistic for $r \gg r_s$], then $\rho_0 \partial_r(\lambda f_2) = \partial_r(\lambda \rho_0 f)$, and one can write Eq. (7) as

$$\frac{\partial_r(-g_{tt})}{(-g_{tt})} = -\frac{2}{\rho + p_{\mathrm{eff}}} \partial_r p_{\mathrm{eff}}, \tag{9}$$

where $p_{\mathrm{eff}} = \lambda \rho_0 f_2(R(r))$.

Notice also that if g_{tt} does not depend on r or θ, neither does R (provided $\lambda F_2 \neq 0$). In particular, taking the Minkowski case $g_{tt} = -1$ immediately yields $R =$ constant, independently of the remaining metric coefficients (however, remember we are considering a static situation, so this result does not apply to the Friedmann-Robertson-Walker metric). Moreover, if $\partial_r g_{tt} \neq 0$ (or $\partial_\theta g_{tt} \neq 0$) and $1 + \lambda f_2(R) \neq 0$, one can divide both equations to arrive at

$$\frac{\partial_\theta g_{tt}}{\partial_r g_{tt}} = \frac{\partial_\theta R}{\partial_r R}, \tag{10}$$

which shows an even more straightforward relation between the derivatives of g_{tt} and R.

Before proceeding, two important comments are in order. At first, it does seem that Eq. (7) is flawed, since it forbids the Kerr/Schwarzschild case. That apparent problem arises from implicitly assuming that $\rho \neq 0$ when deriving the result. The vaccum case $\rho = 0$ yields the trivial relation $0 = 0$. Second, although it may seem that Eqs. (7) and (10) are totally independent of ρ, that is not quite correct. Indeed, the matter density influences g_{tt}, which, in turn, influences R. One can go even further and assume that Einstein's equations hold approximately, resulting in $R \simeq \rho/2\kappa$, so that Eq. (10) reads

$$\frac{\partial_\theta g_{tt}}{\partial_r g_{tt}} \simeq \frac{\partial_\theta \rho}{\partial_r \rho}, \tag{11}$$

showing a definite dependence on ρ. Nonetheless, it should be stressed that the above equations represent additional constraints on g_{tt}, R, and ρ.

To continue, we must cast Eq. (7) in a different form. First, let us show that $\frac{\partial}{\partial R} = \frac{\partial r}{\partial R} \frac{\partial}{\partial r} = \frac{\partial \theta}{\partial R} \frac{\partial}{\partial \theta}$. Considering $R \equiv R(r, \theta)$, the chain rule reads $\frac{\partial}{\partial R} = \frac{\partial r}{\partial R} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial R} \frac{\partial}{\partial \theta}$. However, to compute $\frac{\partial r}{\partial R}$ (or $\frac{\partial \theta}{\partial R}$) one must express r as a function of R, which is not possible, in general, since R also depends on θ. To solve this issue, define a new variable $\Theta \equiv \Theta(r, \theta) = \theta$, so that for sufficiently well-behaved R functions one can write $r \equiv r(R, \Theta)$ and $\theta = \theta(R, \Theta) = \Theta$. This implies $\frac{\partial \theta}{\partial R} = 0$, proving the first equality. The second can be proved using the same procedure.

By making use of the result derived above, one can now write

$$\begin{aligned} -\frac{2\lambda F_2(R)}{1 + \lambda f_2(R)} \partial_i R &= -2 \frac{\partial_i(1 + \lambda f_2)}{1 + \lambda f_2} \\ &= -2\partial_i[\ln(1 + \lambda f_2)]. \end{aligned} \tag{12}$$

Inserting Eq. (12) into Eq. (7) yields

$$\partial_i[-g_{tt}(1 + \lambda f_2)^2] = 0, \qquad i = r, \theta, \tag{13}$$

which finally leads to

$$1 + \lambda f_2(R(r, \theta)) = \frac{C}{\sqrt{-g_{tt}}}, \tag{14}$$

where C is an integration constant. Note that g_{tt} must be a function of λ satisfying $g_{tt}(\lambda = 0) = -1$, which fixes $C = 1$. Although Eq. (14) does not specify how the coupling depends on R, it determines how the function f_2 varies with r and θ, provided one knows the functional form of g_{tt}, an information that is potentially more useful. As an example, it immediately follows that $f_2 \to \infty$ as $g_{tt} \to 0$. Moreover, since g_{tt} is related to the Newtonian potential Φ, one expects that the strength of the coupling is associated with the motion of test particles.

