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Green's-Function Approach to Inverse Compton Scattering

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The emission, absorption, and transfer of radiation produced through the Compton interaction of relativistic electrons with microwave photons in a homogeneous source is considered. It is shown that the intensity of Compton-scattered radiation can be conveniently expressed in terms of a Green's function, and this enables the radiation-transfer problem for arbitrarily many scatterings to be formalized in terms of a series of appropriately defined scattering vertex functions and absorption propagators. A rigorous discussion of the cross section for inverse Compton scattering is presented which is valid both in the extreme relativistic ($\gamma \gg 1$) and weakly relativistic limits ($\gamma \simeq 1$). Using these results the spectral behavior of the intensity of Compton-scattered radiation is evaluated explicitly for once- and twice-scattered photons; the qualitative features of the multiple-scattering problem are described. At sufficiently high frequencies the spectral behavior of the first- and all higher-order scatterings is the same.

I. INTRODUCTION

The discovery of compact and highly luminous sources of cosmic radio radiation has stimulated a continuing interest in the interactions that are expected to occur when an energetic electron passes through a tenuous gas of low-energy photons and particles. In astrophysical situations where the energy density of the radiation field is sufficiently high, Compton collisions between fast electrons and the ambient microwave photons will provide both a substantial drain on the electron distribution and a concomitant supply of high-energy photons. This process, the so-called "inverse Compton effect"—in which the energy of the microwave photons are increased by a factor on the order of γ^2 (where γ is the Lorentz factor of the relativistic electrons) and the energy is provided at the expense of the projectile particles—has been studied by a number of authors with a view toward astrophysical applications.[1-6] Several approximations have been incorporated in all of these calculations, the most important of which are (1) that the projectile electrons are highly relativistic, $\gamma \gg 1$ and $\beta = 1$, and (2) that the frequency of the microwave photons that are to be scattered is very much larger than ν_p, the electron plasma frequency. The effect of the former simplification is to allow useful approximations to be made for the scattering angle, whereas the effect of the latter simplification is to obviate the need to consider the fact that the electron appears less relativistic to an observer located in the medium than to one outside the medium. However, if a substantial fraction of the microwave emission from astrophysical sources is attributed to coherent plasma processes that give rise to emission near the plasma frequency, as has recently been suggested by Lerche and Papadopoulos,[7-11] rather than (or in addition to) incoherent synchrotron radiation, then a study of the Compton interactions between the "enhanced bremsstrahlung" radiation fields and weakly relativistic electrons requires that the inverse Compton-scattering process be calculated in a regime where the usual simplifications noted above are not applicable.

Consequently, we are motivated to investigate the spectrum of Compton-scattered radiation that arises when a distribution of weakly relativistic electrons interacts with a coambient isotropic distribution of photons whose frequencies are very near the plasma frequency. This study is carried out below according to the following outline: In Sec. II a general description is provided for the transfer of Compton-scattered radiation in a plasma; the total scattering cross section is evaluated explicitly in Sec. III for arbitrary γ and for ν_0 (the frequency of the target photons) arbitrarily close to the plasma frequency; in Sec. IV the general features of the single-scattering process applicable to astronomical sources are described; in Sec. V the spectrum of twice-scattered photons is considered; and finally the gross features of the entire spectrum of Compton-scattered radiation including all multiple scatterings are described in Sec. VI.

II. THE TRANSFER OF COMPTON-SCATTERED RADIATION

In a particular cosmic source of finite extent that emits Compton-scattered radiation, every volume element in the source can be thought of as a radiation emitter. From our location enormous distances from the source we are concerned only with the radiant intensity that flows in a small solid angle $d\Omega$ about the line of sight. This flux of radiation at a particular frequency ν per unit solid angle $d\Omega$ and per unit frequency interval $d\nu$ we denote by I_ν. Here I_ν is comprised of the emission from each volume element less that part of the flux that is absorbed within the source. Mathematically we express the increase in specific intensity per unit distance along the line of sight in terms of the emissivity ϵ_ν, while the fractional loss of intensity is accounted for by the absorption coefficient κ_ν. The resultant change in the radiant flux per unit distance through the source dI_ν/ds can be described by means of the radiation-transfer equation

$$\frac{dI_\nu(s)}{ds} = -\kappa_\nu I_\nu(s) + \epsilon_\nu(s). \tag{1}$$

Here

$$\kappa_\nu(s) = \int\int \sigma(\nu_f', \nu, \gamma) N(\gamma) d\nu_f' d\gamma, \tag{2}$$

$$\epsilon_\nu(s) = \int\int \sigma(\nu, \nu_i', \gamma) N(\gamma) I_{\nu_i'}(s) d\nu_i' d\gamma, \tag{3}$$

where $N(\gamma)$ represents the number of electrons having energies within an interval $d\gamma$ about some energy $\gamma = E/mc^2$; $\sigma(\nu_f, \nu_i, \gamma)$ is the total inverse Compton-scattering cross section; and ν_i, ν_f are the initial and final frequencies, respectively, of the scattered photon. (In the above relations, and hereafter, the subscripts i and f refer to quantities measured in the initial and final scattering states, respectively, whereas the superscripts R and L introduced below will refer to quantities measured in the rest frame of the electron and the laboratory frame, respectively.) In writing Eq. (2) we are considering only Compton absorption and have neglected other absorptive processes. The limitations imposed by this restriction are discussed in Sec. IV.

The intensity $I_\nu(s)$ can be conveniently expressed in terms of a Green's function $G_\nu(s, s')$ by noticing that

$$\left(\frac{d}{ds} + \kappa_\nu\right) G_\nu(s, s') = \delta(s - s') \tag{4}$$

implies

$$I_\nu(s) = I_\nu(0) G_\nu(s, 0) + \int_0^s \epsilon_\nu(s') G_\nu(s, s') ds' \tag{5}$$

as the complete solution to the radiation-transfer equation (1). Fourier-transforming Eq. (4) we obtain

$$G_\nu(s, s') = \int_{-\infty}^{\infty} \frac{e^{ik(s-s')}}{2\pi i(k - i\kappa_\nu)} dk = e^{-\kappa_\nu(s-s')} \tag{6}$$

as the Green's function appropriate for this particular calculation. With the aid of this formalism the salient features of the scattering problem under consideration may be illustrated in a straightforward manner. Notice that if we integrate the expression for ϵ_ν [Eq. (3)] first over γ (changes in the order of integration are discussed in Sec. V), a vertex function $\tilde{\Lambda}(\nu, \nu')$ may be defined such that

$$\epsilon_\nu(s) = \int I_{\nu'}(s) \tilde{\Lambda}(\nu, \nu') d\nu'. \tag{7}$$

This enables a simple physical interpretation to be attached to the solution to the radiative-transfer equation (5). By successive self-substitutions,

$$I_\nu(s) = G_\nu(s, 0) I_\nu(0) + \int\int I_{\nu'}(s') \tilde{\Lambda}(\nu, \nu') G_\nu(s, s') d\nu' ds' \tag{8}$$

$$\begin{aligned} &= G_\nu(s, 0) I_\nu(0) + \int\int G_\nu(s, s') \tilde{\Lambda}(\nu, \nu') G_{\nu'}(s', 0) I_{\nu'}(0) d\nu' ds' \\ &\quad + \int\int\int\int G_\nu(s, s') \tilde{\Lambda}(\nu, \nu') G_{\nu'}(s', s'') \tilde{\Lambda}(\nu', \nu'') G_{\nu''}(s'', 0) I_{\nu''}(0) d\nu'' ds'' d\nu' ds' \\ &\quad + \text{higher-order terms in } \tilde{\Lambda}(\nu, \nu'). \end{aligned} \tag{9}$$

