

Noncommutative quantum mechanics of a harmonic oscillator under linearized gravitational waves

Anirban Saha* and Sunandan Gangopadhyay†

Department of Physics and Astrophysics, West Bengal State University, Barasat, North 24 Paraganas, West Bengal, India

Swarup Saha

Debipur, Duttapukur 743248, India

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We consider the quantum dynamics of a harmonic oscillator in noncommutative space under the influence of linearized gravitational waves (GWs) in the long-wavelength and low-velocity limit. Following the prescription in Saha and Gangopadhyay [*Phys. Lett. B* **681**, 96 (2009)] we quantize the system. The Hamiltonian of the system is solved by using standard algebraic iterative methods. The solution shows signatures of the coordinate noncommutativity via alterations in the oscillation frequency of the harmonic oscillator system from its commutative counterpart. Moreover, it is found that the response of the harmonic oscillator to periodic GWs, when their frequencies match, will oscillate with a time scale imposed by the noncommutative parameter. We expect this noncommutative signature to show up as some noise source in the GW detection experiments since the recent phenomenological upper bounds set on the spatial noncommutative parameter imply a length scale comparable to the length variations due to the passage of gravitational waves, detectable in the present-day GW detectors.

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Gravitational waves (GWs) are tiny vibrations in the spacetime structure itself. The present-day scenario of GW detection experiments primarily consists of ground-based (LIGO [1], VIRGO [2], GEO [3], TAMA [4], etc.) and space-based (LISA [5]) interferometers. The key idea here is to measure the relative optical phase shift between the light paths in two perpendicular km-length arm cavities where this phase shift is due to the relative displacement, induced by a passing GW, of the two mirrors hung at the end of each arm cavity. However, the history of experimental GW physics began with resonant-mass detectors, pioneered by Weber in the 1960s. In the following decades, although the sensitivity of resonant-mass detectors has improved considerably, it is clear that they could only allow the detection of relatively strong GW signals in our Galaxy or in the immediate galactic neighborhood. Nevertheless, building the interferometric detectors takes many years of preparation and considerable finance whereas the resonant detectors, being relatively small-scale instruments, are more easily realizable. Besides, the study of resonant-mass detectors is instructive in itself because it focuses on how a GW interacts with an elastic matter causing vibrations with amplitudes many orders smaller than the size of a nucleus. In a resonant bar it is possible to measure vibrations which correspond to just a

few tens of phonons [6], and variations ΔL of their length L , with $\frac{\Delta L}{L} \sim 10^{-19}$. It is, therefore, at the quantum mechanical level, that experimental evidence for the GWs is likely to appear [7].

Interestingly, in recent developments of noncommutative (NC) quantum mechanics [8–12] and NC quantum field theory [13–16], where the coordinates x^μ satisfy the NC algebra

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (1)$$

the upper bounds on various NC parameters appearing in the literature [17–23] are quite close to this length scale. A wide range of theories have been constructed in a NC framework including various gauge theories [13], gravity [24], and even encompassing certain possible phenomenological consequences [25–30]. The upper bound on the value of the coordinate commutator θ^{ij} found in [17] is $\lesssim (10 \text{ TeV})^{-2}$ which corresponds to $4 \times 10^{-40} \text{ m}^2$ for $\hbar = c = 1$, whereas such upper bounds on time-space NC parameter θ^{0i} is $\lesssim 9.51 \times 10^{-18} \text{ m}^2$. However, recent studies in NC quantum mechanics revealed that the NC parameters associated with different particles are not the same [31] and this bound could be as high as $\theta \lesssim (4 \text{ GeV})^{-2} - (30 \text{ MeV})^{-2}$ [21]. These upper bounds correspond to the length scale $\sim 10^{-20} - 10^{-17} \text{ m}$.