Spherical symmetry

Having discussed the model in a general axially symmetric framework, we are now in conditions to add up further assumptions so that it is possible to evaluate its applicability to situations of physical interest.

Spherical symmetry is perhaps the simplest case to consider. In that case, we immediately have the identities

$$g_{t\phi} = 0, \tag{15}$$

$$g_{\phi\phi} = r^2 \sin(\theta), \tag{16}$$

$$g_{\theta\theta} = r^2, \tag{17}$$

$$\partial_\theta g_{\mu\nu} = 0. \tag{18}$$

Furthermore, we can assume that g_{tt} and g_{rr} are functions of r only. Although these restrictions greatly simplify calculations, Eq. (2) still seems to be too involved to allow for a straightforward mathematical treatment, even considering the general relativistic choice $f_1 = 2\kappa R$ for the geometric part of the Lagrangian. As such, we will try to pursue a different approach here, motivated chiefly by comparisons with GR. Though less rigorous, it allows establishing some tentative results, possibly shedding some light into the physical meaning of the nonminimal coupling.

Consider the case of a spherical black hole of mass $M_{\rm BH}$ surrounded by a pressureless perfect fluid of density ρ. We could argue that the space-time structure in the vicinity of a BH is dominated by its presence, and try to describe it using the familiar metric $g_{tt} = -1 + \frac{r_s}{r}$, $g_{rr} = -1/g_{tt}$, and $g_{\theta\theta}$, $g_{\phi\phi}$ as above, where $r_s = 2GM_{\rm BH}$ is the usual Schwarzschild radius. However, this choice has $g_{tt} > 0$ for some r, violating the condition imposed upon g_{tt} at the very inception of this model. Naturally, there are a number of different ways to tackle this issue. A rather straightforward alternative is to add a constant $r_c \geq r_s^*$ to the denominator, resulting in $g_{tt} = -1 + \frac{r_s^*}{r+r_c}$ and $1 + \lambda f_2 = \sqrt{1 + \frac{r_s^*}{r+r_c-r_s^*}}$. Here $r_s^* = \alpha r_s$, where $\alpha \simeq 3$. This modification stems from the fact that for $r \gg r_c$ one recovers a spherically symmetric vacuum metric, i.e., $g_{tt} \to -1 + \frac{r_s^*}{r}$. However, one must now take into account the mass of fluid present near the black hole, so that $r_s = 2GM_{\rm BH} \to r_s^* = 2GM_{\rm total}$, where $M_{\rm total} = M_{\rm BH} + M_f = \alpha M_{\rm BH}$, and M_f is the fluid's mass. We shall show later that, in first approximation, $M_f \simeq 2M_{\rm BH}$, so that $\alpha \simeq 3$. Finally, notice that r_c controls the onset of a vacuum metric. Again, we shall show that about 90% of the fluid's mass is concentrated inside a sphere of radius $3r_s^*$, suggesting that we should take $r_c = r_s^*$. Although this choice makes the coupling singular at $r = 0$, this is not much of a problem, as the curvature scalar is also singular there, and it could be taken $r_c = r_s^* + \epsilon$ and later assume $\epsilon \to 0$. This leads to [8]

