$N = 4$ supersymmetric Yang-Mills multiplet in nonadjoint representations

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We formulate a theory for an $N = 4$ supersymmetric Yang-Mills multiplet in a nonadjoint representation R of $SO(\mathcal{N})$, as an important application of our recently proposed model for $N = 1$ supersymmetry. This system is obtained by dimensional reduction from an $N = 1$ supersymmetric Yang-Mills multiplet in a nonadjoint representation in ten dimensions. The consistency with supersymmetry requires that a nonadjoint representation R with the indices $i, j, \ldots$ satisfy the three conditions $\eta^{ij} = \delta^{ij}$, $(T^I)^{ij} = -(T^I)^{ji}$, and $(T^I)^{[ij|}(T^I)^{|k]l} = 0$ for the metric η^{ij} and the generators T^I, which are the same as the $N = 1$ case.

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I. INTRODUCTION

The importance of $N = 4$ extended supersymmetry in four dimensions (4D) [1] is associated with its all-order finiteness [2], and also its natural link with superstring theories in 4D [3]. Moreover, there is an important duality between $N = 4$ supersymmetric Yang-Mills theory in 4D and IIB string theory in 10D compactified on $\mathrm{AdS}_5 \times S_5$ [4].[1]

In the conventional formulation of $N = 1$ supersymmetry in 4D, a vector multiplet is supposed to be in the adjoint representation, such as $(A_\mu{}^I, \lambda^I)$ carrying the common adjoint index I [1]. However, we have recently shown [6] that this is not necessarily the case, by constructing an explicit $N = 1$ Yang-Mills multiplet in a nonadjoint representation. We have shown that the multiplet $(B_\mu{}^i, \chi^i)$ with the nonadjoint real representation index i can consistently couple to the conventional Yang-Mills multiplet $(A_\mu{}^I, \lambda^I)$. Such a nonadjoint real representation R should satisfy certain conditions [6] for the system to be consistent with supersymmetry [cf. (2.1) below].

In this paper, we show that the $N = 1$ formulation in [6] can be further generalized to extended $N = 4$ supersymmetry. In addition to the conventional $N = 4$ supersymmetric Yang-Mills multiplet $(A_\mu{}^I, \lambda_{(i)}{}^I, A_\alpha{}^I, \tilde{A}_\alpha{}^I)$ $[\alpha = 1, 2, 3; (i) = 1, 2, 3, 4]$, we can consider the additional vector multiplet $(B_\mu{}^i, \chi_{(i)}{}^i, B_\alpha{}^i, \tilde{B}_\alpha{}^i)$ carrying the index i for a particular nonadjoint real representation R. As explained in the case of $N = 1$ [6], we have to maintain the conventional Yang-Mills multiplet in the adjoint representation, once we introduce the extra vector multiplet in the nonadjoint representation R.

It seems to be a prevailing notion that $N = 4$ supersymmetric Yang-Mills theory in 4D has the "unique" field content all in the adjoint representation of a certain gauge group. For example, the first sentence of Sec. 3 in [5] states that "The Lagrangian for the $N = 4$ super-Yang-Mills theory is unique." In our present paper, we establish a counterexample against the prevailing notion of the "uniqueness" of $N = 4$ supersymmetric Yang-Mills theory in 4D.

II. THE LAGRANGIAN

As has been mentioned, our system has two $N = 4$ vector multiplets $(A_\mu{}^I, \lambda_{(i)}{}^I, A_\alpha{}^I, \tilde{A}_\alpha{}^I)$ and $(B_\mu{}^i, \chi_{(i)}{}^i, B_\alpha{}^i, \tilde{B}_\alpha{}^i)$. The indices $(i), (j), \ldots = 1, 2, 3, 4$ are for $N = 4$ supersymmetry, while the indices $\alpha, \beta, \ldots = 1, 2, 3$ are used for the three scalars and three pseudoscalars [1]. The former multiplet is the conventional $N = 4$ supersymmetric Yang-Mills vector multiplet [1] with the adjoint index I of the gauge group $SO(\mathcal{N})$. The latter multiplet is our new vector multiplet carrying the indices $i, j, \ldots$ for the nonadjoint representation R of $SO(\mathcal{N})$, which satisfies the conditions

