

Spiky strings, giant magnons, and β deformationsN. P. Bobev^{1,*} and R. C. Rashkov^{2,†}¹*Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089-0484, USA*²*Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria*

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We study rigid string solutions rotating on the S^3 subspace of the β -deformed $\text{AdS}_5 \times S^5$ background found by Lunin and Maldacena. For particular values of the parameters of the solutions we find the known giant magnon and single spike strings. We present a single spike string solution on the deformed S^3 and find how the deformation affects the dispersion relation. The possible relation of this string solution to spin chains and the connection of the solutions on the undeformed S^3 to the sine-Gordon model are briefly discussed.

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I. INTRODUCTION

The idea for the correspondence between the large N limit of gauge theories and string theory was proposed over 30 years ago [1] and an explicit realization of it was provided when Maldacena conjectured the AdS/CFT correspondence [2]. Since then this has become a major research area and many fascinating discoveries were made in the last decade.

Recent advances on both the string and the gauge theory sides of the correspondence have indicated that type IIB string theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory in four dimensions may be integrable in the planar limit. The techniques of integrable systems have therefore become useful in studying the AdS/CFT correspondence in detail. One of the conjectures of the correspondence is the duality between the spectrum of anomalous dimensions of gauge invariant operators in the gauge theory and the energy spectrum of the string theory.

Assuming that these theories are integrable, the dynamics should be encoded in an appropriate scattering matrix S . This can be interpreted from both sides of the correspondence as follows. On the string side, in the strong-coupling limit the S matrix can be interpreted as describing the two-body scattering of elementary excitations on the world sheet. When their world-sheet momenta become large, these excitations can be described as special types of solitonic solutions, or giant magnons, and the interpolating region is described by the dynamics of the so-called near-flat-space regime [3,4]. On the gauge theory side, the action of the dilatation operator on single-trace operators is the same as that of a Hamiltonian acting on the states of a certain spin chain [5]. This turns out to be of great advantage because one can diagonalize the matrix of anomalous dimensions by using the algebraic Bethe ansatz technique (see [6] for a nice review on the algebraic Bethe ansatz). In

this picture the dynamics involves diffractionless scattering encoded by an S matrix. Proving that the gauge and string theories are identical in the planar limit therefore amounts to showing that the underlying physics of both theories is governed by the same two-body scattering matrix. On the other hand, in several papers the relation between strings and spin chains was established at the level of effective action; see for instance [7–10] and references therein. These ideas opened the way for a remarkable interplay between spin chains, gauge theories, string theory,¹ and integrability (the integrability of classical strings on $\text{AdS}_5 \times S^5$ was proven in [14]).

Recently Hofman and Maldacena [3] were able to map spin chain “magnon” excitations to specific rotating semiclassical string states on $R \times S^2$. These strings move around the equator of the S^2 and have very large energy and angular momentum. The momentum of the spin chain magnon was interpreted as a deficit angle of the string configuration. This result was soon generalized to magnon bound states [15–18], dual to strings on $R \times S^3$ with two nonvanishing angular momenta. Moreover, in [18] a different giant magnon state with two spins was found; it is dual to a string moving on $\text{AdS}_3 \times S^1$, i.e. it has spin in both the AdS and the spherical part of the background (see also [19]). These classical string solutions were further generalized to include dynamics on the whole S^5 [20,21], and in fact a method to construct classical string solutions describing the superposition of arbitrary scattering and bound states was found [22]. The semiclassical quantization of the giant magnon solution was performed in [23].

The relation between energy and angular momentum for the one spin giant magnon found in [3] is

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p}{2} \right| \quad (1.1)$$

where p is the magnon momentum which on the string side is interpreted as a difference in the angle ϕ (see [3] for

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details). In the multispin cases the $E - J$ relations were studied both on the string [16–18] and spin chain sides [15]. A natural way to extend this analysis and find giant magnon solutions in backgrounds which, via the AdS/CFT correspondence, are dual to less supersymmetric gauge theories is to look for similar giant magnon solutions in the Lunin-Maldacena background [24,25].