$$g_{tt} = -1 + \frac{r_s^*}{r + r_s^*}, \tag{19}$$

and hence

$$1 + \lambda f_2 = \sqrt{1 + \frac{r_s^*}{r}}. \tag{20}$$

Having fixed g_{tt}, the only relevant quantity still amiss is g_{rr}. Bearing in mind the discussion presented above, one expects that sufficiently far away from the black hole a Schwarzschild metric should reasonably describe the system (provided $f_1 = 2\kappa R$, an assumption we are considering through the rest of this work). This motivates the Ansatz $g_{rr} = -1/g_{tt}$, or

$$g_{rr} = \frac{1}{1 - \frac{r_s^*}{r+r_s^*}}. \tag{21}$$

It may be that both g_{tt} and g_{rr} deviate considerably from an exact solution at some points. However, they should hold reasonably well in the outer region, and we expect that the added constant may approximately reproduce the dynamics of the inner region too. Moreover, the assumed simple forms allow for a straightforward calculation of the curvature scalar

$$R = \frac{2r_s^{*3}}{r^2(r + r_s^*)^3}. \tag{22}$$

As previously pointed out, there is a singularity at $r = 0$, and the horizon condition $g_{tt}(r_H) = 0$ yields $r_H = 0$, that is, the event horizon coincides with the singularity.

Before proceeding to an explicit calculation of ρ, it is instructive to analyze the physical consequences of a nonminimal coupling.

The trace of Eq. (2) (for $f_1 = 2\kappa R$) reads

$$\left(1 - \frac{\lambda}{\kappa} F_2 \mathcal{L}_m\right) R = \frac{3}{\kappa} \Box(\lambda F_2 \mathcal{L}_m) - \frac{1}{2\kappa}(1 + \lambda f_2) T. \tag{23}$$

If one neglects the terms containing derivatives of the coupling, Eq. (23) can be cast in the approximate form

$$R \simeq \frac{1}{2\kappa}(1 + \lambda f_2)\rho. \tag{24}$$

Consider now that the nonminimal coupling (NMC) does not disturb the metric significantly, so that we may take $R^{\rm GR} \simeq R^{\rm NMC}$, where $R^{\rm GR}$ and $R^{\rm NMC}$ stand for the curvature scalar of general relativity (GR) and NMC theory, respectively. Dividing Eq. (24) by the corresponding GR relation leads to

$$\frac{\rho^{\rm GR}}{\rho^{\rm NMC}} = 1 + \lambda f_2. \tag{25}$$

Equation (25) suggests the following physical picture: if at some point $1 + \lambda f_2 > 1$, then $\rho^{\rm GR} > \rho^{\rm NMC}$, meaning GR's space-time can accommodate a greater density of matter when compared to the NMC case. From an energetic point of view, and recalling that we are dealing with a static equilibrium situation, the amount of matter that minimizes the potential is inferior in the NMC scenario. The contrary holds if $1 + \lambda f_2 < 1$. To summarize, the NMC is energetically favorable (in the sense that it accommodates more matter than GR in equilibrium conditions) if $1 + \lambda f_2 < 1$, and unfavorable in case $1 + \lambda f_2 > 1$. From Eq. (20) one immediately has $1 + \lambda f_2 > 1$ for all r, and indeed we shall show that $M_f^{\mathrm{GR}} \gg M_f^{\mathrm{NMC}}$, reinforcing the physical reasoning presented above. Finally, note that although we have considered an example with unfavorable coupling, it is not difficult to envisage a scenario where $-g_{tt} > 1$ and hence $1 + \lambda f_2 < 1$. However, it is not possible to recover the asymptotic Schwarzschild vacuum in that case, making it a rather odd choice.

If one inserts Eqs. (19) and (22) into (23) and neglects the Laplacian term, the fluid's density can be easily computed,

$$\rho(r) = \frac{\rho_{cp}}{(\frac{r}{r_s^*})^{3/2}(1 + \frac{r}{r_s^*})^{7/2}}\left(1 + \frac{r_s^*}{5r + r_s^*}\right), \tag{26}$$

where $\rho_{cp} = \frac{4\kappa}{r_s^{*2}}$. In order to justify neglecting the term $\frac{3}{\kappa}\Box(\lambda F_2 \mathcal{L}_m)$, we estimate its contribution. From Fig. 1, it is immediately clear that in the region $r < r_s^*$, where most of the matter is concentrated [9], the term $-\frac{1}{2\kappa}(1 + \lambda f_2)T$ dominates. As such, even though they are approximately equal for $r > r_s^*$, the contribution of the Laplacian term is minute, and, in first approximation, negligible.