$$\eta^{ij} = \delta^{ij}, \qquad (T^I)^{ij} = -(T^I)^{ji}, \tag{2.1a}$$

$$(T^I)^{[ij|}(T^I)^{|k]l} \equiv 0, \tag{2.1b}$$

where η^{ij} and $(T^I)^{ij}$ are the metric of the representation R, and the representation matrix of the generators of $SO(\mathcal{N})$, respectively.

We can obtain our Lagrangian for $N = 4$ supersymmetry for these two multiplets by the simple dimensional reduction [7] of $N = 1$ supersymmetric Yang-Mills theory in 10D in the nonadjoint representation, as outlined in [6]. Our Lagrangian thus obtained in 4D is

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[1]For reviews, see, e.g., Ref. [5].

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(\mathcal{F}_{\mu\nu}{}^I)^2 - \frac{1}{4}(G_{\mu\nu}{}^i)^2 + \frac{1}{2}(\bar{\lambda}^I \not{\mathcal{D}}\lambda^I) + \frac{1}{2}(\bar{\chi}^i \not{\mathcal{D}}\chi^i) - \frac{1}{2}(\mathcal{D}_\mu A_\alpha{}^I)^2 - \frac{1}{2}(\mathcal{D}_\mu \tilde{A}_\alpha{}^I)^2 - \frac{1}{2}(\mathcal{D}_\mu B_\alpha{}^i)^2 - \frac{1}{2}(\mathcal{D}_\mu \tilde{B}_\alpha{}^i)^2 \\ &- \frac{i}{2} g f^{IJK}(\bar{\lambda}^I \alpha_\alpha \lambda^J) A_\alpha{}^K - \frac{1}{2} g f^{IJK}(\bar{\lambda}^I \gamma_5 \beta_\alpha \lambda^J)\tilde{A}_\alpha{}^K + ig(T^I)^{ij}(\bar{\lambda}^I \alpha_\alpha \chi^i) B_\alpha{}^j + g(T^I)^{ij}(\bar{\lambda}^I \gamma_5 \beta_\alpha \chi^i)\tilde{B}_\alpha{}^j \\ &+ \frac{i}{2}(T^I)^{ij}(\bar{\chi}^i \alpha_\alpha \chi^j) A_\alpha{}^I + \frac{1}{2}(T^I)^{ij}(\bar{\chi}^i \gamma_5 \beta_\alpha \chi^j)\tilde{A}_\alpha{}^I - \frac{1}{4} g^2 [f^{IJK} A_\alpha{}^J A_\beta{}^K - (T^I)^{ij} B_\alpha{}^i B_\beta{}^j]^2 \\ &- \frac{1}{4} g^2 [f^{IJK} \tilde{A}_\alpha{}^J \tilde{A}_\beta{}^K - (T^I)^{ij} \tilde{B}_\alpha{}^i \tilde{B}_\beta{}^j]^2 - \frac{1}{2} g^2 [f^{IJK} A_\alpha{}^J \tilde{A}_\beta{}^K - (T^I)^{ij} B_\alpha{}^i \tilde{B}_\beta{}^j]^2 - \frac{1}{4} g^2 [(T^I)^{ij}(A_\alpha{}^I B_\beta{}^j - A_\beta{}^I B_\alpha{}^j)]^2 \\ &- \frac{1}{4} g^2 [(T^I)^{ij}(\tilde{A}_\alpha{}^I \tilde{B}_\beta{}^j - \tilde{A}_\beta{}^I \tilde{B}_\alpha{}^j)]^2 - \frac{1}{2} g^2 [(T^I)^{ij}(A_\alpha{}^I \tilde{B}_\beta{}^j - \tilde{A}_\beta{}^I B_\alpha{}^j)]^2. \end{aligned} \tag{2.2}$$