Here the vertex function is essentially determined by the integral

$$\tilde{\Lambda} \sim \int \sigma(\nu, \nu', \gamma) N(\gamma) d\gamma. \tag{10}$$

The integrands in Eq. (9) can be represented diagrammatically as shown in Fig. 1. In this figure G_ν is the inverse Compton absorption propagator at a particular frequency ν and $\tilde{\Lambda}(\nu, \nu')$ is the interaction vertex function describing the Compton scattering of a photon from a frequency ν' to a frequency ν. The first term in Eq. (9) (shown

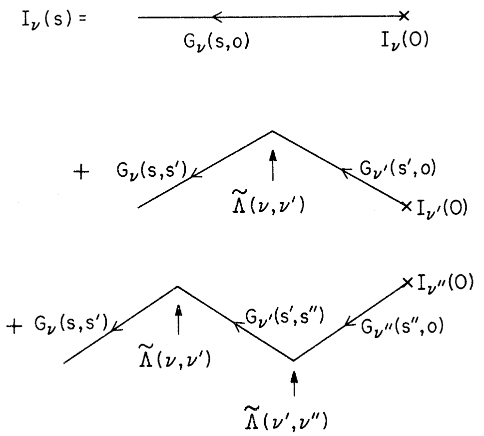

FIG. 1. Diagrammatic representation of the transfer of Compton-scattered radiation. The individual terms correspond to the integrals of Eq. (9).

schematically as the top diagram in Fig. 1) represents pure absorption of the incident flux $I_\nu(0)$ with no accompanying scattering. The second term expresses the contribution to the total flux wherein the incident flux is Compton-scattered once with absorption occurring both before and after the scattering; the net path traversed by this radiation through the macroscopic source is equal to the distance s. The third term represents double scattering of the initial flux $I_\nu(0)$ with two vertex functions and three absorption propagators. The higher-order terms in $\tilde{\Lambda}$ describe multiple-scattering processes with n vertex functions $\tilde{\Lambda}$ and $(n+1)$ absorption propagators G_ν.

Recall that $I_\nu(s)$ is that fraction of the flux received that arises from a volume of emission located at a depth s in the macroscopic source; consequently the total flux received, $\bar{I}_\nu$, from all emitters in the source is a sum over all such volume elements. That is, the total flux received is

$$\bar{I}_\nu = \int_0^d I_\nu(s)ds\,, \tag{11}$$

if the macroscopic source has a dimension d along the line of sight.

III. CROSS SECTION FOR INVERSE COMPTON SCATTERING BY ELECTRONS OF ARBITRARY γ (>1)

In this section we wish to determine the cross section for the scattering of relativistic electrons with arbitrary Lorentz factors (>1) on low-energy photons. We first summarize a derivation of the inverse-Compton-scattering cross section given by Pacholczyk[6] for scattering in the limit $\gamma \gg 1$, and we shall discuss the error introduced in his results by neglecting lower-energy electrons.

To begin it is most convenient to transform to the rest frame (R) of the electron and consider the scattering of photons on stationary electrons, i.e., ordinary Compton scattering. This calculation must then be referred back to the laboratory frame (L) since we are interested in processes in which energy is transferred from the electrons to the photons, viz., precisely the "inverse" of the usual Compton-scattering problem. The Lorentz transformation that must be performed is shown in Fig. 2 where the variables to be transformed are referenced to the initial and final scattering states by the subscripts i and f, respectively. It can be seen in Fig. 2 that the transformation of all the necessary physical parameters can be represented geometrically through a simple Lorentz transformation of an ellipse in the laboratory frame to a circle in the rest frame of the electron.[12] If we now employ the usual formulas for Lorentz transfor-

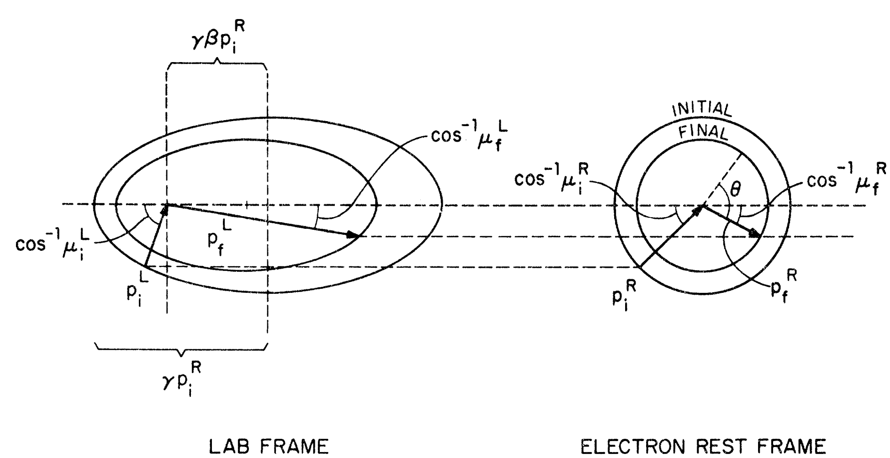

FIG. 2. Geometrical illustration of the Lorentz transformation from the rest frame of the relativistic electron to the laboratory frame.

mation of angles, solid angles, frequencies, and intensities, we can obtain expressions for the (Lorentz-invariant) number of scattered photons in the electron rest frame and the lab frame. By equating these expressions we derive the transformation of the differential cross section $d\Sigma/d\Omega_f$ for Compton scattering.

For simplicity we shall restrict ourselves to Compton scatterings in which the electron loses only a very small fraction of its energy in a single collision, that is scatterings such that in the rest frame $h\nu_i^R \ll mc^2$ or in the laboratory frame $h\nu_i^L\gamma \ll mc^2$. In this limit (the Thomson limit in the electron rest frame) the energy conservation condition reduces to

$$\nu_f^L = \nu_i^L(1-\beta\mu_i^L)(1+\beta\mu_f^R)\gamma^2\,, \tag{12}$$

where μ is the cosine of the angle between the direction of motion of the electron and the direction to the observer.

The total Compton-scattering cross section in the laboratory frame may be obtained by summing the differential cross section directly over all initial and final directions that are allowed by (12),

$$\sigma(\nu_f^L,\nu_i^L,\gamma) = \frac{3}{16}\sigma_T\frac{1}{\nu_i^L\beta\gamma^2}\int[1+(\lambda^R)^2]d\mu_i^L\,\frac{d\chi}{2\pi}\bigg|_{\nu_f^L=\nu_i^L\gamma^2(1-\beta\mu_i^L)(1+\beta\mu_f^R)}\,, \tag{13}$$

and we have set $d\Omega_i^L = d\mu_i^L d\chi$, $d\Omega_f^L = d\mu_f^L d\chi$ (χ invariant).

To express λ^R in terms of μ_i^L,χ in the integrand we consider relations among the available angular parameters. There are two such relations: the cosine law in the electron rest frame (μ_i^R and μ_i^L are Lorentz-related) as in Fig. 3, and the energy conservation condition (12). Using these expressions we choose μ_i^L and χ to be independent variables and determine $\lambda^R(\mu_i^L,\chi)$.