Equation (26) can be exactly integrated to yield the mass of fluid inside a sphere of radius r. However, the resulting expression is quite cumbersome and not particularly enlightening. In order to cast the equation in a slightly more manageable form, one may note that the last term, $1 + \frac{r_s^*}{5r+r_s^*}$, is dimensionless and bounded between 1 and 2. This suggests the tentative substitution $1 + \frac{r_s^*}{5r+r_s^*} \to \beta$, where $\beta \in]1, 2[$ is a constant that can be fixed later by comparison with the exact solution. Performing this replacement, Eq. (26) reads approximately

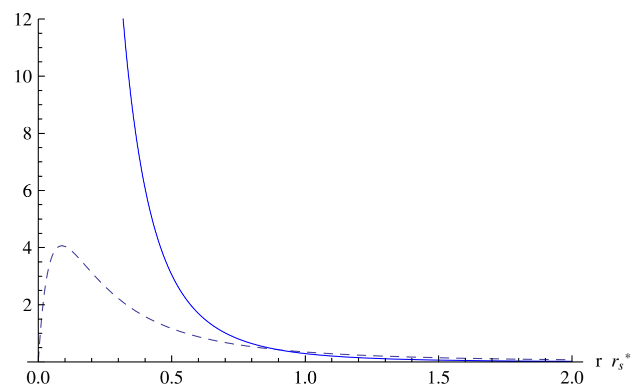

FIG. 1 (color online). Contribution of the various terms on the right-hand side of Eq. (23) as a function of r/r_s^*: the Laplacian one $\frac{3}{\kappa}\Box\lambda F_2 \mathcal{L}_m$, in dashed line, and the remaining trace term $-\frac{1}{2\kappa}(1 + \lambda f_2)T$, in solid line. Although they are approximately equal for $r > r_s^*$, the trace term clearly dominates in the inner region, where most of the matter is concentrated.

$$\rho_\beta(r) \simeq \beta \frac{\rho_{cp}}{(\frac{r}{r_s^*})^{3/2}(1 + \frac{r}{r_s^*})^{7/2}}, \tag{27}$$

where the subscript β was added to differentiate it from the exact case. The interesting feature is that one has now written the density in the form of a well-known profile type, specifically the generalized spherical cusped profile [10–13], having density scale $\beta\frac{4\kappa}{r_s^{*2}}$ and typical length r_s^*. Integrating the result over a sphere of radius r, we find that the mass of fluid enclosed in such volume to be

$$M_f(r) = \beta \frac{32\pi\kappa r_s^*}{15} \frac{r^{3/2}(2r + 5r_s^*)}{(r + r_s^*)^{5/2}}, \tag{28}$$

and taking the limit $r \to +\infty$, the total fluid mass reads

$$M_f = \beta \frac{64\pi\kappa r_s^*}{15} = \beta \frac{8\alpha}{15} M_{\mathrm{BH}}, \tag{29}$$

where we have used $\kappa = \frac{1}{16\pi G} = \frac{M_{\mathrm{BH}}}{8\pi r_s}$. Inserting Eq. (29) into Eq. (28), it can be readily shown that $M_f(r_s^*) \simeq 0.6 M_f$ and $M_f(3r_s^*) \simeq 0.9 M_f$, proving the previous claims about the matter distribution. To conclude the calculation, we must determine α and β. From $M_f + M_{\mathrm{BH}} = \alpha M_{\mathrm{BH}}$ it is immediate that

$$\alpha = \frac{1}{1 - \frac{8\beta}{15}}. \tag{30}$$

The remaining constant, β, can be fitted using data from the exact solution, yielding the result $\beta \simeq 1.28$. Figure 2 illustrates a graphic comparison between the exact solution and the approximate one. The difference between the two never exceeds 7% of the total mass. Using this result, we finally obtain

$$\alpha \simeq 3.15, \qquad M_f \simeq 2.15 M_{\mathrm{BH}}. \tag{31}$$

Equation (31) predicts that the total fluid mass in the vicinity of a spherical, static black hole should be roughly twice that of the black hole itself. Notice that if one had

