

Evolution of a black hole-inhabited brane close to reconnection

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Last moments of a mini black hole escaping from a brane are studied. It is argued that at the point of reconnection, where the piece of the brane attached to the black hole separates from the rest, the world sheet of the brane becomes isotropic (lightlike). The degenerate mode of evolution, with the world sheet isotropic everywhere, is investigated. In particular, it is shown that the brane approaches the reconnection point from below if it reconnects within a certain limit distance, and from above if it reconnects beyond that distance. The rate of relaxation to the degenerate mode is established. If the dimension of the brane is p , the nondegeneracy, measured by the determinant of the relevant part of the induced metric tensor, falls down as (latitudinal angle) $^{2(p-1)}$.

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I. INTRODUCTION

One of the predictions of the theories with large extra dimensions [1] is that there can exist mini black holes with masses of order 1 TeV that can, in principle, be observed at LHC [2]. In [3] it was pointed out that if a mini black hole has been produced in a collision of two high-energy particles, it can escape from the brane (that is, from our Universe) after emitting a hard quantum of Hawking radiation into the bulk. The description of the brane-black hole system in [3] was quantum mechanical. A classical description, which appeared first in [4] in the context of the problem of a static domain wall interacting with a Schwarzschild black hole in 3 dimensions, was applied to the escape of the black hole from the brane in [5,6]. In [5] it was established, by solving numerically the equation of motion of the brane in the field of the black hole, that the brane develops a neck that eventually shrinks to a point. Then the piece of the brane attached to the black hole cuts loose and the rest of the brane reconnects. In [6] it was argued that the scenario applies to the brane with codimension 1 no matter what the velocity of the black hole, but for greater codimensions the velocity must exceed some critical value. Here we continue this study. In Sec. II we write down the equation of motion of the brane, in Sec. III we inspect the behavior of a special class of solutions, which we call degenerate, close to reconnection, in Sec. IV we investigate the relation between degenerate and true solutions and in Sec. V we discuss the results.

II. EQUATION OF MOTION

Consider a static n -dimensional black hole with a planar p -dimensional brane attached to a given great sphere of the

horizon, and suppose that the black hole is knocked out of the brane perpendicularly to it. The metric of an isolated black hole is [7]

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

$$f = 1 - r^{2-n},$$

where $d\Omega^2$ is the metric of a unit $(n-1)$ -dimensional sphere and we use a system of units in which $c = 1$ and r_s (the Schwarzschild radius) $= 1$. The metric (1) is applicable, strictly speaking, only if the space the black hole is living in is infinite and asymptotically flat, but can be used also if some dimensions are compactified or warped, provided the size of the black hole is much less than the scale on which that happens. To describe the motion of the brane we pass to the rest frame of the black hole and suppose that the brane has no gravitational field of its own. Then the bulk space has metric (1). Furthermore, we assume that the brane is infinitely thin. The dynamics of the brane is then governed by the Dirac-Nambu-Goto action $S = -T \times$ the volume of the world sheet, where T is the tension of the brane and the volume is computed from the metric of the bulk space induced on the world sheet of the brane. Finally we use the symmetry of the problem. Let θ be the latitudinal angle measured from the axis that points opposite to the direction in which the black hole is knocked out. The brane is obviously symmetric with respect to that axis, therefore its world sheet can be described, at least locally, by the function $\theta(r, t)$.

After computing the induced metric, inserting it into the definition of the volume and integrating over the angles, we find [5]

$$S = -T_o \int \sqrt{F} (r \sin \theta)^{p-1} dr dt, \quad (2)$$

$$F = 1 + r^2 f \dot{\theta}^2 - r^2 f^{-1} \theta'^2,$$

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where o is the volume of a unit $(p - 1)$ -dimensional sphere and the prime and the overdot denote differentiation with respect to r and t , respectively. Denote

$$q = 2(p - 1), \quad h = r^{-2}f, \\ \mathcal{F} = hF = h + f^2\theta'^2 - \dot{\theta}^2.$$

By varying S , we obtain the equation of motion of the brane:

$$\Phi \equiv \tan\theta(\dot{\theta}\dot{\mathcal{F}} - f^2\theta'\mathcal{F}' - 2\Theta\mathcal{F}) - qh\mathcal{F} = 0, \quad (3)$$

where

$$\Theta = \ddot{\theta} - f^2\theta'' - f^2\mathcal{R}\theta', \quad \mathcal{R} = \frac{p}{r} + \frac{3f'}{2f}. \quad (4)$$

The equation is linear in second derivatives of θ , with the coefficients proportional to the components of the contravariant metric tensor on the world sheet. If the brane is to describe our Universe, its world sheet has to be *timelike* (lay within the light cones) everywhere, with a possible exception of the point of reconnection. For such world sheet, the signature of the relevant part of the world sheet metric tensor is $(-+)$, hence the equation is hyperbolic.

The character of the world sheet is given by the sign of the determinant of the world sheet metric tensor: the world sheet is timelike if the determinant is negative, isotropic if it is zero and spacelike if it is positive. The determinant is proportional to $-F = -\mathcal{F}/h$, therefore the three cases mentioned above correspond to $\mathcal{F} > 0$, $\mathcal{F} = 0$ and $\mathcal{F} < 0$, respectively. Reconnection of the brane takes place on the upper part of the axis of symmetry of the brane (the half line $\theta = 0$), where the full determinant is zero since it contains an extra factor $(r \sin\theta)^q$. However, this is just a coordinate effect and the character of the world sheet is again given by the sign of $\mathcal{F}$. To see that, note that the character of the world sheet is complementary to that of the normal vector $n = \nabla[\theta(r, t) - \theta]$ (the world sheet is timelike if n is spacelike, isotropic if n is isotropic and spacelike if n is timelike), and the square of the normal vector equals $r^{-2}F$.

The behavior of the brane at reconnection depends on whether it is a string ($p = 1$) or a higher dimensional brane ($p > 1$). For a string, the second term in Eq. (3) is absent, and the expression in the brackets in the first term is zero everywhere including the reconnection point. In this case the world sheet stays timelike at reconnection. For higher dimensional branes, if the expression in the brackets does not diverge at reconnection, $\mathcal{F}$ must vanish there. As a result, the world sheet becomes *isotropic* (tangential to the light cone). We will show that this indeed happens if the function θ satisfies some natural requirements when approaching zero.

Rewrite Φ so that all derivatives $\ddot{\theta}$ are absorbed into $\dot{\mathcal{F}}$ and all remaining derivatives $\dot{\theta}'$ are absorbed into $\mathcal{F}'$. In this way we obtain

$$\Phi \propto \tan\theta(a\dot{\theta}\dot{\mathcal{F}} + f^2b\theta'\mathcal{F}' + 2f^2\Xi\mathcal{F}) - qh\dot{\theta}^2\mathcal{F}, \quad (5)$$

where

$$a = h + f^2\theta'^2, \\ b = \mathcal{F} - \dot{\theta}^2, \\ \Xi = \dot{\theta}^2(\theta'' + \mathcal{R}\theta') - f^2\theta'^2(\theta'' + \mathcal{R}_0\theta') - \frac{1}{2}h'\theta', \quad (6) \\ \mathcal{R}_0 = \frac{f'}{f}.$$