Consider a three-dimensional space described by λ^R, μ_i^L, and χ as orthogonal axes. In this space the integration (13) reduces to an evaluation of the integrand throughout a two-dimensional surface determined by $\lambda^R(\mu_i^L,\chi)$. Furthermore, this surface of integration is restricted by $|\lambda^R|\le 1$, $|\mu_i^L|\le 1$, $|\mu_f^R|\le 1$, and $0\le\chi\le 2\pi$. The situation is illustrated in Fig. 4. Owing to Eq. (12) $|\mu_f^R|\le 1$ is actually a further restriction on μ_i^L:

$$p\le\mu_i^L\le q\,, \tag{14}$$

where

$$p = \frac{(1-\beta-\nu_f^L/\nu_i^L\gamma^2)}{\beta(1-\beta)}\,, \tag{15}$$

$$q = \frac{(1+\beta-\nu_f^L/\nu_i^L\gamma^2)}{\beta(1+\beta)}\,. \tag{16}$$

The above conditions on the integration variable μ_i^L and χ define 2 regions for the cross-section integral (13) that are of physical interest: (a) When

$$\nu_i^L\left(\frac{1-\beta}{1+\beta}\right)\le\nu_f^L\le\nu_i^L\,, \tag{17}$$

the proper limits in (13) are $[1,p]$ where $p\le 1$; (b) when

$$\nu_i^L\le\nu_f\le\nu_i^L\gamma^2(1+\beta)^2\,, \tag{18}$$

the integration limits are $[q,-1]$ where $q\ge 1$.

Here it can be seen that interval (a) exists only

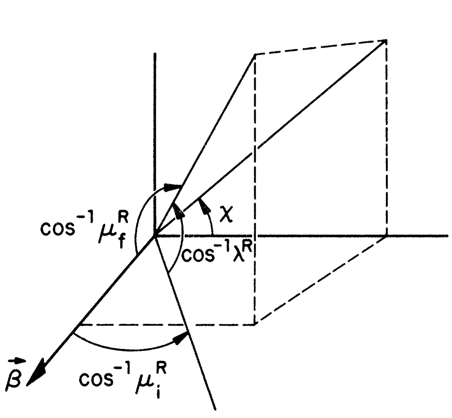

FIG. 3. Angles in the rest frame of the electron.

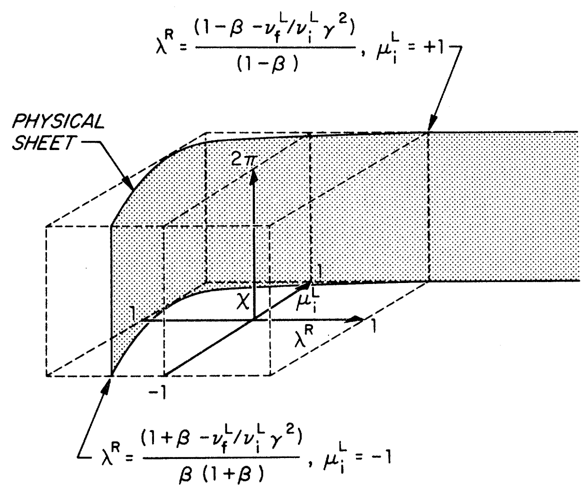

FIG. 4. The physical sheet relevant for evaluation of the inverse-Compton-scattering cross section. The contours are along planes parallel to the λ^R axis.

for weakly relativistic electrons, $\beta < 1$, and it collapses to zero as $\beta \to 1$. On the other hand, interval (b) exists for arbitrary electron energy and is the result given by Pacholczyk.[6] In what follows we shall neglect contributions from region (a) since these contributions are negligible in most astrophysical circumstances. Bear in mind, however, that this is an approximation whose error is dependent on the width of interval (a).

Carrying out the integration (13) over the limits defined for region (b) by Eq. (18), we obtain the resultant total Compton-scattering cross section in the limit $\beta \to 1$,

$$\sigma(\nu_f^L, \nu_i^L, \gamma) = \frac{3\sigma_T}{16\nu_i^L\gamma^2}\left\{4 + \frac{\nu_f^L}{\nu_i^L\gamma^2}\left[1 + 2\ln\left(\frac{\nu_f^L}{4\nu_i^L\gamma^2}\right) - \frac{1}{2}\left(\frac{\nu_f^L}{\nu_i^L\gamma^2}\right)^2\right]\right\}. \tag{19}$$

The physical domain of the variable μ_i^L is

$$-1 \leqslant \mu_i^L \leqslant q \to 1 - \nu_f^L/2\nu_i^L\gamma^2, \tag{20}$$

and in region (b) this limit is

$$\nu_i^L \leqslant \nu_f^L \leqslant \nu_i^L(1+\beta)^2\gamma^2 \to 4\nu_i^L\gamma^2. \tag{21}$$

IV. COMPTON SCATTERING IN ASTRONOMICAL SOURCES

A. General Features: Specification of the Radiation Field

Recall that our motivation for calculating the total Compton-scattering cross section is to obtain an expression for the vertex function $\tilde{\Lambda}(\nu, \nu')$ applicable to inverse Compton scattering [cf. Eq. (7)]. Once the appropriate $\tilde{\Lambda}(\nu, \nu')$ is determined, it can be applied in (9) to provide the intensity spectrum of Compton-scattered radiation that we ultimately desire.

The spectrum of scattered radiation (9) depends, of course, not only on the scattering cross section [here we intend to use the limiting form given by Eq. (19) which is valid throughout the domain specified by Eq. (21)], but it also depends on the initial frequency distribution of microwave photons, that is on the distribution of target photons. In the calculations that follow we intend to choose an initial-photon spectrum which is a simple δ function at twice the electron plasma frequency ν_p,

$$I_\nu(0) = \rho\delta(\nu - \nu_0), \tag{22}$$

where

$$\nu_0 = 2\nu_p = (4n_0e^2/\pi m_e)^{1/2}. \tag{23}$$

This particular choice of $I_\nu(0)$ is stimulated by the work of Lerche and Papadopoulos,[7-11] who demonstrated that when a tenuous, isotropic, relativistic electron flux permeates a homogeneous thermal plasma, a substantial enhancement of the subluminous wave emission around $2\nu_p$ is to be expected. This process is a relativistic generalization of the case first considered by Tidman and Dupree,[13] where enhancement was predicted when an energetic but yet nonrelativistic electron distribution coexisted with a thermal plasma. These predictions have subsequently been verified in the laboratory.[14] Basically, the radiation mechanism rests on the conversion of longitudinal plasma waves into radiation through mode-mode scattering. In the case to be considered here, the distribution of relativistic electrons amplifies the wave-field part of the longitudinal fluctuation spectrum by means of Čerenkov emission of plasma oscillations. Radiation is emitted at $2\nu_p$ following collisions of components of the fluctuation spectra with low-frequency ion-density fluctuations and collisions of the individual fluctuation-spectrum components with each other.

B. The General Vertex Function for Compton Scattering

In order to evaluate the expression for the intensity of Compton-scattered radiation it is necessary to describe further the vertex function $\tilde{\Lambda}(\nu, \nu')$ defined in Sec. II. Recall that it is essentially a sum of the cross section for Compton scattering over the distribution of relativistic electrons present in the plasma

$$\tilde{\Lambda}(\nu, \nu') = \int \sigma(\nu, \nu', \gamma)N(\gamma)d\gamma, \tag{24}$$

where $N(\gamma)d\gamma$ denotes the number of relativistic electrons per unit volume having Lorentz factors between γ and $\gamma + d\gamma$. We shall assume here and henceforth that $N(\gamma)$ is comprised of particles having γ in the range $\gamma_0 \leqslant \gamma \leqslant \infty$, that is, that there is a low-energy cutoff to the spectrum at $\gamma = \gamma_0$. With this representation the absorption coefficient κ_ν and the Compton emissivity ϵ_ν become

$$\kappa_\nu = \int \tilde{\Lambda}(\nu_f', \nu)d\nu_f', \tag{25}$$

$$\epsilon_\nu = \int I_{\nu_i'}\tilde{\Lambda}(\nu, \nu_i')d\nu_i'. \tag{26}$$

However, in arriving at the forms (25) and (26) from (2) and (3) the order of integration of $d\nu$ and $d\nu'$ has been interchanged. Hence the limits of integration are likely to be changed; this problem must be investigated before we can proceed.