The 4×4 antisymmetric matrices α and β satisfy the conditions for $\alpha, \beta, \ldots = 1, 2, 3$:

$$\alpha_\alpha \alpha_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\alpha_\gamma, \qquad \beta_\alpha \beta_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\beta_\gamma, \qquad [\alpha_\alpha, \beta_\beta] = 0, \tag{2.3}$$

which are $SO(3)$ matrices, and used for the global $SO(4) \approx SO(3) \times SO(3)$ [1]. Similarly to [6], our field strengths and covariant derivatives are defined by

$$\mathcal{F}_{\mu\nu}{}^I \equiv 2\partial_{[\mu} A_{\nu]}{}^I + g f^{IJK} A_\mu{}^J A_\nu{}^K - g(T^I)^{ij} B_\mu{}^i B_\nu{}^j, \tag{2.4a}$$
$$G_{\mu\nu}{}^i \equiv 2\partial_{[\mu} B_{\nu]}{}^i + 2g(T^I)^{ij} A_{[\mu}{}^I B_{\nu]}{}^j, \tag{2.4b}$$
$$\mathcal{D}_\mu \chi^i \equiv \partial_\mu \chi^i + g(T^I)^{ij} A_\mu{}^I \chi^j - g(T^I)^{ij} B_\mu{}^j \lambda^I, \tag{2.4c}$$
$$\mathcal{D}_\mu \lambda^I \equiv \partial_\mu \lambda^I + g f^{IJK} A_\mu{}^J \lambda^K - g(T^I)^{ij} B_\mu{}^i \chi^j, \tag{2.4d}$$
$$\mathcal{D}_\mu A_\alpha{}^I \equiv \partial_\mu A_\alpha{}^I + g f^{IJK} A_\mu{}^J A_\alpha{}^K - g(T^I)^{ij} B_\mu{}^i B_\alpha{}^j, \tag{2.4e}$$
$$\mathcal{D}_\mu \tilde{A}_\alpha{}^I \equiv \partial_\mu \tilde{A}_\alpha{}^I + g f^{IJK} A_\mu{}^J \tilde{A}_\alpha{}^K - g(T^I)^{ij} B_\mu{}^i \tilde{B}_\alpha{}^j, \tag{2.4f}$$
$$\mathcal{D}_\mu B_\alpha{}^i \equiv \partial_\mu B_\alpha{}^i + g(T^I)^{ij} A_\mu{}^I B_\alpha{}^j - g(T^I)^{ij} A_\alpha{}^I B_\mu{}^j, \tag{2.4g}$$
$$\mathcal{D}_\mu \tilde{B}_\alpha{}^i \equiv \partial_\mu \tilde{B}_\alpha{}^i + g(T^I)^{ij} A_\mu{}^I \tilde{B}_\alpha{}^j - g(T^I)^{ij} \tilde{A}_\alpha{}^I B_\mu{}^j. \tag{2.4h}$$

Our action $I \equiv \int d^4x \mathcal{L}$ is invariant under supersymmetry,

$$\delta_Q A_\mu{}^I = +(\bar{\epsilon}\gamma_\mu \lambda^I), \qquad \delta_Q B_\mu{}^i = +(\bar{\epsilon}\gamma_\mu \chi^i), \tag{2.5a}$$
$$\delta_Q A_\alpha{}^I = +i(\bar{\epsilon}\alpha_\alpha \lambda^I), \qquad \delta_Q B_\alpha{}^I = +i(\bar{\epsilon}\alpha_\alpha \chi^i), \tag{2.5b}$$
$$\delta_Q \tilde{A}_\alpha{}^I = +(\bar{\epsilon}\gamma_5 \beta_\alpha \lambda^I), \qquad \delta_Q \tilde{B}_\alpha{}^I = +(\bar{\epsilon}\gamma_5 \beta_\alpha \chi^i), \tag{2.5c}$$