Denote by t_0 the time of reconnection and by r_0 the radial coordinate of the point on the axis at which the reconnection takes place. We are interested in the behavior of $\mathcal{F}$ near the point (r_0, t_0) . Consider a function θ that is smooth in the vicinity of the reconnection point, except possibly at this point itself, and suppose the first two derivatives of θ with respect to r are well-behaved at reconnection, $\theta' \rightarrow 0$ and $\theta'' \rightarrow$ positive number at (r_0, t_0) . Suppose furthermore that θ falls to zero linearly with t , $\dot{\theta} \rightarrow$ negative number at (r_0, t_0) . (Note that $\dot{\theta}$ is necessarily finite at reconnection since the function $\mathcal{F}$, which is by assumption positive outside the point (r_0, t_0) , contains the term $-\dot{\theta}^2$. We have added only the requirements that $\dot{\theta}$ has a limit at (r_0, t_0) and that that limit is nonzero. Note also that the assumption about nonzero $\dot{\theta}$ close to reconnection has been already used in (5), where we have suppressed the general factor $\dot{\theta}^{-2}$ on the right-hand side.) The assumptions seem plausible, if for nothing else because of the shape of the curves in [5]. A principal consequence is that $\mathcal{F}$ has a nonnegative limit at (r_0, t_0) , so that all we have to show is that the limit is zero. Our starting point will be equation $\Phi = 0$ with Φ given in (5), regarded as a first order differential equation for $\mathcal{F}$. The equation is quasilinear and can be solved by the method of characteristics [8]. For the three variables t , r and $\mathcal{F}$ as functions of λ we have three ordinary differential equations of first order:

$$d_\lambda t = a\dot{\theta}, \quad d_\lambda r = f^2b\theta', \\ d_\lambda \mathcal{F} = -(2\Xi - qh\cot\theta\dot{\theta}^2)\mathcal{F}, \quad (7)$$

where d_λ denotes derivative with respect to λ . The curves in the 3-dimensional space $(r, t, \mathcal{F})$ defined by these equations are called *characteristics* of equation $\Phi = 0$. Close to the point (r_0, t_0) we have

$$d_\lambda t \sim h_0\theta_{10}, \quad d_\lambda r \sim 0, \quad d_\lambda \mathcal{F} \sim qh_0\theta_{10}^2\theta^{-1}\mathcal{F},$$

where h_0 and θ_{10} are the values of h and $\dot{\theta}$ at reconnection. Denote $\delta t = t - t_0$. For the characteristic that reaches the reconnection point we have $\theta \sim \theta_{10}\delta t$, and after inserting this into the third equation and dividing the third equation by the first, we obtain $d_t \mathcal{F} \sim q\mathcal{F}/\delta t$ and $\mathcal{F} \sim \text{const}(-\delta t)^q$. The proof is completed.

To show that the world sheet becomes isotropic at reconnection we needed some *ad hoc* assumptions about

its form. Also, we did not address the question whether the world sheet stays timelike *outside* the reconnection point. The assumptions can be relaxed and the missing proof supplied with the help of a conservation law known from the string theory. Let us start with an observation that a higher dimensional brane can be replaced by a string living in the $2 + 1$ dimensional space (r, θ, t) , with an effective metric $(r \sin \theta)^{q/2} \times$ the true metric of the space (r, θ, t) (equal to $-f dt^2 + f^{-1} dr^2 + r^2 d\theta^2$). In a static gravitational field, it is convenient to introduce static gauge with the coordinate σ numerating the points of the string chosen in such a way that the string moves perpendicularly to itself. The energy stored in infinitesimal segments of the string is then conserved not only in a sum, but also separately. The energy is proportional to $\sqrt{-g_{00}} dl / \sqrt{1 - v^2}$, where dl is the length of the segment and v is its velocity. (For the derivation of the formula in flat space, see [9].) The world sheet is timelike if $v < 1$, so that the strip of the world sheet corresponding to the given segment of the string remains timelike for all times if it was timelike at the beginning. This holds unless the string develops *cusps*, where $v \rightarrow 1$ simultaneously with $dl \rightarrow 0$ (the segment at the tip of the cusp is contracted to a point, and moves perpendicularly to the cusp with the velocity of light). A string in the effective metric we are interested in behaves similarly as a cusp at reconnection: both $\sqrt{-g_{00, \text{eff}}}$ and dl_{eff} approach zero there because of the collapse of the metric, therefore v_{eff} must approach one. However, the velocity remains unchanged as we return from the effective metric to the true one. As a result, the velocity v of the brane must approach 1 as the brane moves towards the reconnection point.

In [5], the initial velocity of the brane is chosen as $v = (1 - 1/r)v_\infty$, where v_∞ is the velocity of the brane at infinity (the velocity with which the black hole is knocked out of the brane with the sign minus), and the calculation is performed for three different values of v_∞ from the interval $0 < v_\infty \leq 1$. (The value $v_\infty = 1$ has been obviously included into the list in order to cover the case $v_\infty < 1$, $1 - v_\infty \ll 1$.) For such choice of v , $\mathcal{F} = h(1 - v^2)$ is positive, and the world sheet is timelike, at the moment when the black hole receives an initial push that puts it into motion. Consequently, if the brane does not develop cusps, $\mathcal{F}$ stays positive and the world sheet stays timelike up to the point of reconnection, where $\mathcal{F}$ vanishes and the world sheet becomes isotropic.

III. PROPERTIES OF DEGENERATE SOLUTIONS

We are interested in the behavior of a brane that is just about to reconnect. If the brane becomes isotropic at the point of reconnection, it should be approximately isotropic close to that point. Thus, it is natural to start with the brane that is *exactly* isotropic. Such brane is described by the equation

$$\mathcal{F} \equiv h + f^2 \theta'^2 - \dot{\theta}^2 = 0. \quad (8)$$

The equation can be viewed as an initial condition imposed on the solutions of equation $\Phi = 0$. Indeed, since Φ is homogeneous in $\mathcal{F}$, the function θ that satisfies $\mathcal{F} = 0$ at a certain moment, and is evolving according to $\Phi = 0$, will satisfy $\mathcal{F} = 0$ ever after. (The brane that was once isotropic remains always isotropic.) On the other hand, one can view (8) as a dynamic equation. Since it is of the first order only, we need half as much initial data for it than for the equation $\Phi = 0$; we get along with the function θ and do not need to know the function $\dot{\theta}$ at the beginning. With regard to this, solutions to $\Phi = 0$ that satisfy also $\mathcal{F} = 0$ can be called *degenerate*.

To solve Eq. (8) close to reconnection, expand θ in the powers of $\delta t = t - t_0$ and $\delta r = r - r_0$,

$$\theta = \theta_{10} \delta t + \frac{1}{2} \theta_{20} \delta t^2 + \theta_{11} \delta t \delta r + \frac{1}{2} \theta_{02} \delta r^2 + \dots \quad (9)$$

The coefficients θ_{00} and θ_{01} are missing because the brane reconnects on the axis, and touches the axis while reconnecting rather than intersects it. At the moment of reconnection, θ reduces to

$$\theta_0 = \frac{1}{2} \theta_{02} \delta r^2 + \frac{1}{6} \theta_{03} \delta r^3 + \dots,$$

where θ_{02} is supposed to be nonzero and positive. The coefficients θ_{0n} entering this expression must be fixed in advance. By choosing them, we provide Eq. (8) with an initial condition. (Except that we must evolve the solution backwards in time, so that the condition is final rather than initial.) All coefficients θ_{mn} with $m > 0$ can be expressed in terms of the coefficients θ_{0n} by solving Eq. (8) order by order. The first few coefficients are

$$\begin{aligned} \theta_{10} &= -\sqrt{h_0}, & \theta_{20} &= 0, \\ \theta_{11} &= \frac{h_1}{2\theta_{10}}, & \theta_{12} &= \frac{\frac{1}{2}h_2 + f_0^2 \theta_{02}^2 - \theta_{11}^2}{\theta_{10}}, \end{aligned} \quad (10)$$

where we have denoted the value of f at $r = r_0$ by f_0 and the value of the n th derivative of h at $r = r_0$ by h_n . The square root in the expression for θ_{10} is taken with the sign minus in order to ensure that θ decreases before reconnection. Of the three coefficients of third order, θ_{30} , θ_{21} and θ_{12} , we list only the last since the other two will not be needed in what follows.