(i) The absorption coefficient. The integration domain appropriate for evaluation of the absorption

coefficient is illustrated in Fig. 5. A convenient representation of κ_ν can be obtained from this figure if we define

$$\Lambda(\nu', \nu)_\gamma^\infty = \int_{\gamma_0}^\infty \sigma(\nu', \nu, \gamma) N(\gamma) d\gamma \tag{27}$$

and

$$\Lambda(\nu', \nu)_{(\nu'/\nu)^{1/2}/2}^\infty = \int_{(\nu'/\nu)^{1/2}/2}^\infty \sigma(\nu', \nu, \gamma) N(\gamma) d\gamma \tag{28}$$

so that

$$\kappa_\nu = \int_\nu^{4\gamma_0^2\nu} \Lambda(\nu', \nu)_{\gamma_0}^\infty d\nu' + \int_{4\gamma_0^2\nu}^\infty \Lambda(\nu', \nu)_{(\nu'/\nu)^{1/2}/2}^\infty d\nu' . \tag{29}$$

Comparing (29) with (24)–(26) we conclude that

$$\tilde{\Lambda}(\nu', \nu) = \Lambda(\nu', \nu)_{\gamma_0}^\infty \tag{30}$$

when $\nu \leqslant \nu' \leqslant 4\gamma_0^2\nu$ or $\nu'/4\gamma_0^2 \leqslant \nu \leqslant \nu'$, and

$$\tilde{\Lambda}(\nu', \nu) = \Lambda(\nu', \nu)_{(\nu'/\nu)^{1/2}/2}^\infty \tag{31}$$

when $4\gamma_0^2\nu \leqslant \nu' \leqslant \infty$ or $0 \leqslant \nu \leqslant \nu'/4\gamma_0^2$. These equations may be evaluated once the form of the electron distribution is established. If we adopt, arbitrarily, a power-law spectrum of relativistic electrons

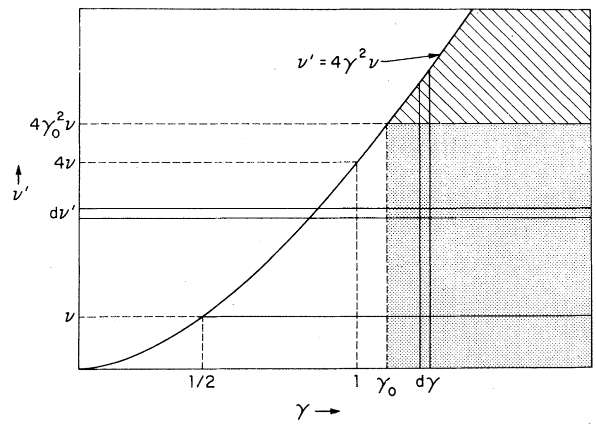

FIG. 5. Integration domain for the absorption coefficient κ_ν. The shaded area under the curve defines the integration domain for the first term in Eq. (29) and the area crossed by diagonal lines represents the integration domain for the second term in Eq. (29).

$$N(\gamma)d\gamma = N_0\gamma^{-\alpha}d\gamma , \quad \gamma_0 \leqslant \gamma \leqslant \infty \tag{32}$$

then

$$\Lambda(\nu', \nu)_{\gamma_0}^\infty = \frac{3}{32} N_0 \sigma_T \left\{ \frac{8}{(\alpha+1)\gamma_0^{\alpha+1}} + \frac{2\nu'}{\nu} \frac{1}{(\alpha+3)\gamma_0^{\alpha+3}} - \left(\frac{\nu'}{\nu}\right)^2 \frac{1}{(\alpha+5)\gamma_0^{\alpha+5}} + \frac{16\gamma_0^{-(\alpha+3)}}{(\alpha+3)} \left(\frac{\nu'}{4\nu}\right) \left[\ln\left(\frac{\nu'}{4\nu\gamma_0^2}\right) - \frac{2}{(\alpha+3)} \right] \right\}, \tag{33}$$

$$\Lambda(\nu', \nu)_{(\nu'/\nu)^{1/2}/2}^\infty = F(\alpha) \frac{1}{\nu} \left(\frac{\nu}{\nu'}\right)^{(\alpha+1)/2} , \tag{34}$$

where

$$F(\alpha) = \frac{3}{32} N_0 \sigma_T 2^{\alpha+4} \left[\frac{1}{(\alpha+1)} + \frac{1}{(\alpha+3)} - \frac{2}{(\alpha+5)} - \frac{4}{(\alpha+3)^2} \right] . \tag{35}$$

In this case the absorption coefficient may be expressed as

$$\kappa_\nu = N_0 \sigma_T \left\{ \frac{1}{(\alpha-1)\gamma_0^{\alpha-1}} - \frac{3}{4(\alpha+1)\gamma_0^{\alpha+1}} + \frac{1}{32(\alpha+5)\gamma_0^{\alpha+5}} - \frac{3}{16(\alpha+3)\gamma_0^{\alpha+3}} \left[\ln \frac{1}{4\gamma_0^2} - \frac{2}{(\alpha+3)} \right] \right\} \tag{36}$$

so that κ_ν is independent of ν. Therefore, the absorption propagator

$$G_\nu(s, s') = e^{-\kappa_\nu(s-s')} \tag{37}$$

is independent of frequency (again, this applies only when Compton absorption alone is considered). This allows a considerable simplification to be made in the calculations that follow because the integration in Eq. (9) over frequencies can be separated from the integration over emission path lengths.

(ii) The Compton emissivity. Precisely the same procedures can be used to express the Compton emissivity in a convenient form. The integration domain applicable for evaluation of the emissivity is shown in Fig. 6:

$$\epsilon_\nu = \int_{\nu/4\gamma_0^2}^\nu I_{\nu'} \Lambda(\nu, \nu')_{\gamma_0}^\infty d\nu' + \int_0^{\nu/4\gamma_0^2} I_{\nu'} \Lambda(\nu, \nu')_{(\nu/\nu')^{1/2}/2}^\infty d\nu' . \tag{38}$$

C. Single Scattering

The second term in the expansion (9) for the intensity of Compton-scattered radiation represents the contribution to the intensity at a frequency ν due to single scattering,

$$I_{\nu'}(s)=\int\int G_\nu(s,s')\tilde{\Lambda}(\nu,\nu')G_{\nu'}(s,0)I_{\nu'}(0)d\nu'\,ds'. \tag{39}$$

Recall that previously [Eq. (37)] we showed that G_ν does not depend on ν so that in this expression we can separate the spatial integration from the frequency integration. Considering again the δ-function distribution of initial-photon frequencies (22),

$$\begin{aligned}I_\nu(s)&=\int e^{-\kappa_\nu(s-s')}\tilde{\Lambda}(\nu,\nu')e^{-\kappa_\nu s'}\\&\quad\times\delta(\nu'-2\nu_p)d\nu'\,ds'\\&=e^{-\kappa_\nu s}s\tilde{\Lambda}(\nu,2\nu_p).\end{aligned} \tag{40}$$

This expression represents the intensity due to single scattering in a source of unit volume located a distance s along the line of sight into the electron plasma. To obtain the total Compton intensity we must sum all contributions from all parts of the radiating source; using (40) as the desired integrand in (11) the resultant intensity from single scatterings in a source of dimension d is

$$\bar{I}_\nu=\frac{1-e^{-\kappa_\nu d}}{\kappa_\nu}\left(d+\frac{1}{\kappa_\nu}\right)\tilde{\Lambda}(\nu,2\nu_p). \tag{41}$$

In astrophysical applications, typically $\kappa_\nu s\gg 1$ so that $\bar{I}_\nu$ does not depend on the normalization of the electron spectrum; specifically $\bar{I}_\nu$ does not depend on the product $N_0\sigma_T$. This simply means that as $N_0\sigma_T$ increases (i.e., as the energy density in relativistic electrons increases) the radiation emitted increases ($\tilde{\Lambda}$ increases), but in addition the absorption also increases: The two opposing effects exactly cancel each other. Naturally for multiple scatterings we expect the parameter $N_0\sigma_T$ to play a more important role.