$$\begin{aligned}\delta_Q \lambda^I = &+\frac{1}{2}(\gamma^{\mu\nu}\epsilon)\mathcal{F}_{\mu\nu}{}^I + i(\alpha_\alpha \gamma^\mu \epsilon)\mathcal{D}_\mu A_\alpha{}^I - (\beta_\alpha \gamma_5 \gamma^\mu \epsilon)\mathcal{D}_\mu \tilde{A}_\alpha{}^I + \frac{i}{2} g\epsilon_{\alpha\beta\gamma}(\alpha_\gamma \epsilon)[f^{IJK} A_\alpha{}^J A_\beta{}^K - (T^I)^{ij} B_\alpha{}^i B_\beta{}^j] \\ &+ \frac{i}{2} g\epsilon_{\alpha\beta\gamma}(\beta_\gamma \epsilon)[f^{IJK} \tilde{A}_\alpha{}^J \tilde{A}_\beta{}^K - (T^I)^{ij} \tilde{B}_\alpha{}^i \tilde{B}_\beta{}^j] - ig(\alpha_\alpha \beta_\beta \gamma_5 \epsilon)[f^{IJK} A_\alpha{}^J \tilde{A}_\beta{}^K - (T^I)^{ij} B_\alpha{}^i \tilde{B}_\beta{}^j], \end{aligned} \tag{2.5d}$$

$$\begin{aligned}\delta_Q \chi^i = &+\frac{1}{2}(\gamma^{\mu\nu}\epsilon) G_{\mu\nu}{}^i + i(\alpha_\alpha \gamma^\mu \epsilon)\mathcal{D}_\mu B_\alpha{}^i - (\beta_\alpha \gamma_5 \gamma^\mu \epsilon)\mathcal{D}_\mu \tilde{B}_\alpha{}^i + ig\epsilon_{\alpha\beta\gamma}(T^I)^{ij}(\alpha_\gamma \epsilon) A_\alpha{}^I B_\beta{}^j \\ &+ ig\epsilon_{\alpha\beta\gamma}(T^I)^{ij}(\beta_\gamma \epsilon)\tilde{A}_\alpha{}^I \tilde{B}_\beta{}^j - ig(\alpha_\alpha \beta_\beta \gamma_5 \epsilon)(T^I)^{ij}(A_\alpha{}^I \tilde{B}_\beta{}^j - \tilde{A}_\beta{}^I B_\alpha{}^j). \end{aligned} \tag{2.5e}$$

The supersymmetric invariance of our action $\delta_Q I = 0$ can be confirmed in the usual way. The crucial relationships are the conditions (2.1), as well as the Bianchi identities

$$\mathcal{D}_{[\mu} \mathcal{F}_{\nu\rho]}{}^I \equiv \partial_{[\mu} \mathcal{F}_{\nu\rho]}{}^I + g f^{IJK} A_{[\mu}{}^J \mathcal{F}_{\nu\rho]}{}^K - g(T^I)^{ij} B_{[\mu}{}^j G_{\nu\rho]}{}^j \equiv 0, \tag{2.6}$$

$$\mathcal{D}_{[\mu} G_{\nu\rho]}{}^i \equiv \partial_{[\mu} G_{\nu\rho]}{}^i + g(T^I)^{ij} A_{[\mu}{}^I G_{\nu\rho]}{}^j - g(T^I)^{ij} B_{[\mu}{}^j \mathcal{F}_{\nu\rho]}{}^I \equiv 0. \tag{2.6b}$$