Expansion of θ can be used to determine, for example, in what direction the brane approaches the reconnection point or by what rate it is either broadening or narrowing in the process, given the location of the reconnection point and the final shape of the brane. In this way we learn how an isotropic brane can possibly move just before reconnection. The resulting picture is applicable, as we will see, to a timelike brane as well.

Define the *neck* of the brane as the $(p - 1)$ -dimensional sphere at which the brane touches the cone $\theta = \text{const}$. The location of the neck is shown in Fig. 1 on the left. Define, furthermore, the angle ω at which the brane approaches the

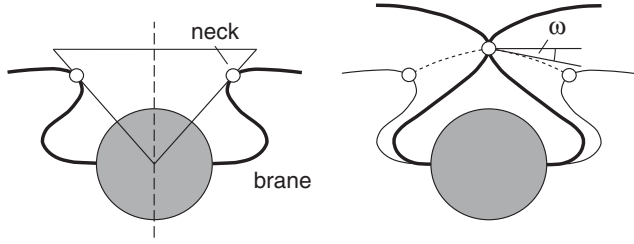


FIG. 1. Motion of the brane close to reconnection (schematically).

reconnection point as the angle between the direction in which the neck arrives at the axis and the plane perpendicular to the axis. This is depicted in Fig. 1 on the right. If we are interested in the behavior of the brane as visualized in pictures like this, it is more instructive to compute the *apparent* angle (the angle referring to the Euclidean geometry on the plane) than the true one. Thus, we define

$$\tan \omega = - \left. \frac{\delta r_N}{r_N \theta_N} \right|_{\delta t=0},$$

where r_N and θ_N is the radial and latitudinal coordinate of the neck. (The true value differs from this by the factor $f_0^{-1/2}$.) The location of the neck is given by the condition

$$\theta'|_{r=r_N} = 0.$$

From this we obtain r_N as a function of t , and by inserting it into $\theta(r, t)$ we find θ_N as a function of t . To compute $\tan \omega$, we need to know only the leading term in the expansions of δr_N and θ_N in δt . By the procedure described above we obtain

$$\delta r_N = - \frac{\theta_{11}}{\theta_{02}} \delta t, \quad \theta_N = \theta_{10} \delta t, \quad (11)$$

and after inserting this into the definition of $\tan \omega$ we find

$$\tan \omega = \frac{\theta_{11}}{r_0 \theta_{10} \theta_{02}}. \quad (12)$$

With the help of (10), this can be rewritten as $\tan \omega = \text{positive number} \times h_1 = \text{positive number} \times (-2 + n r_0^{-n+2})$. As a result, the sign of ω depends on the radius r_0 ,

$$\omega \begin{cases} > 0 & \text{for } r_0 < r_{0\text{crit}}, \\ < 0 & \text{for } r_0 > r_{0\text{crit}} \end{cases}, \quad (13)$$

with the critical radius defined as

$$r_{0\text{crit}} = \left(\frac{n}{2} \right)^{1/(n-2)}. \quad (14)$$

The brane arrives at the axis from below if the reconnection takes place inside the sphere with the radius $r_{0\text{crit}}$, and from above if the reconnection takes place outside that sphere.

The shape of the brane close to reconnection can be characterized by its apparent curvature at the neck:

$$k = r_N \theta_N''.$$

(The true value differs from this by the factor f_N .) Let us determine how this quantity varies with time just before reconnection. After inserting $\delta r = \delta r_N + \epsilon$ into the expansion of θ and collecting the terms of zeroth and first order in δt that are proportional to ϵ^2 , we find

$$\theta = \theta_N + \frac{1}{2}(\theta_{02} + \theta_{12} \delta t + \theta_{03} \delta r_N) \epsilon^2,$$

so that

$$\begin{aligned} k &= (r_0 + \delta r_N)(\theta_{02} + \theta_{12} \delta t + \theta_{03} \delta r_N) \\ &= k_0 + r_0 \theta_{12} \delta t + (\theta_{02} + r_0 \theta_{03}) \delta r_N, \end{aligned}$$

where $k_0 = r_0 \theta_{02}$ is the apparent curvature of the brane at the point of reconnection. If we insert here δr_N from the first Eq. (11), we obtain

$$k = k_0 + \left[r_0 \theta_{12} - \left(1 + r_0 \frac{\theta_{03}}{\theta_{02}} \right) \theta_{11} \right] \delta t.$$

Thus, the time derivative of the curvature at the point of reconnection is

$$\dot{k}_0 = r_0 \theta_{12} - \left(1 + r_0 \frac{\theta_{03}}{\theta_{02}} \right) \theta_{11}. \quad (15)$$

In Fig. 2 we plot the angle ω and the time derivative of the curvature $\dot{k}_0$ as functions of the radius r_0 . Both parameters depend on the curvature k_0 , and $\dot{k}_0$ depends also on the parameter of asymmetry $l_0 = r_0 \theta_{03}$. In the right panel, solid lines correspond to $l_0 = 0$ and dashed lines correspond to $l_0 = -5$ for $k_0 = 2$ and $l_0 = -25$ for $k_0 = 5$. The brane is supposed to live in a 4-dimensional space. (Its own dimension does not affect its motion in case it is isotropic.) For such brane, the critical radius is $r_{0\text{crit}} = \sqrt{2}$. Both ω and $\dot{k}_0$ decrease monotonically with r_0 , but pass through zero, in general, at different points: ω at $r_{0\text{crit}}$ and $\dot{k}_0$ at a point depending on k_0 and l_0 . In a space with one extra dimension, the latter point is located at $r_{0\text{crit}}$ for $k_0 = 2$. Thus, the brane that arrives at the axis with the curvature 2 (in ordinary units $2/r_s$) is narrowing just before touching the axis if the reconnection takes place

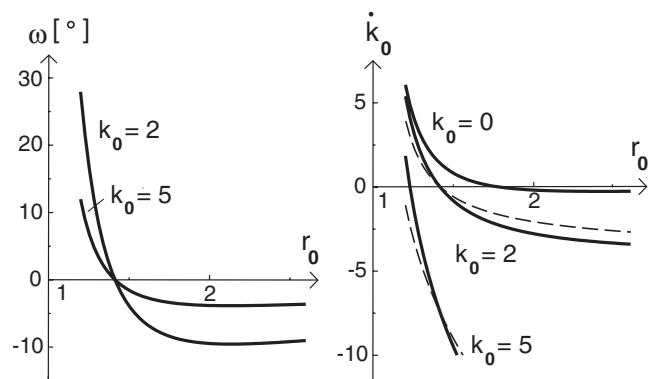


FIG. 2. Parameters of the brane at reconnection.

inside the sphere with the radius r_{crit} , and broadening if the reconnection takes place outside that sphere.