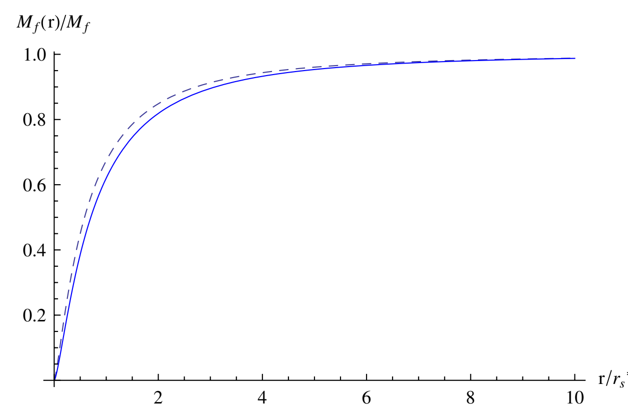

FIG. 2 (color online). Comparison between the exact $M_f(r)$ solution (dashed line) and the approximate one (solid line) for $\beta = 1.28$. The difference between the two never exceeds 7% of the total mass.

used the general relativistic equation $R = \rho/2\kappa$, the condition on α would be (assuming $\beta \simeq 1$) $\alpha = 1 + \alpha$. Although this equation has no solution, one can think that in the limit $\alpha \gg 1$ the equality is approximately verified, leading to $M_f^{\rm GR} \gg M_{\rm BH}^{\rm GR}$, and, consequently, $M_f^{\rm GR} \gg M_f^{\rm NMC}$, suggesting, as previously remarked, that the NMC's space-time accommodates much less matter in equilibrium than its general relativistic counterpart. This strengthens the idea that the NMC acts as an effective fluid pressure.

III. GENERALIZATION OF THE NEWTONIAN POTENTIAL FOR A PERFECT FLUID

Consider the nonconservation Eq. (4) and the stress-energy tensor of a perfect fluid

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}. \tag{32}$$

Introducing the projection tensor $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, Eq. (4) can be used to show that the equation of motion for a fluid element is given by [2]

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = f^\mu, \tag{33}$$

where the right-hand side reads

$$f^\mu = \frac{1}{p+\rho}\left[\frac{\lambda F_2}{1+\lambda f_2}(\mathcal{L}_m - p)\nabla_\nu R - \nabla_\nu p\right] h^{\mu\nu}. \tag{34}$$

Regarding Eq. (34), the second term on the right is just the usual pressure gradient acceleration. However, the first is an extra force that arises from the NMC. If one inserts $\mathcal{L}_m = -\rho$ and uses the previously proven identity [14] $\lambda F_2 \nabla_\nu R = \partial_\nu(1 + \lambda f_2) = \partial_\nu(\lambda f_2)$, it can be cast in the form

$$f^\mu_{\rm NMC} = -\frac{\partial_\nu(\lambda f_2)}{1+\lambda f_2} h^{\mu\nu}. \tag{35}$$

Once again, multiplying numerator and denominator by ρ and assuming a near constant density, one ends up with an NMC force that resembles the pressure gradient case of GR.

