Effective Homogeneity of Some Inhomogeneous Cosmologies*

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The Hamiltonian methods used to quantize spherically symmetric, dust-filled systems in general relativity are applied to the locally rotationally symmetric inhomogeneous generalizations of Kantowski-Sachs and Bianchi type-I models. As in the case of spherical symmetry, the ∞ degrees of freedom decouple. Moreover, both systems behave as if they were effectively homogeneous since inhomogeneities are "homogeneously propagated."

Dust-filled systems in general relativity have been studied in great detail by Ellis.[1] In particular, he has considered locally rotationally symmetric[2] space-times, and has found solutions to the field equations which generalize the Kantowski-Sachs and Bianchi type-I cosmological models in that they have inhomogeneous space sections. The purpose of this note is to apply the methods used in quantizing spherically symmetric dust-filled geometries[3] to these new inhomogeneous models. As in paper I, the canonical form given by Schutz[4] to the action principle for a perfect fluid is specialized to pressure-free dust. The specialization to spherical symmetry is now replaced by a restriction to local rotational symmetry. The conclusion will be that, although inhomogeneous, these geometries effectively behave as if they were homogeneous, since they are completely described by a finite numer of degrees of freedom.

We are interested in what Ellis[1] calls case IIb. The spatial line element may then be written as[5]

$$dl^2 = X^2(t, x^1)(dx^1)^2 + Y^2(t)\left[(dx^2)^2 + q^2(x^2)(dx^3)^2\right] \ , \qquad (1)$$

where $q^2(x^2) = \sin^2(k^{1/2}x^2)$ (with k an arbitrary constant) corresponds to the generalized Kantowski-Sachs geometry and $q^2(x^2) = 1$ corresponds to the generalized Bianchi type-I model.

Consider first the generalized Kantowski-Sachs model, with $k = 1$ for convenience. The case $k \neq 1$ is straightforward and does not add any new features. With the definition

$$\pi^{ik} =: \mathrm{diag}\left(\tfrac{1}{2}X^{-1}\pi_X,\ \tfrac{1}{4}Y^{-1}\pi_Y,\ \tfrac{1}{4}Y^{-1}\pi_Y \sin^{-2}(x^2)\right)$$

the action functional for the system [cf. expression (1) of I] becomes

$$S = 4\pi\int dx^1 dt\left[p^\phi\dot\phi + \pi_X\dot X + \pi_Y\dot Y - N(\mathcal{H}^0 + \mathcal{E}) - N_1(\mathcal{H}^1 + \mathcal{P}^1)\right] \ , \qquad (2)$$

where the integrations over x^2 and x^3 have been explicitly carried out, and with

$$\mathcal{H}^0 = X^{-1}Y^{-2}\left(\tfrac{1}{8}X^2\pi_X{}^2 - \tfrac{1}{4}XY\pi_X\pi_Y - 2X^2Y^2\right) \ , \qquad (3)$$

$$\mathcal{H}^1 = -X\pi_X{}' \ , \qquad (4)$$

$$\mathcal{E} = p^\phi\left[1 - X^{-2}(\phi')^2\right]^{1/2} \ , \qquad (5)$$

$$\mathcal{P}^1 = X^{-2}p^\phi\phi' \ . \qquad (6)$$

As a consequence of the symmetry, one has $\mathcal{H}^2 + \mathcal{P}^2 = \mathcal{H}^3 + \mathcal{P}^3 \equiv 0$. Here, dot and prime denote differentiation with respect to time and x^1, respectively.

Following I, we make the coordinate condition $\phi = t$ and solve the constraints $\mathcal{H}^0 + \mathcal{E} = 0$ and $\mathcal{H}^1 + \mathcal{P}^1 = 0$. Since $\phi' = 0$, the second of these implies $\pi_X{}' = 0$. The first one then reads

$$-p^\phi = \tfrac{1}{8}Y^{-2}X\pi_X{}^2 - \tfrac{1}{4}Y^{-1}\pi_X\pi_Y - 2X \ , \qquad (7)$$

and the action (2) becomes

$$S = 4\pi\int dx^1\, dt\,(\pi_X\dot X + \pi_Y\dot Y - \mathcal{H}) \ , \qquad (8)$$

with $\mathcal{H} = -p^\phi$. This Hamiltonian, like the Hamiltonian for the Tolman-Bondi models,[3] is time-independent and does not involve derivatives of the canonical variables. The ∞ degrees of freedom again decouple from one another.