To see how an isotropic brane actually approaches the point of reconnection, let us solve equation $\mathcal{F} = 0$ numerically. We can use once again the method of characteristics, this time for a *nonlinear* differential equation [8]. To simplify formulas, replace the Schwarzschild coordinate r by the “tortoise coordinate”

$$R = \int f^{-1} dr.$$

Then

$$\mathcal{F} = h + \hat{\theta}^2 - \dot{\theta}^2, \quad (16)$$

where the hat denotes differentiation with respect to R . We want to solve equation $\mathcal{F} = 0$ with the initial condition $\theta = \theta_i$ at some moment t_i . Introduce an auxiliary Hamiltonian obtained by replacing the derivatives $\hat{\theta}$ and $\dot{\theta}$ in $\mathcal{F}$ by the momenta π_R and π_t , and adding an extra factor $1/2$ for convenience:

$$H = \frac{1}{2}(h + \pi_R^2 - \pi_t^2).$$

The Hamiltonian lives in the phase space $x^A = (R, t)$, $\pi_A = (\pi_R, \pi_t)$. The solution to equation $\mathcal{F} = 0$ is given by the Hamilton equations for R and t and an additional equation for θ as functions of the “time” λ :

$$\begin{aligned} d_\lambda R &= \pi_R, & d_\lambda \pi_R &= -\frac{1}{2}\hat{h}, & d_\lambda t &= -\pi_t, \\ d_\lambda \pi_t &= 0, & d_\lambda \theta &= \pi \cdot d_\lambda x = \pi_R^2 - \pi_t^2, \end{aligned} \quad (17)$$

with the initial conditions

$$\begin{aligned} R &= R_*, & \pi_R &= \hat{\theta}_*, & t &= t_i, \\ \pi_t &= \dot{\theta}_* = -\sqrt{h_* + \hat{\theta}_*^2}, & \theta &= \theta_*. \end{aligned} \quad (18)$$

The star refers to the value of the function at R_* , or if it depends on both R and t , at (R_*, t_i) ; hence $\theta_* = \theta_i(R_*)$ and $\hat{\theta}_* = \hat{\theta}_i(R_*)$. The curves given by Eqs. (17) are characteristics of equation $\mathcal{F} = 0$. From the initial conditions it follows that $H = 0$ at the starting point of the characteristic, and since H does not depend explicitly on λ , it is conserved and we have $H = 0$ along the whole characteristic. This allows us to simplify the last equation in (17) to $d_\lambda \theta = -h$. In addition to that, since π_t is constant, we could use equation $H = 0$ to compute π_R . However, the resulting expression is not very helpful in numerical calculations because of the ambiguity of its sign, therefore it is preferable to compute π_R from the differential equation. By combining equations for R , θ and π_R with that for t , and returning from R to r , we arrive at the equations

$$\frac{dr}{dt} = -\frac{f\pi_R}{\pi_t}, \quad \frac{d\pi_R}{dt} = \frac{fh'}{2\pi_t}, \quad \frac{d\theta}{dt} = \frac{h}{\pi_t}. \quad (19)$$

The value of π_t and the initial value of π_R are given in (18), with $\hat{\theta}_*$ replaced by $f_*\theta'_*$.

The curves in the $2 + 1$ dimensional space (r, θ, t) we have constructed are in fact null geodesics. It is seen most easily if we multiply the h term in the Hamiltonian by π_θ^2 , since then the Hamiltonian transforms into $H = \frac{1}{2}g_{\text{eff}}^{\mu\nu}\pi_\mu\pi_\nu$ with the effective metric $g_{\text{eff}}^{\mu\nu} = fg^{\mu\nu}$. (If we are interested in *null* geodesics, metric can be rescaled by an arbitrary function.) The previous theory is completely reproduced if we add one more initial condition $\pi_\theta = -1$ to the conditions (18). We can also notice that equation $\mathcal{F} = 0$ is *eikonal equation* in the space (t, r, θ) , with the eikonal defined as $\theta(r, t) - \theta$. Thus, the world sheet of the brane can be viewed as *wave* (hypersurface of constant phase), and the curves we have constructed as *rays* (curves that are at the same time normal and tangential to the wave). The fact that rays are null geodesics is a well-known result of geometrical optics in curved spacetime [10].

In Fig. 3 we depict two branes with a parabolic shape at the time 0.6 before reconnection, ending up at the point $r_0 = 1.25$ (upper panel) and $r_0 = 1.75$ (lower panel) with the curvature $k_0 = 5$ and 3, respectively. Both branes are asymmetric at reconnection (have nonzero l_0). Heavy lines are graphs of $\theta(r, t)$ at different times, dotted lines are characteristics used in the computation and dashed lines are paths of the neck of the brane. The first brane approaches the axis from below while the second brane approaches it from above, at the angle ω equal to 7.1° and -5.6° , respectively. The further from the horizon the more divergent the characteristics, therefore the second brane has a smaller range of allowed curvatures at reconnection than the first brane. For example, if we raised the curvature of the second brane to $k_0 = 5$ and evolved it backwards in time, the procedure would collapse at the

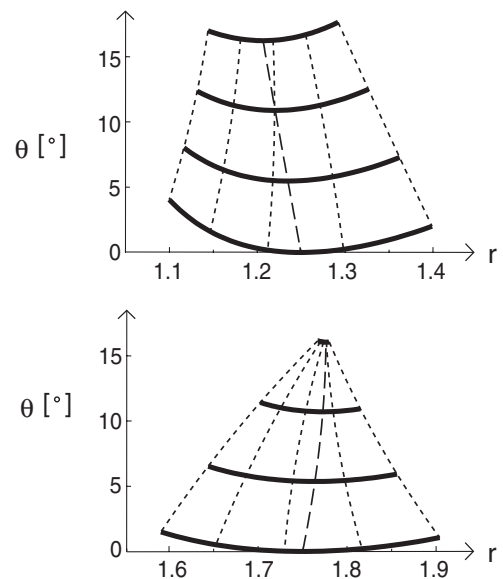


FIG. 3. Degenerate evolution of the brane.

time about 0.3 before reconnection. The characteristics would start to intersect and the curves would develop cusps. It must be stressed, however, that no pathology arises if the brane moves *forward* in time. If an inward bent, shrinking isotropic brane appears near the black hole, it stays smooth until its neck shrinks to a point.

IV. RELAXATION OF THE TRUE SOLUTION TO THE DEGENERATE ONE

Let us turn to the brane whose world sheet is timelike before reconnection. We have seen that such brane becomes isotropic when reconnecting; thus, $\mathcal{F}$ relaxes to zero as θ approaches zero. Now we address the question how *fast* it relaxes.

Let us write $\mathcal{F}$ for $\theta \sim 0$ as $\mathcal{F} \sim a\theta^Q$, where α is a function of r and Q is a positive constant. The possibility to express $\mathcal{F}$ in this way is not self-evident; it is rather an *assumption* which can be accepted only if we prove that it is consistent with Eq. (3). For $\mathcal{F}$ of the given form, the left-hand side of the equation becomes

$$\Phi \sim \tan\theta[\alpha Q\theta^{Q-1}\dot{\theta}^2 - f^2(\alpha'\theta^Q + \alpha Q\theta^{Q-1}\theta') - 2\Theta\alpha\theta^Q] - qh\alpha\theta^Q.$$

If we keep here only the leading term, the expression simplifies to

$$\Phi \sim \alpha\theta^Q[Q(\dot{\theta}^2 - f^2\theta'^2) - qh].$$

This is zero in the leading order if $Q = q$, since then the expression in the square brackets equals $-q\mathcal{F}$ and is of order θ^q . This suggests that $\mathcal{F}$ goes to zero with θ as

$$\mathcal{F} \sim \alpha\theta^q. \quad (20)$$

Denote the value of α at the reconnection point by α_0 . If α_0 is nonzero, we can write the relaxation law for $\mathcal{F}$ as $\mathcal{F} \sim \alpha_0\theta^q$, or $\mathcal{F} \propto \theta^q$. In Sec. II we obtained the asymptotics $\mathcal{F} \propto (-\delta t)^q$, valid along a curve approaching the reconnection point perpendicularly to the axis (so that $\theta \propto -\delta t$). The present formula generalizes this asymptotics to an arbitrary direction.