Assuming the usual Newtonian limit approximations $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ and $|\epsilon_{\mu\nu}| \ll 1$, $\frac{dt}{d\tau} \simeq 1$ and $|\frac{dx^i}{dt}| \ll 1$, the left-hand side of Eq. (33) can be simplified to

$$\frac{du^i}{d\tau} + \Gamma^i_{\alpha\beta} u^\alpha u^\beta \simeq \frac{d^2 x^i}{dt^2} - \frac{1}{2}\partial_i \epsilon_{tt}, \tag{36}$$

the index i running over the space coordinates. Noting additionally that $h^{i\nu} \simeq \eta^{i\nu} = \delta^{i\nu}$, the full Eq. (33) reads approximately

$$\frac{d^2 x^i}{dt^2} = \partial_i\left[\frac{g_{tt}+1}{2} - \ln(|1 + \lambda f_2|)\right] - \frac{1}{\rho + p}\partial_i p, \tag{37}$$

where the identities $\epsilon_{tt} = g_{tt} + 1$ and $\partial_i(\lambda f_2)/(1 + \lambda f_2) = \partial_i \ln(|1 + \lambda f_2|)$ were used. It can be seen that the NMC gives rise to an additional force, and its form suggests the gravitational potential should be generalized to

$$\Phi = \ln(|1 + \lambda f_2|) - \tfrac{1}{2}(g_{tt} + 1) + \Phi_0. \tag{38}$$

The inclusion of an extra term in the potential can lead to a vast number of consequences [15,16]. As a practical example, it is not difficult to envisage a coupling, whose effects would only be noticeable at a galactic level, that could account for the flattening of the rotation curves of galaxies [17]. However, such study is clearly outside the scope of this work. Instead, one may try to further explore the physical meaning of an NMC in a rather general context. To achieve that, consider the usual Newtonian choice $g_{tt} + 1 \simeq \frac{r_{\rm sc}}{r}$, where $r_{\rm sc}$ is a constant that sets the length scale (typically the Schwarzschild radius), and a coupling of the form

$$1 + \lambda f_2^\pm = 1 \pm \left(\frac{r_{\rm sc}}{r + r_{\rm sc}}\right)^n. \tag{39}$$

The different signs are used to study both $1 + \lambda f_2 > 1$ and $1 + \lambda f_2 < 1$ cases. It is also convenient to take $n > 1$ so that the term $\frac{1}{2}(g_{tt} + 1)$ dominates when $r \to +\infty$, although other choices may also be admissible, and indeed, we shall show that the case presented in Sec. II yields a physically meaningful coupling that may be cast in the form of Eq. (39) with $n = 1$ when $r \gg r_{\rm sc}$. Defining $f_r^{\rm GR} = -r_{\rm sc}/2r^2$ and neglecting the pressure gradient, the radial force per unit mass $f_r^\pm$ reads

$$f_r^\pm = f_r^{\rm GR} + \frac{n}{r + r_{\rm sc}} \frac{(\frac{r_{\rm sc}}{r+r_{\rm sc}})^n}{(\frac{r_{\rm sc}}{r+r_{\rm sc}})^n \pm 1}. \tag{40}$$

Notice that the last term is positive (negative) if one takes the + (−) sign. This leads to

$$|f_r^+| < |f_r^{\rm GR}|, \qquad |f_r^-| > |f_r^{\rm GR}|, \tag{41}$$

which tells us that the corrected gravitational force is weaker (stronger) than the Newtonian one when $1 + \lambda f_2 > 1 (1 + \lambda f_2 < 1)$. Figure 3 exhibits this feature graphically for $n = 3$. Notice that Eqs. (41) are valid even if $0 < n \le 1$. This result reinforces the arguments already presented in Sec. II: it seems that a supra-unitary coupling weakens the link between geometry and matter, while a subunitary one strengthens that connection.

To conclude, let us study the application of the generalized gravitational potential derived above to the situation described in the first part of this paper.