With the prospect of the direct detection of GWs of such tiny amplitude as $\sim 10^{-18}$ in the near future, a good possibility of detecting the NC structure of spacetime would be in the GW detection experiments as it may as well detect the signature of noncommutativity. As a first step towards this endeavor, we analyzed the interplay of classical GWs with a free test particle in a NC quantum mechanical

*Visiting Associate in Inter University Centre for Astronomy and Astrophysics, Pune, India.
anirban@iucaa.ernet.in

†Visiting Associate in Satyendra Nath Bose National Center for Basic Sciences, Kolkata, India.
sunandan@bose.res.in; sunandan.gangopadhyay@gmail.com

framework in [32]. Our analysis suggested that investigating a NC harmonic oscillator system interacting with the passing GW is more likely to reveal a NC effect comparable with the effect of the GW. Moreover, a simplistic consideration of the interaction of the GW with the resonant bar shows that the fundamental mode of vibration of a thin cylindrical bar is formally identical to a harmonic oscillator driven by a force exerted by the GW [6]. Therefore, NC quantum mechanical consideration of the interaction of GWs with matter, specifically the harmonic oscillator which is essentially inherent in the resonant-mass detector system, would certainly prove instructive. In light of all these facts, we, in the present paper, formulate the quantum mechanics of a NC harmonic oscillator, interacting with a linearized GW in the long-wavelength and low-velocity limit.

Since it has been demonstrated in various formulations of NC general relativity [24,33,34] that any NC correction in the gravity sector is second order in the NC parameter, therefore, in a first order theory in NC space, the linearized GW remains unaltered by NC effects and any NC correction appearing in the system will be through the matter part only. This is true not only with the canonical (i.e., constant) form of noncommutativity but also for the Lie-algebraic NC spacetime [34]. We shall incorporate the NC effect by writing the NC Hamiltonian for the system and then reexpressing it in terms of the commutative coordinates and their momenta by the well known Bopp-shift transformations [21,23]. This commutative equivalent model will be quantized following [35]. We shall deal with the linearly polarized GWs in this analysis for simplicity.

In a linearized theory of gravity the connection and curvature tensor take the form [36]

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \eta^{\mu\nu} (h_{\beta\nu,\alpha} + h_{\alpha\nu,\beta} - h_{\alpha\beta,\nu}), \quad (2)$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2} (h_{\alpha\nu,\mu\beta} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu} - h_{\alpha\mu,\beta\nu}), \quad (3)$$

where $h_{\mu\nu}$ is the metric perturbation on the flat Minkowski background $\eta_{\mu\nu}$. Next we choose the transverse-traceless gauge

$$h_{0\mu} = 0, \quad h_{\mu\nu}{}^{,\mu} = 0, \quad h^\mu_\mu = 0 \quad (4)$$

to remove all the gauge redundancies of the theory and the GW is characterized by the only nonzero components [6] $h_{11} = -h_{22}$ and $h_{12} = h_{21}$, called the + and $\times$ polarization, respectively. The only nontrivial components of the curvature tensor in the transverse-traceless gauge are [37]

$$R^j_{0,k0} = -\frac{d\Gamma^j_{0k}}{dt} = -\frac{1}{2} \frac{d^2 h_{jk}}{dt^2} \quad (5)$$

and the geodesic deviation equation in the proper detector frame becomes [6]

$$m \frac{d^2 x^j}{dt^2} = -m R^j_{0,k0} x^k - m \varpi^2 x^j. \quad (6)$$

Here t is the coordinate time of the proper detector frame and is the same as its proper time since we are confining ourselves to first order in the metric perturbation. Equation (6) governs the response of a two-dimensional harmonic oscillator with frequency ϖ to the passage of a GW. Here x^j is the proper distance of the pendulum from the origin, m is its mass, and the GW is treated as an external classical field. Note that Eq. (6) can be used to describe the evolution of proper distance in the transverse-traceless-gauge frame as long as the spacial velocities involved are non-relativistic. Also, $|x^j|$ has to be much smaller than the typical length scale over which the gravitational field changes substantially, i.e., the reduced wavelength $\frac{\lambda}{2\pi}$ of a GW. The above conditions are collectively referred to as the *small-velocity and long-wavelength limit*. Thus, with Eq. (6) we can analyze the interaction of a GW with a detector which has a characteristic linear size $L \ll \frac{\lambda}{2\pi}$. Note that this condition is satisfied by resonant bar detectors as well as earth bound interferometers but not by the proposed space-borne interferometers such as LISA [5] or by the Doppler tracking of spacecraft.