The shape of the resultant spectrum (41) can be easily obtained by noting that $\tilde{\Lambda}(\nu,2\nu_p)$ behaves differently on either side of $\nu=4\gamma_0^2\nu_0(\nu_0=2\nu_p)$; this can be seen from Eqs. (30) and (31). For $\nu<4\gamma_0^2\nu_0$, $\tilde{\Lambda}(\nu,2\nu_p)=\Lambda(\nu,2\nu_p)_{\gamma_0}^{\infty}$ and this is an increasing function of ν, ν^2, and $\nu\ln\nu$, whereas when $\nu\geqslant 4\gamma_0^2\nu_0$, $\tilde{\Lambda}(\nu,2\nu_p)=\Lambda(\nu,2\nu_p)_{(\nu/2\nu_p)^{1/2}/2}^{\infty}$ and this function is an inverse power law $\nu^{-(\alpha+1)/2}$. This spectrum is shown explicitly in Fig. 7. In deriving this shape and throughout the above derivation we have assumed that the only photon absorption process is that due to Compton scattering. Near the plasma frequency this simplification is not, of course, valid, because at these frequencies free-free absorption may be very efficient indeed. However, the general characteristics of the spectrum shown in Fig. 7—the frequency at which the maximum intensity occurs, the spectral dependence of the high-frequency ($\nu>4\gamma_0^2\nu_0$) power-law component—will not change as long as

$$\nu_{\rm ff}(\tau=1)\lesssim 4\gamma_0^2\nu_0 \tag{42}$$

or[15]

$$\nu_p\lesssim\frac{\gamma_0^2}{eZR^{1/2}}\left[\frac{2^7c^2(3kT)^3}{\pi^5g^2m}\right], \tag{43}$$

where g is the free-free Gaunt factor. For typical astrophysical plasmas, $T\simeq 10^4$ °K, $Z=1$, $g=1$, and $R_{\rm pc}=R/3.08\times10^{18}$ cm (=1 parsec),

$$\nu_p\lesssim 2760\gamma_0^2R_{\rm pc}^{-1/2}\ \text{Hz}, \tag{44}$$

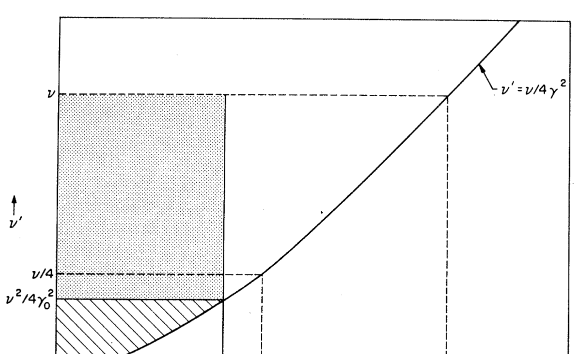

FIG. 6. Integration domain for evaluation of the inverse Compton emissivity ϵ_ν. The shaded area above the curve defines the integration domain for the first term in Eq. (38), while the area crossed by diagonal lines represents the integration domain for the second term in Eq. (38).

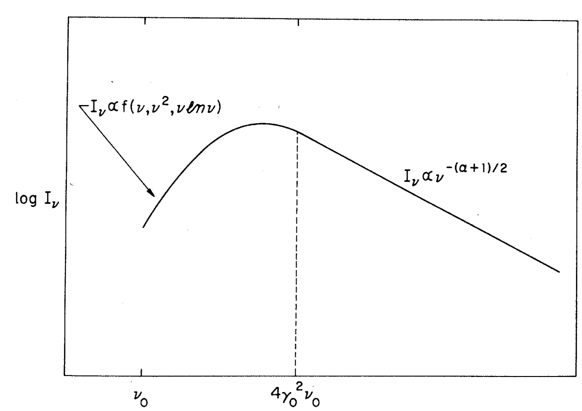

FIG. 7. The spectrum of Compton-scattered radiation resulting from single scattering only.

a condition which is satisfied throughout the radio-frequency domain and at even higher frequencies for all sources except those that are either highly compact or in which the spectrum of relativistic electrons has a low-energy cutoff at rather large $\gamma=\gamma_0$. On the other hand, at frequencies affected by free-free attenuation, $\nu<\nu_{ff}(\tau=1)<4\gamma_0^2\nu_0$, the spectrum of Compton-scattered radiation will be somewhat steeper than shown in Fig. 7; we have not evaluated this quantitatively.

V. DOUBLE SCATTERING

The contribution to the total intensity of Compton-scattered radiation due to double scatterings [the third term of Eq. (9)] is more complex than the single-scattering term but it may be evaluated in precisely the same way. First, the spatial integration follows directly from Eqs. (9) and (11) where again this simple result is provided by the fact that κ_ν is independent of frequency,

$$\int_0^d\int_0^s\int_0^{s'} e^{-\kappa_\nu(s-s')}e^{-\kappa_{\nu'}(s'-s'')}e^{-\kappa_{\nu''}s''}ds''ds'ds = \frac{1-e^{-\kappa_\nu d}}{2\kappa_\nu}\left(d^2+\frac{2d}{\kappa_\nu}+\frac{2}{\kappa_\nu^2}\right). \quad (45)$$

The integration over frequencies is now restricted to a four-dimensional volume which is, in reality, a composition of two areas, each representing one scattering; the relevant integration domain for each of these scatterings is shown in Fig. 8. A more complete discussion of the integration over each of these regions is found in Appendix A. It is sufficient to note here that the total intensity of double-Compton-scattered radiation is represented by a sum of the following four terms, each representing one of the domains illustrated in Fig. 8:

$$\bar{I}_\nu(\text{double})=\frac{1-e^{-\kappa_\nu s}}{2\kappa_\nu}\left(s^2+\frac{2s}{\kappa_\nu}+\frac{2}{\kappa_\nu^2}\right)\times[\bar{I}_\nu(1)+\bar{I}_\nu(2)+\bar{I}_\nu(3)+\bar{I}_\nu(4)], \quad (46)$$

and expressions for the $\bar{I}_\nu(i)$ are given in Appendix A. In addition, the spectral dependence of each of these terms is shown explicitly in Fig. 9.

The resultant spectrum $\bar{I}_\nu$(double) of double-Compton-scattered radiation can be summarized as follows:

(i) $\nu_0\le\nu\le4\gamma_0^2\nu_0$. Here only $\bar{I}_\nu(1)$ contributes to the total intensity and this term behaves as a function of frequency as an increasing polynomial in ν, ν^2, $1/\nu$, and $\int(1/\nu')(\ln\nu')^2d\nu'$.