To complete the discussion, let us justify the assumption that α_0 is nonzero. This is valid generically, and *must* be valid if the brane develops no cusps during its evolution and meets the requirements used in the derivation of the asymptotics $\mathcal{F} \propto (-\delta t)^q$. To prove that, consider again the ordinary differential equation for $\mathcal{F}$ in Sec. II (the last equation in (7)). It is a homogeneous linear equation of first order, hence its general solution can be written as $\mathcal{F} = C\mathcal{F}_{\text{ref}}$, where $\mathcal{F}_{\text{ref}}$ is a reference solution and C is an arbitrary constant. Since $\mathcal{F} \propto (-\delta t)^q$ for $\delta t \sim 0$, the reference solution can be written as $\mathcal{F}_{\text{ref}} = (-\delta t)^q + \text{higher order terms in } \delta t$. If C is zero, $\mathcal{F}$ is zero all the way down the characteristic that terminates at the reconnection point. However, in the nondegenerate mode of evolution with no cusps $\mathcal{F}$ is positive everywhere outside the reconnection

point; and $\mathcal{F}_{\text{ref}}$ is positive at least in *some* interval $\delta t < 0$. Thus, C must be positive, and the value of α at the reconnection point, equal to $C(-\theta_{10})^{-q}$, must be positive, too.

The asymptotics of $\mathcal{F}$ that we have found allows us to determine how $\mathcal{F}$ depends on the variables r and t , provided we know the dependence of θ on them. The simplest possibility is that θ is *regular* in the sense that it can be Taylor-expanded in δr and δt , as we have already assumed in the degenerate case. Then $\mathcal{F}$ is regular, too, and its Taylor expansion begins by the terms proportional to δt^q , $\delta t^{q-1}\delta r^2, \dots, \delta r^{2q}$. However, equation for θ is singular at reconnection, because the coefficients at the highest order derivatives are zero there. Thus, to decide whether θ is regular or not we need to analyze the effect of this singularity on the character of the solution. In what follows we aim to show that θ *can* be regular; that is, that the assumption of regularity of θ does not lead to contradiction. For that purpose we will combine the expansion of θ in δr and δt with the expansion of $\mathcal{F}$ in θ . This turns out to be more effective than to expand all quantities in δr and δt from the very start.

Let us write $\mathcal{F}$ as a power series in θ :

$$\mathcal{F} = \alpha\theta^q + \beta\theta^{q+1} + \dots, \quad (21)$$

where α is as before a function of r and the other coefficients are functions of both r and t . Since θ is itself a function of r and t , the expansion is not unique. However, for our purposes it is sufficient to have a *certain* expansion of this form, as long as it is well-defined and consistent with Eq. (3). To obtain such expansion, we start by replacing h in the last term in Φ by $\mathcal{H} + \mathcal{F}$, where $\mathcal{H} = \dot{\theta}^2 - f^2\theta'^2$. Then we fix θ , so that Eq. (3) becomes a first order differential equation for $\mathcal{F}$. Finally we insert the expansion of $\mathcal{F}$ into (3) and collect the terms containing θ in the powers $q, q+1, q+2, \dots$ *explicitly*. In this way we obtain a system of equations for the expansion coefficients. The first equation is satisfied identically, so that we have no restriction on α . (This could be seen in advance from the considerations at the beginning of this section.) The remaining equations can be solved order by order to obtain expressions for $\beta, \gamma, \dots$ in terms of α . In particular, the coefficient appearing in the next-to-leading order is

$$\beta = \frac{1}{\mathcal{H}}(f^2\theta'\alpha' + 2\Theta\alpha + \alpha^2\delta_{q1}). \quad (22)$$

In this expression, the quadratic term is present only for the nonphysical value $q = 1$ (“one-and-a-half dimensional brane”). For an arbitrary q , the first $q-1$ coefficients starting from β are linear in α , and a higher power of α appears first in the q th coefficient, in front of θ^{2q} .

The previous procedure enables us to express $\mathcal{F}$ for a given θ solely in terms of an arbitrary function of one variable α . This agrees with the fact that for a first order differential equation we need one initial condition. With our expression for $\mathcal{F}$, we can satisfy one constraint

imposed on $\mathcal{F}$ by a proper choice of α . On the other hand, the possibility to fix the initial condition in this way explains why we could have chosen α as a function of r only.

To find θ as a function of r and t , we must supplement Eq. (3) by the definition of $\mathcal{F}$. Thus, we must solve *two* equations,

$$\Phi = 0, \quad \Psi \equiv \dot{\theta}^2 - f^2 \theta'^2 - h + \mathcal{F} = 0, \quad (23)$$

regarded as equations for $\mathcal{F}$ and θ , respectively. The equations look as if they were of first order, but we must keep in mind that Φ contains derivatives of θ up to second order. If we transformed Φ into the form (5), we would get rid of the derivatives $\dot{\theta}$ and θ' , but the global picture would stay the same; therefore we stick to the simpler expression (3) for Φ . The presence of the derivatives of θ in Φ apparently ruins the idea of solving the equations by expanding $\mathcal{F}$ in the powers of θ . If we expand $\mathcal{F}$ and solve the first equation order by order, in the resulting expression there will appear derivatives of θ of all orders (second derivatives in β , third derivatives in γ etc.). After inserting this into the second equation we obtain an equation of infinite order, which is unacceptable, if for nothing else because we need infinitely many initial conditions to fix the solution. Nevertheless, the equation *can* be solved by expanding θ into the powers of δr and δt . The point is that the coefficients of higher order in δt appearing in the expressions for the coefficients of lower order can be calculated in advance, since they have necessarily a lower *total* order (sum of the orders in δr and δt). In other words, algebraic equations for the coefficients can be solved one by one if lined up by their total order. The details can be found in Appendix A.

As noted before, the arbitrary function α appears in the expansion of $\mathcal{F}$ because of the freedom of choice of the initial condition for $\mathcal{F}$. Write $\mathcal{F}$ as $\mathcal{F} = \xi \theta^q$. The initial (in fact, final) conditions for both θ and $\mathcal{F}$ can be imposed at the moment of reconnection, by choosing θ_0 and $\mathcal{F}_0$, or equivalently, θ_0 and ξ_0 . (Here we use the notation introduced after Eq. (9): for a given function u of the variables r and t , u_0 is a function of the variable r equal to u at the moment t_0 , $u_0(r) = u(r, t_0)$.) Once θ_0 is chosen, α must be given uniquely by ξ_0 . In fact, we can obtain expansion coefficients α_n of α by computing them order by order from expansion coefficients ξ_{0n} of ξ_0 . For more detail, see the second part of Appendix A.

Once we have assumed that θ can be expanded in δr and δt , we could have expanded both equations for $\mathcal{F}$ and θ in δr and δt immediately, instead of expanding first the former equation in θ . In such an approach, we would expect that there exists no equations for the expansion coefficients $\mathcal{F}_{qn}$. The reason is that $\mathcal{F}_{qn}$ are in one-to-one correspondence with ξ_{0n} , which in turn are in one-to-one correspondence with α_n ; and since α_n are free, $\mathcal{F}_{qn}$ must be free, too. However, after actually expanding equations for $\mathcal{F}$ and θ in δr and δt , we obtain an infinite string

of equations for $\mathcal{F}_{qn}$. The fact that $\mathcal{F}_{qn}$ are free leads to the conclusion that, after all θ 's appearing in the equations are expressed in terms of θ_{0n} and $\mathcal{F}_{qn}$, the equations must collapse to $0 = 0$ due to massive cancellations. The argument is elaborated in Appendix B.