The Lagrangian for the system, whose time evolution is described by Eq. (6), can be written, up to a total derivative term [35], as

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - m \Gamma^j_{0k} \dot{x}_j x^k - \frac{1}{2} m \varpi^2 x^2. \quad (7)$$

The canonical momentum corresponding to x_j is $p_j = m \dot{x}_j - m \Gamma^j_{0k} x^k$ and hence the Hamiltonian becomes

$$H = \frac{1}{2m} (p_j + m \Gamma^j_{0k} x^k)^2 + \frac{1}{2} m \varpi^2 x_j^2. \quad (8)$$

Assuming that the GW is propagating along the z axis, due to the transverse nature of GWs, Γ^j_{0k} has nonzero components only in the x - y plane and therefore the response of the pendulum to it is essentially confined to that plane. To impose noncommutativity on this plane, we assume that the coordinates follow the algebra (1) and “quantize” the system on this NC plane. To proceed, we replace x^j and p_j in the above Hamiltonian by operators $\hat{x}^j$ and $\hat{p}_j$ satisfying the NC Heisenberg algebra [38]

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = i\theta \epsilon_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \quad (9)$$

It is well known that this can be mapped to the standard ($\theta = 0$) Heisenberg algebra spanned by X_i and P_j using [21,23]

$$\hat{x}_i = X_i - \frac{1}{2\hbar} \theta \epsilon_{ij} P_j, \quad \hat{p}_i = P_i. \quad (10)$$

Using the traceless property of the GW and rewriting the NC version of Eq. (8) in terms of the operators X_i and P_j , we obtain

$$\begin{aligned} \hat{H} = & \frac{P_j^2}{2m} + \frac{1}{2} m \varpi^2 X_j^2 + \Gamma^j_{0k} X_j P_k - \frac{m \varpi^2}{2\hbar} \theta \epsilon_{jm} X^j P_m \\ & - \frac{\theta}{2\hbar} \epsilon_{jm} P_m P_k \Gamma^j_{0k}. \end{aligned} \quad (11)$$

In the above equation the first two terms are for the ordinary harmonic oscillator; the third term, linear in the affine connections, shows the effect of the passing GW on the ordinary harmonic oscillator system; the fourth term is the signature of NC space, a pure NC term linear in the NC parameter; and the final term shows the coupling between the GW and spatial noncommutativity [40]. Since the length-scale range describing various upper bounds on the NC parameter that we have discussed earlier is quiet close to the detectable variations of length ΔL in the GW detection experiments, at least the fourth term in the left-hand side of Eq. (11) should be comparable with the third (purely gravitational) term and show up in the GW experiments as some noise source. This, in essence, is similar to some recent results [41] where the coordinate noncommutativity is shown to generate holographic noise claimed to be detectable in the signals of the interferometric GW detectors GEO600 [42].

Defining raising and lowering operators in terms of the oscillator frequency ϖ

$$X_j = \left(\frac{\hbar}{2m\varpi}\right)^{1/2} (a_j + a_j^\dagger), \quad (12)$$

$$P_j = -i\left(\frac{\hbar m\varpi}{2}\right)^{1/2} (a_j - a_j^\dagger), \quad (13)$$

we write the Hamiltonian (11) as

$$\begin{aligned} \hat{H} = & \hbar\varpi(a_j^\dagger a_j + 1) - \frac{i\hbar}{4} \dot{h}_{jk}(a_j a_k - a_j^\dagger a_k^\dagger) \\ & + \frac{m\varpi\theta}{8} \epsilon_{jm} \dot{h}_{jk}(a_m a_k - a_m^\dagger a_k^\dagger + \text{C.C.}) \\ & - \frac{i}{2} m\varpi^2 \theta \epsilon_{jk} a_j^\dagger a_k, \end{aligned} \quad (14)$$

where C.C. means complex conjugate. Working in the Heisenberg representation, the time evolution of $a_j(t)$ is given by