(ii) $4\gamma_0^2\nu_0\le\nu\le16\gamma_0^4\nu_0$. In this region $\bar{I}_\nu(1)$ decreases to zero while $\bar{I}_\nu(2)$ increases from zero at the lower limit up to a maximum at $16\gamma_0^4\nu_0$. $\bar{I}_\nu(3)$ behaves in this region similar to $\bar{I}_\nu(1)$ in region (i); it is defined by an increasing polynomial in ν, ν^2, and $\nu\ln\nu$.

(iii) $16\gamma_0^4\nu_0\le\nu$. All the terms that are nonzero in this region, that is $\bar{I}_\nu(2)$, $\bar{I}_\nu(3)$, and $\bar{I}_\nu(4)$ all have the same inverse power-law spectral behavior with the same spectral index $\propto\nu^{-(\alpha+1)/2}$. The dominant term is provided by $\bar{I}_\nu(4)$ with its $\nu^{-(\alpha+1)/2}\ln\nu$ dependence.

VI. MULTIPLE SCATTERING

The terms in Eq. (9) that represent contributions to the total intensity of Compton-scattered radiation due to multiple scatterings are simply understood as integrals over frequency of multiple products of the vertex function $\bar{\Lambda}(\nu,\nu')$ with the proper integration limits determined. Note first that the spatial integration of the absorption propagator for n scatterings involves the following integral that may be evaluated directly:

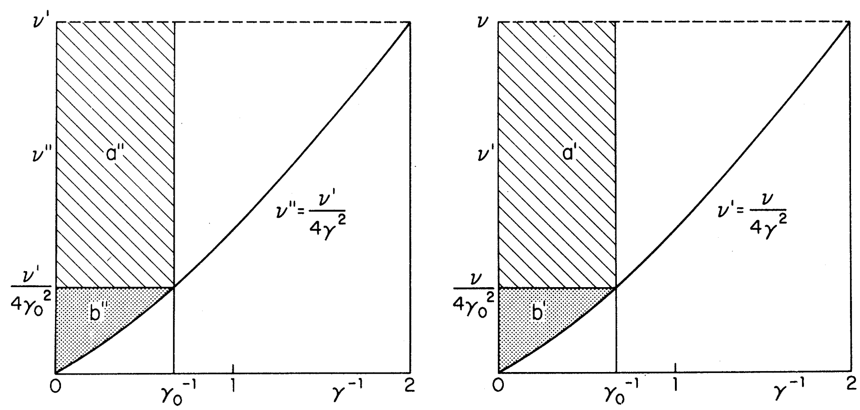

FIG. 8. Illustration of the four-dimensional integration-variable space necessary for the evaluation of the intensity of twice-scattered Compton radiation. The four terms $\bar{I}_\nu(i)$, $i=1,\ldots,4$, of Eq. (46) represent four-dimensional integrations over the following regions: The $\bar{I}_\nu(1)$ integration is over region $a''\times a'$; the $\bar{I}_\nu(2)$ integration is over region $b''\times a'$; the $\bar{I}_\nu(3)$ integration is over $a''\times b'$; and the $\bar{I}_\nu(4)$ integration is over the region $b''\times b'$.

$$\bar{I}_\nu(n) \propto \int_0^d e^{-\kappa_\nu s}\frac{s^n}{n!}ds$$
$$= \frac{1-e^{-\kappa_\nu d}}{\kappa_\nu}\left[d^n + \frac{n d^{n-1}}{\kappa_\nu} + \frac{n(n-1)d^{n-2}}{\kappa_\nu{}^2} + \cdots\right]. \tag{47}$$

The n successive integrations of n vertex functions $\tilde{\Lambda}(\nu, \nu')$ over frequency are considerably more complicated, but we can compute the high-frequency behavior of the multiple-scattering process. This is made possible because the function

$$\Lambda(\nu,\nu')^{\infty}_{(\nu/\nu')^{1/2}/2} = F(\alpha)\,\frac{1}{\nu'}\left(\frac{\nu'}{\nu}\right)^{(\alpha+1)/2} \tag{48}$$

given by Eq. (34) is a relatively simple function of ν'. For the n-scattering process in the frequency region $\nu \geqslant (4\gamma_0{}^2)^n\,\nu_0$ all of the vertex functions $\tilde{\Lambda}(\nu, \nu')$ required in the frequency integrations have the form $\Lambda(\nu,\nu')^{\infty}_{(\nu/\nu')^{1/2}/2}$ so that the Compton intensity resulting from n scatterings is

$$\bar{I}_\nu(n) = \int_{(4\gamma_0{}^2)^{n-1}\nu_0}^{\nu/4\gamma_0{}^2}\int_{(4\gamma_0{}^2)^{n-2}\nu_0}^{\nu_1/4\gamma_0{}^2}\cdots\int_0^{\nu_{n-1}/4\gamma_0{}^2}\rho\delta(\nu_n-\nu_0)\left[F(\alpha)\,\frac{1}{\nu_n}\left(\frac{\nu_n}{\nu_{n-1}}\right)^{(\alpha+1)/2}\cdots F(\alpha)\,\frac{1}{\nu_1}\left(\frac{\nu_1}{\nu}\right)^{(\alpha+1)/2}\right]$$
$$\times\, d\nu_n\, d\nu_{n-1}\cdots d\nu_1 \tag{49}$$

for $\nu \geqslant (4\gamma_0{}^2)^n\,\nu_0$. These integrals can now be evaluated directly by the transformation shown in Appendix B, with the result that the intensity of Compton-scattered radiation after n scatterings is

$$\bar{I}_\nu(n) = \rho F^n(\alpha)\,\frac{1}{(n-1)!}\,\frac{1}{\nu_0}\left(\frac{\nu_0}{\nu}\right)^{(\alpha+1)/2}$$
$$\times\left[\ln\frac{\nu}{(4\gamma_0{}^2)^n\,\nu_0}\right]^{(n-1)}, \tag{50}$$

and this result holds for all $\nu \geqslant (4\gamma_0{}^2)^n\,\nu_0$. The principal result to be emphasized here is that the inverse power-law dependence of the spectrum of scattered radiation at high frequencies is a characteristic preserved through arbitrarily many scatterings. This result will hold until the energy of the photon is high enough to initiate pair-production $\gamma \to e^+ + e^-$.

A discussion of the astrophysical applications of these results is presented elsewhere.

ACKNOWLEDGMENTS

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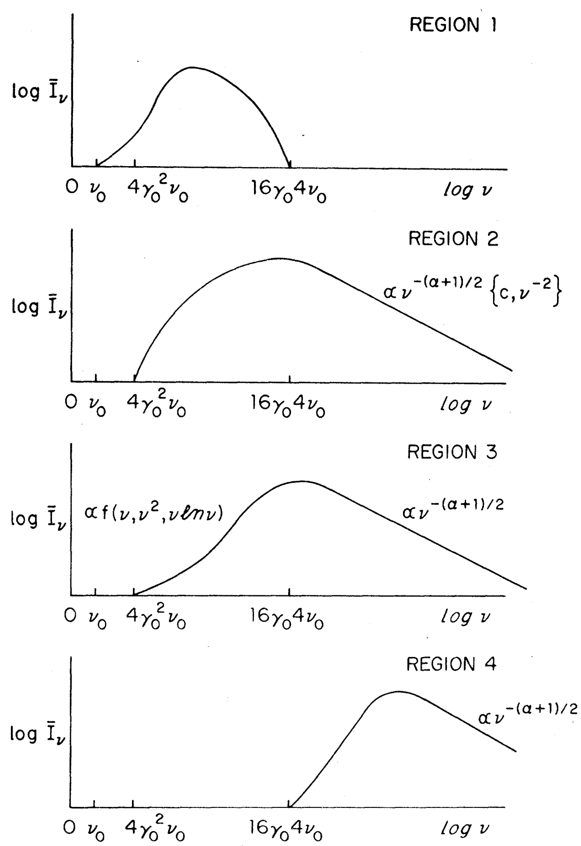

FIG. 9. Qualitative representation of the spectral intensity of twice-scattered Compton radiation evaluated and plotted separately for each of the terms of Eq. (46); these four contributions are discussed separately in Appendix A.