To summarize the previous discussion, if θ expands in δr and δt , we can compute its coefficients in terms of expansion coefficients of θ_0 and ξ_0 ; however, as for now we cannot tell whether the resulting series converges or not. To obtain a tentative answer, consider Eq. (3) with the initial conditions $\theta = \theta_0$ and $\xi = \xi_0$ *outside* the reconnection point. If the initial conditions are formulated on two intervals I_- and I_+ on the axis r to the left and to the right of the point $r = r_0$, the equation can be solved inside two strips I_- and I_+ in the plane (r, t) , bounded from above by the intervals I_- and I_+ shifted to the line $t = t_0$, and from the sides by the diverging characteristics of Eq. (3). If, furthermore, the initial conditions are smooth, the solution must be smooth, too. For θ_0 and ξ_0 that can be both expanded into Taylor series around the point $r = r_0$ on an interval I containing I_- and I_+ , this implies that θ is smooth in the domains I_- and I_+ . On the other hand, θ can be written as the Taylor series we have formally introduced earlier, hence the series must be convergent in both domains. If so, it seems plausible that it is convergent in the strip between them, too.

The first nonzero coefficient $\mathcal{F}_{mn}$ is $\mathcal{F}_{q0}$, followed by $\mathcal{F}_{q-1,2}$, $\mathcal{F}_{q1}$, $\mathcal{F}_{q+1,0}$ etc.; thus, the first coefficient θ_{mn} affected by nondegeneracy is $\theta_{q+1,0}$, followed by θ_{q2} , $\theta_{q+1,1}$, $\theta_{q+2,0}$ etc. In particular, for the nonphysical value $q = 1$ equation $\theta_{20} = 0$ does not hold any longer, but is replaced by $\theta_{20} = -\mathcal{F}_{10}/(2\theta_{10})$. Also, in θ_{12} there appears a new term $-\mathcal{F}_{02}/(2\theta_{10})$ with $\mathcal{F}_{02} = \theta_{02}\mathcal{F}_{10}/\theta_{10}$. All θ_{mn} of lower order than those cited above are the same as for an isotropic brane. Because of that, Eqs. (12) and (15) for α and $\dot{k}_0$, derived for degenerate evolution, stay valid for nondegenerate evolution in almost all cases. The only exception is the second equation in case $q = 1$, which is modified because of the additional term in θ_{12} .

To demonstrate the effect of nondegeneracy on the evolution of the brane, let us solve the approximate equation

$$\Psi_{\text{app}} \equiv \dot{\theta}^2 - f^2 \theta'^2 - h + \alpha \theta^q = 0. \quad (24)$$

For that purpose, we must add a term proportional to θ^q to the Hamiltonian of Sec. III, and modify the equations for π to $d_\lambda \pi = -\partial_x H - \pi \partial_\theta H$ [8]. As a result, equations for π_R and θ , as well as the trivial equation for π_t , acquire new terms in comparison to (19)

$$\begin{aligned} \frac{d\pi_R}{dt} &= \dots - \frac{f\alpha'\theta^q + q\alpha\theta^{q-1}\pi_R}{2\pi_t}, \\ \frac{d\pi_t}{dt} &= -\frac{1}{2}q\alpha\theta^{q-1}, \\ \frac{d\theta}{dt} &= \dots - \frac{\alpha\theta^q}{\pi_t}. \end{aligned}$$

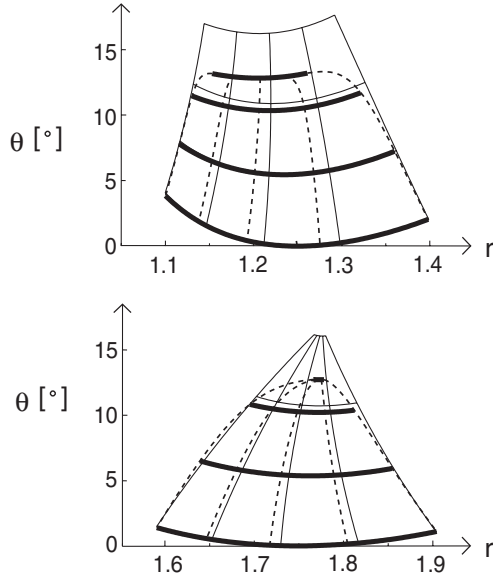


FIG. 4. Nondegenerate evolution of the brane.

The initial conditions stay unchanged, except for the condition for π_t which now reads $\pi_t = -\sqrt{h_* + \hat{\theta}_*^2 - \alpha_* \theta_*^q}$. For numerical calculations, it is convenient to write equation for θ in terms of momenta, as $d_t \theta = -\pi_R^2 / \pi_t + \pi_t$. This equation conserves the value $H = 0$ better than the equation with α -term cited above, since it leads to $d_t H = 0$ rather than $d_t H \propto H$. In Fig. 4 we depict two 3-branes living in a 4-dimensional space with the same properties as those in Fig. 3, but with such a large α that $\dot{\theta}$ at the initial moment is suppressed by the factor $1/10$. For comparison, we added branes with zero α , depicted along with their characteristics by the light lines. The nondegenerate evolution is considerably slower than the degenerate one at the beginning, but at the time about 0.2 before reconnection the two modes of evolution become practically indistinguishable.

V. CONCLUSION

We have investigated how a piece of brane wrapped around a black hole separates from the rest after the black hole is knocked out into an extra dimension. The process is of interest since it reflects nontrivial features of brane dynamics, and can affect signatures of mini black holes produced in high-energy collisions. We have shown that the world sheet becomes isotropic at the reconnection point, and found how fast the brane approaches the degenerate mode of evolution, with the world sheet isotropic everywhere, as it comes close to that point. The rate of relaxation depends on the dimension of the brane: the higher the dimension, the faster the brane becomes isotropic. The relaxation is fast enough to guarantee that the two parameters characterizing the motion of the brane just before reconnection which we have computed for an isotropic brane, the angle at which the brane moves and the

rate of change of its curvature, do not change when we pass to a timelike brane.

The main idea of our approach was that, instead of solving the equation of motion from the very start, we restricted ourselves to the last moments before reconnection. In this way we were able to describe the behavior of the brane at the final stage of its evolution, but we had to resign on determining the characteristics of the process as a whole. In particular, we could not compute the total time it takes the black hole to separate from the brane.

In the simplified description we have adopted from [5], the brane passes through itself at the moment of reconnection and develops an expanding intermediate domain between the piece attached to the black hole and the bulk part where the black hole resided before. To correct this picture, we must assign the brane a finite thickness, while keeping it without self-gravitation. This can be most easily done by identifying the brane with a domain wall composed of a scalar field with double-well potential. In [11] it was shown, by approximating the angular dependence of the scalar field with the help of Chebyshev polynomials, that a thick brane truly splits up at reconnection. Instead of solving the equation for the scalar field, one can pass to an effective theory in which the smearing of the scalar field over a finite domain is taken into account by including curvature corrections into the Lagrangian of the brane. This approach was used to study static brane-black hole systems in [12]. However, the resulting formulas are extremely lengthy even without the time derivatives; thus, to extend our investigation to a thick brane, a more promising approach seems to be that of Ref. [11].