Make now the restriction $0 \leq x^1 \leq 1$ and consider the new variables

$$\overline{X}(t) = \int_0^1 dx^1\, X(t, x^1) \qquad (9)$$

and

$$\overline{\pi}_Y(t) = \int_0^1 dx^1\, \pi_Y(t, x^1) \ . \qquad (10)$$

In terms of them, we have now a system with just *two* degrees of freedom, described by the action

$$S = 4\pi\int dt\left(\pi_{\overline{X}}\frac{d\overline{X}}{dt} + \overline{\pi}_Y\frac{dY}{dt} - H\right), \qquad (11)$$

with $H = \frac{1}{8}Y^{-2}\overline{X}\pi_{\overline{X}}{}^2 - \frac{1}{4}Y^{-1}\pi_{\overline{X}}\overline{\pi}_Y - 2\overline{X}$, which is the

Hamiltonian for the homogeneous Kantowski-Sachs universe with dust.

Consider now the generalized Bianchi type-I model. In this case we define

$$\pi^{ik} =: \mathrm{diag}(\tfrac{1}{2} X^{-1}\pi_X, \tfrac{1}{4} Y^{-1}\pi_Y, \tfrac{1}{4} Y^{-1}\pi_Y)$$

and we obtain

$$\mathcal{H}^0 = X^{-1} Y^{-2} (\tfrac{1}{8} X^2 \pi_X{}^2 - \tfrac{1}{4} XY\pi_X\pi_Y) \ . \tag{12}$$

The expressions for $\mathcal{H}^1$, $\mathcal{E}$, and $\mathcal{P}^1$ are just (4), (5), and (6), respectively. Making again the transformations (9) and (10) one obtains (11), this time with $H = \frac{1}{8} Y^{-2}\overline{X}\pi_{\overline{X}}{}^2 - \frac{1}{4} Y^{-1}\pi_{\overline{X}}\overline{\pi}_Y$, which is the Hamiltonian for the homogeneous dust-filled Bianchi type-I model.

It is surprising to find an inhomogeneous system which turns out to have only a finite number of true dynamical degrees of freedom. This is of course not the case in general, and depends crucially on the way in which the space-dependent functions enter the integrand in (8). (In this case, X and π_Y enter linearly.) This behavior can be understood by noticing that the inhomogeneities carried by $X(t, x^1)$ are, loosely speaking, homogeneously propagated due to the constraint $\pi_X{}' = 0$. The inhomogeneities of this geometry can be prescribed initially and will be maintained throughout its evolution, so they do not enter the dynamical description of the system. To get a picture of the situation, remember that the homogeneous Kantowski-Sachs universe is a three-cylinder of finite length getting longer and thinner until it collapses to zero volume. The inhomogeneities considered here consist in introducing (for $K=1$) spherically symmetric bumps on this cylinder. These bumps then preserve their shape as the cylinder evolves, thus behaving for all practical purposes as a homogeneous system.

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[1]G. F. R. Ellis, J. Math. Phys. 8, 1171 (1967).

[2]A space-time is called locally rotationally symmetric if at each point p in an open neighborhood U of a point p_0 there exists a nondiscrete subgroup g of the Lorentz group in the tangent space T_p which leaves invariant the curvature tensor and all its covariant derivatives to the third order (cf. Ref. 1).

[3]F. Lund, Phys. Rev. D 8, 3253 (1973), hereafter referred to as I.

[4]B. F. Schutz, Jr., Phys. Rev. D4, 3559 (1971).

[5]It is also possible to consider metrics of the form

$$dl^2 = X^2(t, x^1)(dx^1)^2 + Y^2(t, x^1)\,[(dx^2)^2 + \sin^2(k^{1/2}x^2)(dx^3)^2].$$

They are, however, trivial generalizations of the Tolman-Bondi models considered in I and do not add anything new to them.