APPENDIX A: FREQUENCY INTEGRATION OF THE DOUBLE-SCATTERING PROCESS

The total intensity of double-Compton-scattered radiation is expressed above as a sum of four contributions (46) that represent integrations over the volumes described in Fig. 8. We will discuss each of these contributions separately.

(i) Region 1. In this region we consider the integral

$$\bar{I}_\nu(1) \propto \int_{\nu/4\gamma_0{}^2}^{\nu}\Lambda(\nu,\nu')^{\infty}_{\gamma_0}\int_{\nu'/4\gamma_0{}^2}^{\nu}(\nu',\nu'')^{\infty}_{\gamma_0}$$
$$\times\,\rho\delta(\nu''-\nu_0)d\nu''d\nu'. \tag{A1}$$

The limits of integration define the relevant surface, viz.,

$$\nu \geq \nu' \geq \nu/4\gamma_0^2$$

and (A2)

$$4\gamma_0^2\nu_0 \geq \nu' \geq \nu_0,$$

which may be combined to give limits on ν and ν' individually:

$$\min(\nu, 4\gamma_0^2\nu_0) \geq \nu' \geq \max(\nu_0, \nu/4\gamma_0^2), \quad \text{(A3)}$$

$$16\gamma_0^4\nu_0 \geq \nu \geq \nu_0. \quad \text{(A4)}$$

Therefore,

$$\bar{I}_\nu(1) \propto \rho_0 \int_p^q \Lambda(\nu, \nu')_{\gamma_0}^{\infty} \Lambda(\nu', \nu_0)_{\gamma_0}^{\infty} d\nu' \quad \text{(A5)}$$

so that for different regions of frequency ν when

$$4\gamma_0^2\nu_0 \geq \nu \geq \nu_0$$

and (A3) implies (A6)

$$\nu \geq \nu' \geq \nu_0,$$

the integration limits $(p, q) = (\nu_0, \nu)$, whereas when

$$16\gamma_0^4\nu_0 \geq \nu \geq 4\gamma_0^2\nu_0$$

and (A7)

$$4\gamma_0^2\nu_0 \geq \nu' \geq \nu/4\gamma_0^2,$$

the integration limits are $(q, p) = (4\gamma_0^2\nu_0, \nu/4\gamma_0^2)$. The integral vanishes for $\nu \leq \nu_0$ and $\nu \geq 16\gamma_0^4\nu_0$ as shown in Fig. 8, corresponding to the volume designated by the number 1. The resultant spectrum $\bar{I}_\nu(1)$ illustrated in Fig. 9 is an increasing polynomial in ν, ν^2, $1/\nu$, and $\int(1/\nu')\ln^2\nu' d\nu'$ from $\nu = \nu_0$ to $\nu = 4\gamma_0^2\nu_0$, but it decreases abruptly to zero in the interval $\nu = 4\gamma_0^2\nu_0$ to $\nu = 16\gamma_0^4\nu_0$ because the integration interval (p, q) degenerates to zero at the upper limit.

(ii) Region 2. The integral here is

$$\bar{I}_\nu(2) \propto \int_{\nu/4\gamma_0^2}^{\nu} \Lambda(\nu, \nu')_{\gamma_0}^{\infty} \times \int_0^{\nu'/4\gamma_0^2} \Lambda(\nu', \nu'')_{(\nu'/\nu'')^{1/2}/2}^{\infty} \times \rho\delta(\nu'' - \nu_0) d\nu'' d\nu'. \quad \text{(A8)}$$

This is the region where

$$\nu \geq \nu' \geq \nu/4\gamma_0^2, \quad \nu' \geq 4\gamma_0^2\nu_0, \quad \text{(A9)}$$

which implies

$$\nu \geq \nu' \geq \max(4\gamma_0^2\nu_0, \nu/4\gamma_0^2), \quad \text{(A10)}$$

$$\nu \geq 4\gamma_0^2\nu_0, \quad \text{(A11)}$$

hence

$$\bar{I}_\nu(2) \propto \rho \int_p^q \Lambda(\nu, \nu')_{\gamma_0}^{\infty} \Lambda(\nu', \nu_0)_{(\nu'/\nu_0)^{1/2}/2}^{\infty} d\nu', \quad \text{(A12)}$$

where

$$16\gamma_0^4\nu_0 \geq \nu \geq 4\gamma_0^2\nu_0$$

and from (A3) (A13)

$$\nu \geq \nu' \geq 4\gamma_0^2\nu_0$$

so that $p = 4\gamma_0^2\nu_0$ and $q = \nu$, or

$$\nu \geq 16\gamma_0^4\nu_0$$

and (A14)

$$\nu \geq \nu' \geq \nu/4\gamma_0^2$$

so that $p = \nu/4\gamma_0^2$ and $q = \nu$. The integral vanishes for $\nu \leq 4\gamma_0^2\nu_0$ as shown in Fig. 9. Integrating (A12) it can be seen that $\bar{I}_\nu(2)$ behaves as $\nu^{-(\alpha+1)/2}f(\nu^{-2})$ so that it increases from zero to a maximum between $4\gamma_0^2\nu_0$ and $16\gamma_0^4\nu_0$ as the integration interval (p, q) opens up from zero to $(\nu, \nu/4\gamma_0^2)$; the integral then falls off as $\nu^{-(\alpha+1)/2}$ for $\nu \geq 16\gamma_0^2\nu_0$.

(iii) Region 3.

$$\bar{I}_\nu(3) \propto \int_0^{\nu/4\gamma_0^2} \Lambda(\nu, \nu')_{(\nu/\nu')^{1/2}/2}^{\infty} \times \int_{\nu'/4\gamma_0^2}^{\nu'} \Lambda(\nu', \nu'')_{\gamma_0}^{\infty} \times \rho\delta(\nu'' - \nu_0) d\nu'' d\nu'; \quad \text{(A15)}$$

this is the region indicated by the integration limits

$$\nu/4\gamma_0^2 \geq \nu', \quad 4\gamma_0^2\nu_0 \geq \nu' \geq \nu_0, \quad \text{(A16)}$$

which may be combined to yield

$$\min(\nu/4\gamma_0^2, 4\gamma_0^2\nu_0) \geq \nu' \geq \nu_0; \quad \nu \geq 4\gamma_0^2\nu_0. \quad \text{(A17)}$$

Therefore we have

$$\bar{I}_\nu(3) \propto \rho \int_p^q \Lambda(\nu, \nu')_{(\nu/\nu')^{1/2}/2}^{\infty} \Lambda(\nu', \nu_0)_{\gamma_0}^{\infty} d\nu', \quad \text{(A18)}$$

where in the interval

$$16\gamma_0^4\nu_0 \geq \nu \geq 4\gamma_0^2\nu_0$$

and from (A3) (A19)

$$\nu/4\gamma_0^2 \geq \nu' \geq \nu_0$$

the limits are $(p, q) = (\nu_0, \nu/4\gamma_0^2)$, while in the interval

$$\nu \geq 16\gamma_0^4\nu_0$$
and
$$4\gamma_0^2 \geq \nu' \geq \nu_0 \tag{A20}$$

the limits are $(p, q) = (\nu_0, 4\gamma_0^2\nu_0)$. This integral vanishes for $\nu \leq 4\gamma_0^2\nu_0$ as shown in Fig. 9. It can also be seen in this figure that $\bar{I}_\nu(3)$ behaves as an increasing polynomial in ν between $\nu = 4\gamma_0^2\nu_0$ and $\nu = 16\gamma_0^4\nu_0$, but at still higher frequencies $\nu \geq 16\gamma_0^4\nu_0$, $\bar{I}_\nu(3)$ has the characteristic inverse power-law form $\propto \nu^{-(\alpha+1)/2}$.