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APPENDIX A: TWO-STEP EXPANSION

After expanding $\mathcal{F}$ in θ , we can expand θ in δr and δt and solve the resulting infinite “tower” of algebraic equations order by order. To see how the method works, consider the nonphysical case $q = 1$. Equation for θ reads

$$\dot{\theta}^2 - f^2 \theta'^2 - h = -\alpha \theta - \beta \theta^2 - \gamma \theta^3 - \dots,$$

where β and γ are of the form (only terms with the highest order derivatives of θ are listed)

$$\begin{aligned} \beta &= \frac{2\alpha}{\mathcal{H}} (\ddot{\theta} - f^2 \theta'') + \dots, \\ \gamma &= \frac{1}{2\mathcal{H}} (-\dot{\theta} \dot{\beta} + f^2 \theta' \beta') + \dots \\ &= -\frac{\alpha}{\mathcal{H}^2} [\dot{\theta} \ddot{\theta} - f^2 (\dot{\theta} \dot{\theta}' + \theta' \dot{\theta}'') + f^4 \theta' \theta'''] + \dots \end{aligned}$$

Expand θ into the series (9) and consider terms of order (m, n) in the equation written above; by definition, these

are the terms appearing in front of $\delta t^m \delta r^n$ with the factor $1/(m!n!)$ suppressed. On the left-hand side we have an expression of the form

$$2\theta_{10}\theta_{m+1,n} + \text{terms with } \theta_{\text{lower}}^2 - h_n \delta_{m0},$$

where θ_{lower} are θ 's with the sum of the indices not exceeding $m + n$. In this expression, all θ_{lower} have the first index not exceeding m and all but one θ_{lower} have the second index not exceeding n . The exception is θ with the indices $(m - 1, n + 1)$ appearing in the term $-2mf_0^2\theta_{11}\theta_{m-1,n+1}$. Suppose for a moment that all right-hand sides are zero. Then we can find all θ 's up to the order N by solving the equations “row by row”, first the equations $(0, 0), (0, 1), \dots, (0, N - 1)$, then the equations $(1, 0), (1, 1), \dots, (1, N - 2)$, and so forth. In this way we have obtained the formulas (10) in Sec. III. Let us now return to the equations with nonzero right hand sides. The α -term contributes only “safe” θ 's with the first index not exceeding m and the second index not exceeding n , but the remaining terms are producing also “dangerous” θ 's with one or another index exceeding the corresponding limit value. Denote the coefficients of expansion of a function $u(r, t)$ into the powers of δr and δt by u_{mn} and the coefficients of expansion of a function $v(r)$ into the powers of δr by v_m . The “dangerous” θ 's can be identified from the structure of the terms with the highest order derivatives of θ in $\beta, \gamma, \dots$: $\theta_{\mu\nu}$ with $\mu = m + 1, m + 2, m + 3, \dots$ and maximum ν appear in $\beta_{m-1,n-2}(\theta^2)_{12} \sim \theta_{m+1,n-2}\theta_{10}\theta_{02}$, $\beta_{m,n-4}(\theta^2)_{04} \sim \theta_{m+2,n-4}\theta_{02}^2$, $\gamma_{m,n-6}(\theta^3)_{06} \sim \theta_{10}\theta_{m+3,n-6}\theta_{02}^2, \dots$, and $\theta_{\mu\nu}$ with $\nu = n + 1, n + 2, n + 3, \dots$ and maximum μ appear in $\beta_{m-2,n-1}(\theta^2)_{21} \sim \theta_{m-2,n+1}\theta_{10}\theta_{11}$, $\beta_{m-2,n}(\theta^2)_{20} \sim \theta_{m-2,n+2}\theta_{10}^2$, $\gamma_{m-3,n}(\theta^3)_{30} \sim \theta_{11}\theta_{m-4,n+3}\theta_{10}^3, \dots$. In this way we find that the “dangerous” θ 's are located inside the triangle $\mu > m, \nu \leq n - 2(\mu - m)$, and the trapezoid $\nu > n, \mu \leq m - 2$ for $\nu = n + 1$ and $\mu \leq m - 2(\nu - n - 1)$ for $\nu > n + 1$. For $q > 1$ both domains flatten, the higher q the more. The crucial observation is that the domains are embedded into the triangle $\mu \geq 0, \nu \geq 0, \mu + \nu \leq m + n$; in other words, all “dangerous” θ 's are of the type θ_{lower} . Thanks to that we can solve the equations after grouping them appropriately, this time “triangle after triangle”: first the equation $(0,0)$, then the equations $(0,1)$, $(1,0)$, then the equations $(0,2)$, $(1,1)$, $(2,0)$, and so forth.

The expansion in δr and δt can be also used to determine the free function α in the expansion of $\mathcal{F}$ from the known function $\mathcal{F}_0$, or equivalently, ξ_0 . From the formula $\xi = \alpha + \beta\theta + \gamma\theta^2 + \dots$ we find $\xi_{0n} = \alpha_n + \beta_{0,n-2}\theta_{02} + \dots + \gamma_{0,n-4}\theta_{02}^2 + \dots$, so that for the relation we seek we need to know the maximum order of α 's entering β 's, γ 's etc. with the first index zero. We must keep in mind, however, that α 's appear in β 's, γ 's etc. also implicitly through θ 's, since all θ_{kl} with $k > 0$ can be expressed in terms of θ_{0j} and α_j by solving the corresponding algebraic equations order by order. Suppose again that $q = 1$. The coefficient ξ_{0n} first appears in the equation for

θ_{2n} , which is of the form $2\theta_{10}\theta_{2n} +$ terms with $\theta_{\text{lower}}^2 = -\xi_{0n}\theta_{10}$; hence all θ_{kl} contained in ξ_{0n} are from the triangle $k + l \leq n + 1$. To identify α 's entering ξ_{0n} , it suffices to consider α 's that are present *explicitly* in θ 's from the triangle, after β 's, γ 's etc. are expressed in terms of α 's and θ 's (which are also from the triangle). Let us write equation for θ_{kl} as

$$2\theta_{10}\theta_{kl} + \text{terms with } \theta_{\text{lower}}^2 - h_l \delta_{k1} \\ = -\alpha_l \theta_{k-1,0} - \dots - \beta_{k-3,l} \theta_{10}^2 - \dots - \gamma_{k-4,l} \theta_{10}^3 - \dots$$

The enlisted expansion coefficients of β, γ etc. contain the highest order α 's for high enough k . They appear in $\beta_{k-3,l} \sim (\theta')_{k-3,0}(\alpha')_l \propto \alpha_{l+1}$ for $k > 3$ ($l < n - 2$), in $\gamma_{k-4,l} \sim (\theta')_{k-5,0}(\theta')_{10}(\alpha')_l \propto \alpha_{l+2}$ for $k > 5$ ($l < n - 4$) etc. (Other coefficients contribute at best α 's of the same order. For example, for $3 < k \leq 5$ we have $\gamma_{k-4,l} \sim (\theta \beta)_{k-4,l} \sim \theta_{10}\beta_{k-3,l} \propto \alpha_{l+1}$.) All these α 's are clearly of lower order than α_n . Furthermore, the highest order α appears in $\alpha_{l-2}\theta_{02}$ for $k = 1$ ($l \leq n$), in $\alpha_l\theta_{10}$ for $k = 2$ ($l \leq n - 1$) and in $\alpha_l\theta_{20}$ and $\beta_{0l}\theta_{10}^2$, $\beta_{0l} \sim (\theta')_{01}(\alpha')_{l-1} \propto \alpha_l$, for $k = 3$ ($l \leq n - 2$). These are again α 's of lower order than α_n . If we rise q and use the same procedure, the triangle will be larger but the relative order of α 's will be lower, so that the net result will be the same. Thus, for any q we have $\xi_{0n} = \alpha_n +$ terms with α_{lower} ; and by solving these equations order by order with respect to α_n we obtain $\alpha_n = \xi_{0n} +$ terms with ξ_{lower} .