A class of classical string solutions with spikes may be of interest for the AdS/CFT correspondence. These spiky strings were constructed in the AdS_5 subspace of $\text{AdS}_5 \times S^5$ [26], and it was argued that they correspond to single-trace operators with a large number of derivatives. The spiky strings were generalized to include dynamics in the S^5 part of $\text{AdS}_5 \times S^5$ [27], and in [28] closed strings with “kinks” were considered.²

Recently a new analysis of the class of spiky strings appeared; in [30] infinitely wound string solutions with single spikes on S^2 and S^3 were found. It can be shown that these solutions can be found in a certain limit of the parameters of a general rotating rigid string. Interestingly enough, the giant magnon solution of [3] can be found in a very similar way, and thus the single spike solutions fall into the same class as the giant magnons. While the giant magnon solutions can be interpreted as higher twist operators in the field theory, the single spike solutions do not seem to be directly related to some field theory operators. However, in [30] an interpretation of this solution as a spin chain Hubbard model, which means the antiferromagnetic phase of the corresponding spin chain, was found, but the relation to field theory operators is still unclear.

The single spike solutions have interesting properties; for instance, the energy and “the angle deficit” $\Delta\phi$ are infinite, but their difference is finite,

$$E - T\Delta\phi = \frac{\sqrt{\lambda}}{\pi} \left(\frac{\pi}{2} - \theta_1 \right)$$

where $T = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension. The angular momenta of the solution are finite, and in the case of a single spike solution on S^3 , their relation

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2 \theta_1} \quad (1.2)$$

resembles the dispersion relation of giant magnons on S^3 . The single spike solutions are related to the sine-Gordon model and this could be useful to study their scattering and eventually lead to a better understanding of their properties. It seems interesting to study these single spike solutions in greater detail and generalize them to less symmetric backgrounds. The purpose of this paper is to present single spike solutions on the Lunin-Maldacena [31] background which will be the counterparts of the giant magnon solutions on this background constructed in

[24,25]. We find that the close relation between the single spikes and giant magnons known from the solutions on $\text{AdS}_5 \times S^5$ persists in the deformed background. The relation between the energy and the angular momenta is similar, and the effect of the deformation is similar to the case of the giant magnon.

The paper is organized as follows. First, in Sec. II we present a short review of the Lunin-Maldacena background; then we discuss the dynamics of strings on an S^3 subspace of the deformed S^5 . In Sec. III we consider a vanishing deformation parameter and find classical string solutions on S^2 and S^3 in conformal gauge, and by taking different limits we reproduce the giant magnon and single spike solutions found in [3,30]. In Sec. IV we show the relation of the single spike and giant magnon solutions on S^3 to the sine-Gordon model. Then in Sec. V we generalize the single spike solution of [30] to a similar solution on the γ -deformed S^3 and establish its similarities with the giant magnons on S^3_γ found in [25]. In the last section we present our conclusions and a few possible directions for further studies.

II. RIGID STRINGS ON S^3_γ

A. β -deformed $\text{AdS}_5 \times S^5$

Here we review the β -deformed $\text{AdS}_5 \times S^5$ background found by Lunin and Maldacena [31]. This background is conjectured to be dual to the Leigh-Strassler marginal deformations of $\mathcal{N} = 4$ SYM [32]. We note that this background can be obtained from pure $\text{AdS}_5 \times S^5$ by a series of S- and T-dualities and a shift of an angle variable as described in [33]. The deformation parameter $\beta = \gamma + i\sigma_d$ is, in general, a complex number, but in our analysis we will consider $\sigma_d = 0$; in this case, the deformation is called γ deformation. The resulting supergravity background dual to real β deformations of $\mathcal{N} = 4$ SYM is

$$ds^2 = R^2 \left(ds_{\text{AdS}_5}^2 + \sum_{i=1}^3 (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \tilde{\gamma}^2 G\mu_1^2 \mu_2^2 \mu_3^2 \left(\sum_{i=1}^3 d\phi_i^2 \right) \right). \quad (2.1)$$