(iv) Region 4.

$$\bar{I}_\nu(4) \propto \int_0^{\nu/4\gamma_0^2} \Lambda(\nu, \nu')_{(\nu/\nu')^{1/2}/2}^{\infty} \times \int_0^{\nu'/4\gamma_0^2} \Lambda(\nu', \nu'')_{(\nu'/\nu'')^{1/2}/2}^{\infty} \times \rho\delta(\nu'' - \nu_0) d\nu'' d\nu' ; \tag{A21}$$

in this region the integration limits are

$$\nu/4\gamma_0^2 \geq \nu'$$
and
$$\nu' \geq 4\gamma_0^2\nu_0 \tag{A22}$$

so that

$$\nu/4\gamma_0^2 \geq \nu' \geq 4\gamma_0^2\nu_0, \quad \nu \geq 16\gamma_0^4\nu_0 . \tag{A23}$$

Thus

$$\bar{I}_\nu(4) \propto \rho \int_p^q \Lambda(\nu, \nu')_{(\nu/\nu')^{1/2}/2}^{\infty} \Lambda(\nu', \nu)_{(\nu'/\nu_0)^{1/2}/2}^{\infty} d\nu' , \tag{A24}$$

where the limits are simply $(p, q) = (4\gamma_0^2\nu_0, \nu/4\gamma_0^2)$ and from (A23) the integral vanishes for $\nu \leq 16\gamma_0^4\nu_0$ as shown in Fig. 9. Here $\bar{I}_\nu(4)$ behaves like $\nu^{-(\alpha+1)/2}$, increasing from $16\gamma_0^4\nu_0$ due to the logarithm term, and then eventually falls off at higher frequencies to provide the dominant high-frequency contribution to the intensity of twice-scattered Compton radiation.

APPENDIX B: THE MULTIPLE-SCATTERING INTEGRAL

The product of multiple-scattering integrals shown in Eq. (49) can be simplified as follows: Since

$$\int_{(4\gamma_0^2)^j\nu_0}^{\nu_{n-j-1}/4\gamma_0^2} \left[\ln \frac{\nu_{n-j}}{(4\gamma_0^2)^j \nu_0} \right]^{j-1} \nu_{n-j}^{-1} d\nu_{n-j} = \frac{1}{j} \left[\ln \frac{\nu_{n-j-1}}{(4\gamma_0^2)^{j+1}\nu_0} \right]^j , \tag{B1}$$

that is,

$$\int \frac{(\ln x)^n}{x} dx = \frac{1}{n+1} (\ln x)^{n+1} , \tag{B2}$$

we may write, for example,

$$\int \cdots \int_{16\gamma_0^4\nu_0}^{\nu_{n-3}/4\gamma_0^2} \int_{4\gamma_0^2\nu_0}^{\nu_{n-2}/4\gamma_0^2} \frac{1}{\nu_{n-1}} \frac{1}{\nu_{n-2}} \cdots d\nu_{n-1} d\nu_{n-2} \cdots d\nu_1 = \int \cdots \int_{16\gamma_0^2\nu_0}^{\nu_{n-3}/4\gamma_0^2} \nu_{n-2}^{-1} \ln \frac{\nu_{n-2}}{16\gamma_0^4\nu_0} \cdots d\nu_{n-2} \cdots d\nu_1 \tag{B3}$$

$$= \int \cdots \tfrac{1}{2} \nu_{n-3}^{-1} \left[\ln \frac{\nu_{n-3}}{64\gamma_0^6\nu_0} \right]^2 \cdots d\nu_{n-3} \cdots d\nu_1 . \tag{B4}$$

Repeating this procedure $(n-1)$ times we arrive at the expression (50):

$$\bar{I}_\nu(n) = \rho F^n(\alpha) \frac{1}{(n-1)!} \frac{1}{\nu_0} \left(\frac{\nu_0}{\nu} \right)^{(\alpha+1)/2} \left[\ln \frac{\nu}{(4\gamma_0^2)^n \nu_0} \right]^{n-1} . \tag{B5}$$

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Electromagnetic Waves in an Expanding Universe

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The electromagnetic perturbation equations are solved in a closed Robertson-Walker space-time; the frequency spectrum and the proper modes are found. A simple quantum theory of electromagnetic waves in the background metric is then developed using the proper modes of the static metric conformally connected with the closed Robertson-Walker metric. It is shown that the number of "photons" does not change as a result of expansion (or contraction). The application of these results to the cosmic background radiation is discussed.

I. INTRODUCTION

In an attempt to connect the quantum nature of matter and light with the curvature of space-time Schrödinger[1] pointed out an interesting phenomenon. In an expanding (or contracting) universe, matter could be created or annihilated merely by expansion (or contraction), whereas with light there would be a reflection of electromagnetic waves in space. The question of the backscattering of light by expansion (or contraction) has been investigated further by Parker[2] in a classical context. It is the purpose of this paper to present a simple quantum theory of electromagnetic waves in an expanding model universe given by the Robertson-Walker metric (with positive spatial curvature) and prove that the corresponding "photon" number operator is independent of time. This is a partial generalization of Parker's result that in Robertson-Walker space-times there is no backscattering of light due to expansion (or contraction). As also noted by Parker,[2] the basic reason for the absence of "photon" creation or annihilation is the conformal invariance of Maxwell's equations and the fact that expansion (or contraction) is isotropic in a Robertson-Walker space-time. When the relative expansion rates for different spatial directions vary with time, the number of photons is expected to change also.

The perturbations of the expanding model universe under consideration have been studied before.[3] However, we present an explicitly conformally invariant formalism which is convenient for quantization. The frequency spectrum and the eigenfunctions of the proper modes are found in the comoving coordinate frame, and a simple quantization procedure based on the usual Dirac method is proposed. An important defect of our method is its explicit dependence on the comoving coordinate frame. It is well known[4] that difficulties arise when attempts are made to give generally covariant quantization rules.

Finally, we discuss the application of our results to the cosmic background radiation and show that the frequency spectrum thus obtained is the same as the Planck spectrum for usual frequencies.

II. ELECTROMAGNETIC PERTURBATIONS

The covariant Maxwell's equations for electromagnetic waves in a gravitational field are[5]

$$F^{\mu\nu}{}_{;\nu}=0$$

and (1)

$$F_{\mu\nu;\sigma}+F_{\nu\sigma;\mu}+F_{\sigma\mu;\nu}=0\,.$$

In a given coordinate frame where the background metric is $g_{\mu\nu}$, $-ds^2=g_{\mu\nu}dx^\mu dx^\nu$, these equations can be written as

$$[(-g)^{1/2}F^{\mu\nu}]_{,\nu}=0$$

and (2)

$$F_{\mu\nu,\sigma}+F_{\nu\sigma,\mu}+F_{\sigma\mu,\nu}=0\,,$$

where $g=\mathrm{Det}(g_{\mu\nu})$.