APPENDIX B: ONE-STEP EXPANSION

Equations $\Phi = 0$ and $\Psi = 0$ expanded in δr and δt transform into two “towers” of algebraic equations $\Phi_{mn} = 0$ and $\Psi_{mn} = 0$ for the coefficients $\mathcal{F}_{mn}$ and θ_{mn} . Let us look at the structure of these equations. Rewrite Φ as

$$\Phi = \xi \mathcal{F} - \eta \mathcal{F}' - 2\zeta \mathcal{F} - qh \mathcal{F}, \quad (\text{B1})$$

where

$$\xi = \tan\theta\dot{\theta}, \quad \eta = \tan\theta f^2\theta', \quad \zeta = \tan\theta\Theta. \quad (\text{B2})$$

We can see from these formulas that the equations $\Phi_{mn} = 0$ of N th order (with $m + n = N$) contain only $\mathcal{F}_{kl}$ of N th or lower orders (with $k + l \leq N$). Indeed, because of the factor $\tan\theta$ in the definitions of ξ, η and ζ , the zeroth order term is missing in the expansions of all three functions into the powers of δt and δr , therefore the coefficients $\mathcal{F}_{kl}$ contributed to Φ_{mn} by the first three terms in Φ are of maximum order N, N and $N - 1$, respectively. In fact, the expansion of η starts with the terms of *second* order, namely, with η_{20} and η_{11} , hence the second term in Φ yields coefficients $\mathcal{F}_{kl}$ of maximum order $N - 1$, too. Explicitly,

$$\Phi_{mn} = (m\xi_{10} - qh_0)\mathcal{F}_{mn} + \text{terms proportional to } \mathcal{F}_{\text{lower}}.$$

For ξ_{10} we have

$$\xi_{10} = [\theta\dot{\theta}]_{10} = [(\theta_{10}\delta t + \dots)(\theta_{10} + \dots)]_{10} = \theta_{10}^2 = h_0,$$

(the last equality follows from $\mathcal{F}_{00} = 0$, which is now a consequence of $\Phi_{00} = 0$), hence

$$\Phi_{mn} = (m - q)h_0\mathcal{F}_{mn} + \text{terms proportional to } \mathcal{F}_{\text{lower}}. \quad (\text{B3})$$

From the expression for Φ_{mn} it is evident that if q is not a natural number and Φ_{mn} vanishes for all m and n , $\mathcal{F}_{mn}$ must vanish for all m and n , too. Indeed, if the factor in front of $\mathcal{F}_{mn}$ is nonzero, equation $\Phi_{mn} = 0$ determines the coefficient $\mathcal{F}_{mn}$ in terms of coefficients of lower order; and if *all* factors are nonzero, which is the case for any non-natural q , these lower order coefficients have been already fixed to zero by previous equations. We already know that the true solution to equation $\Phi = 0$ merge with the degenerate one at the point of reconnection. Now we can see that for all non-natural values of q a stronger statement is valid: the only solutions that are regular at the point of reconnection (can be Taylor-expanded in δr and δt) are those that are *exactly* degenerate.

The “regularity means degeneracy” claim for non-natural q ’s can be obtained immediately from the fact that the expansion of $\mathcal{F}$ in θ starts from θ^q . Since $\mathcal{F}$ is by definition quadratic in the first derivatives of θ , and since it expands into non-natural powers of θ , the function θ itself must expand into non-natural powers of δr and δt unless $\mathcal{F}$ is identically zero.

Let us now proceed to the case when q is natural, $q = 1, 2, 3, \dots$ (For completeness, we consider also odd values of q , although only even values are physically relevant.) Everything works as with non-natural q ’s until we arrive at the equation $\Phi_{q0} = 0$. The equation is satisfied identically, therefore the coefficient $\mathcal{F}_{q0}$ is *at this stage* free. However, it can be fixed to zero at the next step, by the equation $\Phi_{q1} = 0$; or if not, then at the next-to-next step, by the equation $\Phi_{q2} = 0$; and so forth. The expressions Φ_{qj} on the left-hand side of these equations can be transformed into linear combinations of j coefficients $\mathcal{F}_{q0}, \mathcal{F}_{q1}, \dots, \mathcal{F}_{q,j-1}$, because other coefficients entering them are either zero or can be expressed, by using “their” equations, as linear combinations of the coefficients we have listed. Thus, we have

$$\Phi_{qj} = C_{1j}\mathcal{F}_{q,j-1} + C_{2j}\mathcal{F}_{q,j-2} + \dots + C_{jj}\mathcal{F}_{q0}. \quad (\text{B4})$$

Coefficients C_{ij} appearing here can be written as

$$C_{ij} = \text{terms of the form } \theta^2, \theta^4, \dots - q \binom{j}{i} h_i, \quad (\text{B5})$$

whereby “terms of the form $\theta^2, \theta^4, \dots$ ” we understand terms proportional to the product of the corresponding number of coefficients θ_{mn} . Some of these terms appear in front of $\mathcal{F}_{q,j-i}$ in the original expression for Φ_{qj} , and the rest are secondary terms contributed by $\mathcal{F}$ ’s of higher order. In the primary terms, higher powers of θ come from the expansion of $\tan\theta$, while in the secondary terms they are produced also by inserting for higher order $\mathcal{F}$ ’s the expressions obtained from “their” equations. Indeed, by putting the expression (B3) equal to zero we obtain $\mathcal{F}_{mn} \sim$ terms of the form $\theta^2, \theta^4, \dots \times \mathcal{F}_{\text{lower}}/h_0$ for any $m \neq q$, therefore each consecutive order of $\mathcal{F}_{mn}$ brings with itself at least one extra factor θ^2 into C_{ij} .

The next step is to compute θ ’s from equations $\Psi_{mn} = 0$. The equations can be written as

$$2\theta_{10}\theta_{m+1,n} + \text{terms with } \theta_{\text{lower}}^2 - h_n\delta_{m0} = -\mathcal{F}_{mn},$$

and they yield, when solved order by order, expressions for θ_{mn} containing θ_{0j} and $\mathcal{F}_{qj}$ only. The expressions can be divided into “degenerate” part $\theta_{mn}^{(0)}$ that does not contain $\mathcal{F}_{qj}$, and optional extra terms proportional to the powers of $\mathcal{F}_{qj}$. First few $\theta^{(0)}$ ’s have been computed at the beginning of Sec. III. After the expressions for θ ’s are inserted into the expressions for C ’s, they also split into two parts, “degenerate” part $C_{ij}^{(0)}$ and optional $\mathcal{F}$ terms. Extra terms first appear in $C_{2q,2q}$. They include, in particular, $\mathcal{F}$ term coming from the primary term $q\xi_{1,2q}\mathcal{F}_{q0}$ in $\Phi_{q,2q}$, with $\xi_{1,2q} = 2\theta_{10}\theta_{1,2q} + \text{terms of the form } \theta_{\text{lower}}^2, \theta_{\text{lower}}^4, \dots$. The term is proportional to $\mathcal{F}_{q0}$, since $\theta_{1,2q}$ can be expressed in terms of $\mathcal{F}_{0,2q}$, which in turn can be expressed in terms of $\mathcal{F}_{q0}$ (most simply by using the relation $\mathcal{F} \sim \alpha\theta^q$). All other extra terms in $C_{2q,2q}$ are proportional to $\mathcal{F}_{q0}$, too.

For $C_{ij}^{(0)}$, as well as for the coefficients at the powers of $\mathcal{F}_{qj}$, if present, we have expressions whose complexity grows progressively with their order. However, after computing the first few of them explicitly, we have found that the terms comprising them mutually cancel. We have checked this, partly with the help of MAPLE, for C_{ij} with the first index running up to $i = 4$. Thus, the first few equations $\Phi_{qj} = 0$ reduce to $0 = 0$. The considerations of Sec. IV suggest that this is true for *all* equations $\Phi_{qj} = 0$. In other words, for natural q ’s there holds a direct opposite to what we have established for non-natural q ’s: regularity does *not* mean degeneracy. On the contrary, it means maximum nondegeneracy in the sense that all C_{ij} are identically zero and all $\mathcal{F}_{qj}$ are free.

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