At the cubic-order level in $\delta_Q \mathcal{L}$, we need the Fierz identity

$$[(\gamma_\mu)_{AB}(\gamma^\mu)_{CD}\delta_{(i)(j)}\delta_{(k)(\ell)} - C_{AB}C_{CD}(\alpha_\alpha)_{(i)(j)}(\alpha_\alpha)_{(k)(\ell)} + (\gamma_5)_{AB}(\gamma_5)_{CD}(\beta_\alpha)_{(i)(j)}(\beta_\alpha)_{(k)(\ell)}] + (2\,\text{perm.}) \equiv 0, \tag{2.7}$$

where $A, B, \ldots = 1, \ldots, 4$ are for the Majorana spinor components in 4D, while "2 perm." stands for the two more sets of terms for the cyclic permutations of $B(j) \to C(k)$, $C(k) \to D(\ell)$, $D(\ell) \to B(j)$, so that the whole expression is totally symmetric with respect to these three pairs of indices. This identity is used both for the $g\lambda\chi^2$- and the $g\lambda^3$-terms in $\delta_Q\mathcal{L}$. The key ingredient at the quartic-order level is the usage of the condition (2.1b) in the sector $g^2\chi B_\alpha B_\beta \tilde{B}_\gamma$ in the variation $\delta_Q\mathcal{L}$.

Our peculiar vector multiplet $(B_\mu{}^i, \lambda^i, B_\alpha{}^i, \tilde{B}_\alpha{}^i)$ carrying the indices i of the representation R of $SO(\mathcal{N})$ must satisfy the conditions in (2.1). A necessary condition of (2.1b) is [6]

$$\frac{2dI_2(R)}{N(N-1)} - 2I_2(R) + N - 2 = 0, \tag{2.8}$$

where $d \equiv \dim(R)$, while the second index $I_2(R)$ is defined by $(T^I T^I)^{ij} = -2I_2(R)\delta^{ij}$, and accordingly $(T^I T^J)^{ii} = -4dI_2(R)\delta^{IJ}/N(N-1)$. As long as these conditions are satisfied, the representation R can be any real representation of $SO(\mathcal{N})$. A trivial example is the $\mathcal{N}$ of $SO(\mathcal{N})$, but this system has the hidden local symmetry; i.e., the system is equivalent to a supersymmetric Yang-Mills theory for the local $SO(\mathcal{N}+1)$ [6]. Nontrivial examples are the $\mathbf{8_C}$ and $\mathbf{8_S}$ representations of $SO(8)$, which are different from the usual $\mathbf{8_V}$ representation.

III. SUMMARY AND CONCLUDING REMARKS

In this paper, we have constructed the system of an $N = 4$ supersymmetric Yang-Mills multiplet in a nonadjoint real representation R. As long as the conditions in (2.1) are satisfied for the representation R, we have found that the extra vector multiplet $(B_\mu{}^i, \chi_{(i)}{}^i, B_\alpha{}^i, \tilde{B}_\alpha{}^i)$ can be coupled to the conventional vector multiplet $(A_\mu{}^I, \lambda_{(i)}{}^I, A_\alpha{}^I, \tilde{A}_\alpha{}^I)$ consistently with supersymmetry. As in the $N = 1$ case, we need at least the conventional $N = 4$ supersymmetric Yang-Mills multiplet in the adjoint representation, once we introduce the vector multiplet in the nonadjoint representation R. The nontrivial examples of such representations are the $\mathbf{8_S}$ and $\mathbf{8_C}$ of $SO(8)$.

According to the prevailing notion, since the $N = 4$ supersymmetry is the maximal extended global supersymmetry in 4D, there is *no* other outside multiplet that can be coupled to the basic $8 + 8$ multiplet $(A_\mu{}^I, \lambda_{(i)}{}^I, A_\alpha{}^I, \tilde{A}_\alpha{}^I)$. In that sense, the field content of $N = 4$ supersymmetric theory is supposed to be unique [5]. However, we already know one counterexample against this wisdom, namely, a vector multiplet gauging scale symmetry presented in [8]. Our theory in this paper has established another counterexample now with the $N = 4$ vector multiplet in the nonadjoint representation R consistently coupled to the conventional $N = 4$ Yang-Mills multiplet. It is amazing that such a tight $N = 4$ maximally extended supermultiplet with $8 + 8$ degrees of freedom can be further coupled to an extra vector multiplet with additional physical degrees of freedom.

As the conventional $N = 4$ supersymmetric Yang-Mills theory is finite [2], so may well be our theory to all orders.

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