$$\begin{aligned} \frac{da_j(t)}{dt} = & -i\varpi a_j + \frac{1}{2} \dot{h}_{jk} a_k^\dagger - \frac{m\varpi^2 \theta}{2\hbar} \epsilon_{jk} a_k \\ & + \frac{im\varpi\theta}{8\hbar} (\epsilon_{lj} \dot{h}_{lk} + \epsilon_{lk} \dot{h}_{lj})(a_k - a_k^\dagger) \end{aligned} \quad (15)$$

and that of $a_j^\dagger(t)$ is the C.C. of the above equation. Next, noting that the raising and lowering operators must satisfy the commutation relations

$$[a_j(t), a_k^\dagger(t)] = \delta_{jk}, \quad [a_j(t) a_k(t)] = 0 = [a_j^\dagger(t), a_k^\dagger(t)], \quad (16)$$

we write them in terms of $a_j(0)$ and $a_j^\dagger(0)$, the free operators at time $t=0$, by the time-dependent Bogoliubov transformations

$$\begin{aligned} a_j(t) &= u_{jk}(t) a_k(0) + v_{jk}(t) a_k^\dagger(0), \\ a_j^\dagger(t) &= a_k^\dagger(0) \bar{u}_{kj}(t) + a_k(0) \bar{v}_{kj}(t), \end{aligned} \quad (17)$$

where the bar denotes the C.C. and u_{jk} and v_{jk} are the generalized Bogoliubov coefficients. They are 2×2 complex matrices which, due to Eq. (16), must satisfy $uv^T = u^T v$, $uu^\dagger - vv^\dagger = I$, written in matrix form where T denotes transpose, $\dagger$ denotes complex conjugate transpose, and I is the identity matrix. Since $a_j(t=0) = a_j(0)$, $u_{jk}(t)$ and $v_{jk}(t)$ have the boundary conditions

$$u_{jk}(0) = I, \quad v_{jk}(0) = 0. \quad (18)$$

Then, from Eq. (15) and its C.C., we get the following equations of motion in terms of $\xi = u - v^\dagger$ and $\bar{\xi} = u + v^\dagger$:

$$\frac{d\xi_{jk}}{dt} = -i\varpi \xi_{jk} - \frac{1}{2} \dot{h}_{jl} \xi_{lk} - \frac{m\varpi^2 \theta}{2\hbar} \epsilon_{jl} \xi_{lk}, \quad (19)$$

$$\frac{d\bar{\xi}_{jk}}{dt} = -i\varpi \bar{\xi}_{jk} + \frac{1}{2} \dot{h}_{jl} \bar{\xi}_{lk} + \Theta_{jl} \bar{\xi}_{lk} - \frac{m\varpi^2 \theta}{2\hbar} \epsilon_{jl} \bar{\xi}_{lk}, \quad (20)$$

where Θ_{jl} is the term reflecting the interplay of noncommutativity with the GW

$$\Theta_{jl} = \frac{im\varpi\theta}{4\hbar} (\dot{h}_{jm} \epsilon_{ml} - \epsilon_{jm} \dot{h}_{ml}). \quad (21)$$

Equations (19) and (20) are difficult to solve analytically for general h_{jk} . However, our goal, in the present paper, is to investigate whether we get comparable effects of spatial noncommutativity and gravitational waves on the harmonic oscillator system in the simplest of settings. Therefore we shall solve Eqs. (19) and (20) for the special case of linearly polarized GWs.

In the two-dimensional plane, the GW, which is a 2×2 matrix h_{jk} , is most conveniently written in terms of the Pauli spin matrices as

$$h_{jk}(t) = 2f(t)(\epsilon_\times \sigma_{jk}^1 + \epsilon_+ \sigma_{jk}^3) = 2f(t)\epsilon_A \sigma_{jk}^A. \quad (22)$$