Inserting Eq. (14) into Eq. (38), one has

$$\Phi = -\tfrac{1}{2}[g_{tt} + 1 + \ln(-g_{tt})]. \tag{42}$$

If g_{tt} is written as $g_{tt} = -1 + \xi(r)$, where $\xi(r)$ is a positive decreasing function of r obeying $\xi(r) < 1$ (a condition imposed by $g_{tt} < 0$), then

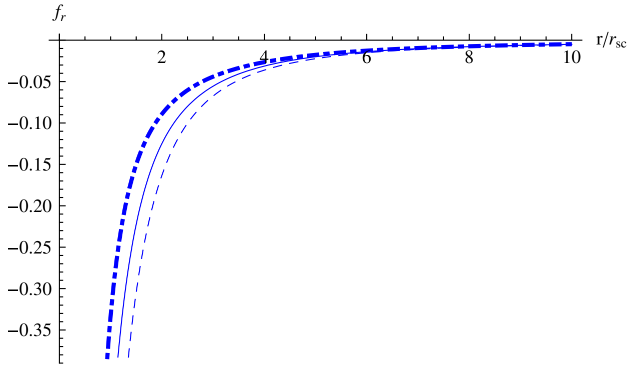

FIG. 3 (color online). Comparison between $f_r^{\rm GR}$ (solid line), f_r^+ (dotted-dashed) and f_r^- (dashed line) for $n = 3$. It is visible that $|f_r^+| < |f_r^{\rm GR}|$, meaning the corrected gravitational force is slightly weaker than the Newtonian one when $1 + \lambda f_2 > 1$. By the same token, a coupling with $1 + \lambda f_2 < 1$ yields a stronger gravitational pull. Once again, this suggests that a supra-unitary coupling is disfavorable (in the sense that it weakens the connection between geometry and matter), while a subunitary one is favorable.

$$\partial_r \Phi = \frac{\xi'(r)}{2} \frac{\xi(r)}{1 - \xi(r)}. \tag{43}$$

This gives rise to a positive (repulsive) radial force. To understand this result, recall, first, that we are dealing with a situation of equilibrium. Indeed, using Eqs. (19) and (21) one can easily verify that Eq. (33) is exactly satisfied when $u^\mu = (u^t, 0, 0, 0)$ with $(u^t)^2 = -(g_{tt})^{-1}$. However, if this equilibrium is disturbed by the addition of a fluid element, for example, that element is repelled to the outskirts. This is in agreement with the interpretation of the NMC as an effective pressure. In a general relativistic nonvacuum static situation, the gravitational pull is balanced by the pressure gradient to avoid collapse, and a similar situation occurs: the fluid particle will be attracted through vacuum, but once it steps inside the matter permeated region the pressure will exert a repulsive force that pushes the element outwards. Since one is assuming space is completely filled up with matter (i.e., $\rho(r)$ reaches out to infinity), the force experienced by a test fluid element is always repulsive. Naturally, in a more refined and realistic model one could assume that matter is concentrated within a spherical shell, with vacuum on the outside. Nevertheless, the example considered is sufficient to convincingly suggest that the effects of a nonminimal coupling mimic those of an effective pressure.

IV. CONCLUSION

In this work, we have explored the consequences of a modified theory of gravity with a nonminimal coupling between geometry and matter as suggested by action Eq. (1). In order to do so, a general axially symmetric, static situation was considered in first place, assuming a matter distribution modeled by a pressureless perfect fluid. This led to Eq. (7), and, ultimately, to Eq. (14), which shows that the coupling, $1 + \lambda f_2$, and g_{tt} are closely related through a rather simple functional form.

The study of a spherically symmetric black hole surrounded by pressureless matter was then examined. Since an exact Schwarzschild solution could not be considered due to restrictions on g_{tt}, one assumed metric coefficients that resembled the vacuum case when $r \to +\infty$, justified by admitting that sufficiently far from the black hole one should have an "almost Schwarzschild" solution, while being significantly different in the inner regions, that is, $r \sim r_s$. This permitted an explicit calculation of the matter density ρ, which was shown to approximately fit into the generalized spherical cusped profile widely present in the literature. The total mass of fluid was then determined and shown to be about twice the mass of the black hole. A brief comparison with the general relativistic case, which predicted a much larger amount of mass, suggested that the nonminimal coupling could be regarded as an effective pressure, and stipulated the conditions under which this coupling could be regarded as favorable or unfavorable to the connection between geometry and matter: $1 + \lambda f_2 < 1$ and $1 + \lambda f_2 > 1$, respectively.