Note that the index A runs from 1–3; however, no contribution from σ^2 is included. $2f(t)$ is the amplitude of the GW whereas $\epsilon_\times(t)$ and $\epsilon_+(t)$ represent the two possible polarization states of the GW and satisfy the condition $\epsilon_\times^2 + \epsilon_+^2 = 1$ for all t . In the case of linearly polarized GWs, however, the polarization states ϵ_A are independent of time and $f(t)$ is arbitrary. To set a suitable boundary condition we shall assume that the GW hits the particle at $t=0$ so that

$$f(t) = 0, \quad \text{for } t \leq 0. \quad (23)$$

We now move on to solve Eqs. (19) and (20) by noting that any 2×2 complex matrix can be written as a linear combination of the Pauli spin matrices and the identity matrix. Hence we make the ansatz

$$\xi_{jk}(t) = A I_{jk} + B_1 \sigma_{jk}^1 + B_2 \sigma_{jk}^2 + B_3 \sigma_{jk}^3, \quad (24)$$

$$\xi_{jk}(t) = CI_{jk} + D_1\sigma_{jk}^1 + D_2\sigma_{jk}^2 + D_3\sigma_{jk}^3. \quad (25)$$

Substituting for h_{jk} , ζ_{jk} , and ξ_{jk} from Eqs. (22), (24), and (25), in Eqs. (19) and (20), and comparing the coefficients of the I and σ matrices, we get a set of first order differential equations for A , B_1 , B_2 , B_3 , C , D_1 , D_2 , D_3 :

$$\begin{aligned} \dot{A} &= -i\varpi C - \dot{f}(\varepsilon_1 B_1 + \varepsilon_3 B_3) - i\Lambda B_2, \\ \dot{B}_1 &= -i\varpi D_1 - \dot{f}(\varepsilon_1 A - i\varepsilon_3 B_2) + \Lambda B_3, \\ \dot{B}_2 &= -i\varpi D_2 - i\dot{f}(\varepsilon_3 B_1 - \varepsilon_1 B_3) - i\Lambda A, \\ \dot{B}_3 &= -i\varpi D_3 - \dot{f}(\varepsilon_3 A + i\varepsilon_1 B_2) - \Lambda B_1, \\ \dot{C} &= -i\varpi A + \dot{f}(\varepsilon_1 D_1 + \varepsilon_3 D_3) + 4i\lambda\dot{f}(\varepsilon_3 B_1 - \varepsilon_1 B_3) \\ &\quad - i\Lambda D_2, \\ \dot{D}_1 &= -i\varpi B_1 + \dot{f}(\varepsilon_1 C - i\varepsilon_3 D_2) + 4\lambda\dot{f}(i\varepsilon_3 A - \varepsilon_1 B_2) \\ &\quad + \Lambda D_3, \\ \dot{D}_2 &= -i\varpi B_2 + i\dot{f}(\varepsilon_3 D_1 - \varepsilon_1 D_3) + 4\lambda\dot{f}(\varepsilon_1 B_1 + \varepsilon_3 B_3) \\ &\quad - i\Lambda C, \\ \dot{D}_3 &= -i\varpi B_3 + \dot{f}(\varepsilon_3 C + i\varepsilon_1 D_2) - 4\lambda\dot{f}(i\varepsilon_1 A + \varepsilon_3 B_2) \\ &\quad - \Lambda D_1, \end{aligned} \quad (26)$$

where Λ and λ are given by

$$\Lambda = \frac{m\varpi^2\theta}{2\hbar}, \quad \lambda = \frac{m\varpi\theta}{4\hbar} \quad (27)$$

and the dot represents the derivative with respect to time t . Noting that $|f(t)| \ll 1$, the above set of equations can be solved iteratively about its $f(t) = 0$ solution by applying the appropriate boundary conditions (18) and (23). We therefore obtain to first order in the gravitational wave amplitude and also in the NC parameter

$$A(t) = C(t) = e^{-i\varpi t} \cos(\Lambda t), \quad (28)$$

$$B_2(t) = D_2(t) = -ie^{-i\varpi t} \sin(\Lambda t), \quad (29)$$

$$\begin{aligned} B_1(t) &= -D_1(t) \\ &= -e^{-i\varpi t} \left[(\varepsilon_1 \cos(\Lambda t) - \varepsilon_3 \sin(\Lambda t))f(t) \right. \\ &\quad + 2i\varpi\varepsilon_1 \int_0^t dt' e^{i\varpi(t-t')} \cos(\Lambda t') f(t') \\ &\quad - 2i\varpi\varepsilon_3 \int_0^t dt' e^{i\varpi(t-t')} \sin(\Lambda t') f(t') \left. \right] \\ &\quad + \varpi^2 \left(\varepsilon_1 \int_0^t g_1(t') dt' - \varepsilon_3 \int_0^t g_2(t') dt' \right), \end{aligned} \quad (30)$$

$$\begin{aligned} B_3(t) &= -D_3(t) \\ &= -e^{-i\varpi t} \left[(\varepsilon_3 \cos(\Lambda t) + \varepsilon_1 \sin(\Lambda t))f(t) \right. \\ &\quad + 2i\varpi\varepsilon_3 \int_0^t dt' e^{i\varpi(t-t')} \cos(\Lambda t') f(t') \\ &\quad + 2i\varpi\varepsilon_1 \int_0^t dt' e^{i\varpi(t-t')} \sin(\Lambda t') f(t') \left. \right] \\ &\quad + \varpi^2 \left(\varepsilon_3 \int_0^t g_1(t') dt' + \varepsilon_1 \int_0^t g_2(t') dt' \right), \end{aligned} \quad (31)$$

where

$$\begin{aligned} g_1(t) &= \int_0^t dt' e^{-i\varpi t'} \cos(\Lambda t') f(t'), \\ g_2(t) &= \int_0^t dt' e^{-i\varpi t'} \sin(\Lambda t') f(t'). \end{aligned} \quad (32)$$

The system has now been essentially solved once we specify the initial expectation values of the pendulum's position $\vec{r}_0 = (x_0, y_0)$ and momentum $\vec{p}_0 = (p_{x_0}, p_{y_0})$ when the GW just hits the system at time $t = 0$. The time evolution of the coordinates can be calculated employing the following scheme. Combining the expressions for A , B_1 , B_2 , B_3 , C , D_1 , D_2 , D_3 , we can write the solutions for ζ and ξ using Eqs. (24) and (25) which in turn give u and v . Using Eq. (17), we can now combine u and v into $a_j(t)$ and $a_j^\dagger(t)$. From the initial position and momentum expectation values, i.e., $\langle \vec{r}_0 \rangle = (X_1(0), X_2(0))$ and $\langle \vec{p}_0 \rangle = (P_1(0), P_2(0))$, we get the raising and lowering operator $a_j(0)$ and $a_j^\dagger(0)$ at time $t = 0$. We then use them in Eq. (17) to find $a_j(t)$ and $a_j^\dagger(t)$ at a general time t and these yield the time evolution of the expectation values of position coordinates $\langle X_1(t) \rangle$ and $\langle X_2(t) \rangle$ of the pendulum. The general expression of $\langle X_1(t) \rangle$ thus obtained is given by

$$\begin{aligned} \langle X_1(t) \rangle &= [\text{Re}(A) + \text{Re}(D_3^*)]X_1(0) + [\text{Im}(D_2) \\ &\quad + \text{Re}(D_1^*)]X_2(0) + [-\text{Im}(A) + \text{Im}(D_3^*)] \frac{P_1(0)}{m\varpi} \\ &\quad + [\text{Re}(D_2) + \text{Im}(D_1^*)] \frac{P_2(0)}{m\varpi}, \end{aligned} \quad (33)$$

where Re and Im denote the real and imaginary parts of the A , D_1 , D_2 , and D_3 . Substituting their values from Eqs. (28)–(31) will give the expression for $\langle X_1(t) \rangle$ for a general GW amplitude $f(t)$. Since it will be difficult to see through this very complicated expression, we assume a monochromatic GW waveform oscillating with frequency ϖ' , $f(t) = f_0 e^{i\varpi' t}$ which, upon substitution in Eq. (33), gives