Finally, in Sec. III it was demonstrated that when one considers the Newtonian limit in a nonminimal coupling theory, an additional logarithmic term appears in the classical gravitational potential. It seems rather clear that such modification may have a wide range of implications. At the very least, it can be used to place constraints on the nonminimal coupling through comparison with Solar System tests [4]. However, such procedure would be well outside the scope of this paper, and, as such, will be pursued elsewhere. Instead, an argument was presented that solidifies the idea that a supra-unitary coupling weakens the association between gravity and matter, whereas a subunitary one strengthens that relation. To conclude, the case presented in Sec. II was inspected under the modified gravitational potential, leading, once again, to the idea that the nonminimal coupling plays the role of an effective pressure.

Although not particularly focused on any fracturing issue of contemporary cosmology (the accelerated expansion of the Universe [18], the flattening of the rotation curves of galaxies [17], or the mimicking of the cosmological constant [19], which were previously considered in the context of NMC theories), the present work shows that a theory with a nonminimal coupling between curvature and matter can yield some new and physically relevant results, especially when applied to astronomical objects whose nature is still poorly known, such as black holes.

ACKNOWLEDGMENTS

The work of O. B. is partially supported by the FCT project PTDC/FIS/111362/2009.

[1] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).
[2] O. Bertolami, C. G. Boehmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007).
[3] O. Bertolami, T. Harko, F. S. N. Lobo, and J. Páramos, arXiv:0811.2876.
[4] O. Bertolami and J. Páramos, Classical Quantum Gravity **25**, 245017 (2008).
[5] O. Bertolami, F. S. N. Lobo, and J. Páramos, Phys. Rev. D **78**, 064036 (2008).
[6] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).
[7] In fact, the same relation holds even for the axially symmetric case. See below.
[8] There is a minor issue we have been ignoring so far: as pointed out earlier, g_{tt} must depend on λ, so that one recovers $g_{tt} = -1$ when $\lambda \rightarrow 0$. Making the substitution $r_s^* \rightarrow \lambda r_s^*$ is one way to deal with this problem. However, if $\lambda \neq 0$, it can be absorbed into f_2, which is the same as taking $\lambda = 1$, so that no problem arises whatsoever. We will assume that $\lambda \neq 0$ throughout the rest of this paper.
[9] Take, for example, the ratio $\rho(r_s^*/2)/\rho(r_s^*) \simeq 8.5$
[10] P. Salucci and M. Persic, ASP Conf. Series **117**, 1 (1997).
[11] D. Merritt, J. F. Navarro, A. Ludlow, and A. Jenkins, Astrophys. J. **624**, L85 (2005).
[12] A. Graham, D. Merritt, B. Moore, J. Diemand, and B. Terzic, Astron. J. **132**, 2701 (2006).
[13] G. van de Ven, R. Mandelbaum, and C. R. Keeton, Mon. Not. R. Astron. Soc. **398**, 607 (2009).
[14] In fact, the result was proven only for $R \equiv R(r, \theta)$. However, its generalization to the case $R \equiv R(x^0, x^1, x^2, x^3)$ is straightforward.
[15] S. Capozziello, V. F. Cardone, and A. Troisi, Mon. Not. R. Astron. Soc. **375**, 1423 (2007).
[16] C. G. Boehmer, T. Harko, and F. S. N. Lobo, Astropart. Phys. **29**, 386 (2008).
[17] O. Bertolami and J. Páramos, J. Cosmol. Astropart. Phys. 03 (2010) 009.
[18] O. Bertolami, P. Frazão, and J. Páramos, Phys. Rev. D **81**, 104046 (2010).
[19] O. Bertolami and J. Páramos, Phys. Rev. D **84**, 064022 (2011).