$$\begin{aligned}
\langle X_1(t) \rangle = & \frac{(\cos \varpi_- t + \cos \varpi_+ t)}{2} X_1(0) + \frac{(\sin \varpi_- t + \sin \varpi_+ t)}{2m\varpi} P_1(0) + \frac{(\cos \varpi_+ t - \cos \varpi_- t)}{2} X_2(0) \\
& + \frac{(\sin \varpi_+ t - \sin \varpi_- t)}{2m\varpi} P_2(0) + \left(1 + \frac{\varpi^2 \Delta \varpi t}{\Delta \varpi^2 - \Lambda^2}\right) f_0 \{ \varepsilon_3 X_1(0) + \varepsilon_1 X_2(0) \} - \left(\frac{\varpi^2 \Lambda t}{\Delta \varpi^2 - \Lambda^2}\right) f_0 \{ \varepsilon_1 X_1(0) + \varepsilon_3 X_2(0) \} \\
& - (\varepsilon_1 + \varepsilon_3) f_0 \left[\left(\frac{\varpi'_+}{\Delta \varpi_-}\right)^2 \sin \frac{\Delta \varpi_- t}{2} \left(X_1(0) \sin \frac{\Delta \varpi_- t}{2} - \frac{P_1(0)}{m\varpi} \cos \frac{\Delta \varpi_- t}{2} \right) + \left(\frac{\varpi'_-}{\Delta \varpi_+}\right)^2 \sin \frac{\Delta \varpi_+ t}{2} \left(X_2(0) \sin \frac{\Delta \varpi_+ t}{2} \right. \right. \\
& \left. \left. - \frac{P_2(0)}{m\varpi} \cos \frac{\Delta \varpi_+ t}{2} \right) \right] + (\varepsilon_1 - \varepsilon_3) f_0 \left[\left(\frac{\varpi'_-}{\Delta \varpi_+}\right)^2 \sin \frac{\Delta \varpi_+ t}{2} \left(X_1(0) \sin \frac{\Delta \varpi_+ t}{2} - \frac{P_1(0)}{m\varpi} \cos \frac{\Delta \varpi_+ t}{2} \right) \right. \\
& \left. \left. - \left(\frac{\varpi'_+}{\Delta \varpi_-}\right)^2 \sin \frac{\Delta \varpi_- t}{2} \left(X_2(0) \sin \frac{\Delta \varpi_- t}{2} - \frac{P_2(0)}{m\varpi} \cos \frac{\Delta \varpi_- t}{2} \right) \right], \tag{34}
\end{aligned}$$

where $\Delta \varpi = \varpi - \varpi'$, $\varpi_{\pm} = \varpi \pm \Lambda$, $\varpi'_{\pm} = \varpi' \pm \Lambda$, and $\Delta \varpi_{\pm} = \Delta \varpi \pm \Lambda$. This result implies that the presence of noncommutativity alters the response of the harmonic oscillator to a periodic GW from its commutative counterpart. When the frequency of the GW is very close to that of the harmonic oscillator ($\Delta \varpi \approx 0$), the oscillatory terms present in the solution representing the response of the system to the GW will oscillate with frequency Λ with a large amplitude. It should be possible to detect this effect. Now putting $\theta = 0$, i.e., $\Lambda = 0$, gives us the classical result of the GW interacting with a harmonic oscillator in the low-velocity, long-wavelength limit, whereas putting $f_0 = 0$, i.e., in the absence of the gravitational wave, the solution assumes the form of a NC harmonic oscillator. Interestingly, the presence of the $\frac{1}{\hbar}$ factor in Λ , i.e., in the NC correction terms, even after the computation of the expectation value indicates that the NC effect is inherently

quantum mechanical in nature. A similar expression for $\langle X_2(t) \rangle$ can also be obtained. Further realistic scenarios can be obtained if we use various forms of periodic GW signals with more than one frequency and do similar computations. Now that we have studied the interaction of a single NC harmonic oscillator with a GW, a further advancement will be to extend it for a macroscopic piece of elastic matter. In fact in a resonant-mass detector it is possible to detect vibrations which are incredibly small, with an amplitude many orders of magnitude smaller than the size of a nucleus. In that context studying the effect of non-commutativity may prove interesting. Work in this direction is in progress and will be taken up in subsequent papers.

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