This background also includes a dilaton field as well as Ramond-Ramond and Neveu-Schwarz–Neveu-Schwarz form fields. The relevant form for our classical string analysis will be the antisymmetric B field:

$$B = R^2 \tilde{\gamma} G (\mu_1^2 \mu_2^2 d\phi_1 d\phi_2 + \mu_2^2 \mu_3^2 d\phi_2 d\phi_3 + \mu_1^2 \mu_3^2 d\phi_1 d\phi_3). \quad (2.2)$$

In the above formulas we have defined

²For other spiky string solutions see [29].

$$\begin{aligned}\tilde{\gamma} &= R^2 \gamma, & R^2 &= \sqrt{4\pi g_s N} = \sqrt{\lambda}, \\ G &= \frac{1}{1 + \tilde{\gamma}^2(\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_1^2 \mu_3^2)}, \\ \mu_1 &= \sin\theta \cos\psi, & \mu_2 &= \cos\theta, \\ \mu_3 &= \sin\theta \sin\psi,\end{aligned}\quad (2.3)$$

where $(\theta, \psi, \phi_1, \phi_2, \phi_3)$ are the usual S^5 variables. This is a deformation of the $\text{AdS}_5 \times S^5$ background governed by a single real deformation parameter $\tilde{\gamma}$ and thus provides a useful setting for the extension of the classical strings/spin chain/gauge theory duality to less supersymmetric cases.

B. Rigid strings on S_γ^3

Let us consider the motion of a rigid string on S_γ^3 . This space can be thought of as a subspace of the γ deformation of $\text{AdS}_5 \times S^5$ presented above,

$$\mu_3 = 0, \quad \phi_3 = 0, \quad \text{i.e. } \psi = 0, \quad \phi_3 = 0. \quad (2.4)$$

The relevant part of the γ -deformed $\text{AdS}_5 \times S^5$ is

$$ds^2 = -dt^2 + d\theta^2 + G \sin^2 \theta d\phi_1^2 + G \cos^2 \theta d\phi_2^2 \quad (2.5)$$

where $G = \frac{1}{1 + \tilde{\gamma}^2 \sin^2 \theta \cos^2 \theta}$, and due to the series of T -dualities, there is a nonzero component of the B field,

$$B_{\phi_1 \phi_2} = \tilde{\gamma} G \sin^2 \theta \cos^2 \theta. \quad (2.6)$$

We will work in conformal gauge and thus use the Polyakov action ($T = \frac{\sqrt{\lambda}}{2\pi}$),

$$\begin{aligned}S &= \frac{T}{2} \int d^2 \sigma [-(\partial_\tau t)^2 + (\partial_\tau \theta)^2 - (\partial_\sigma \theta)^2 \\ &\quad + G \sin^2 \theta ((\partial_\tau \phi_1)^2 - (\partial_\sigma \phi_1)^2) \\ &\quad + G \cos^2 \theta ((\partial_\tau \phi_2)^2 - (\partial_\sigma \phi_2)^2) \\ &\quad + 2\gamma G \sin^2 \theta \cos^2 \theta (\partial_\tau \phi_1 \partial_\sigma \phi_2 - \partial_\sigma \phi_1 \partial_\tau \phi_2)]\end{aligned} \quad (2.7)$$

which is supplemented by the Virasoro constraints

$$\begin{aligned}g_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu &= 0, \\ g_{\mu\nu} (\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu) &= 0.\end{aligned} \quad (2.8)$$

Here $g_{\mu\nu}$ is the metric (2.5) and $X^\mu = \{t, \theta, \phi_1, \phi_2\}$. The ansatz

$$\begin{aligned}t &= \kappa \tau, & \theta &= \theta(y), & \phi_1 &= \omega_1 \tau + \tilde{\phi}_1(y), \\ \phi_2 &= \omega_2 \tau + \tilde{\phi}_2(y)\end{aligned} \quad (2.9)$$

describes the motion of rigid strings on the deformed 3-sphere; here we have defined a new variable, $y = \alpha \sigma + \beta \tau$. One can substitute the above ansatz in the equations of motion and use one of the Virasoro constraints to find three first order differential equations for the unknown functions:

$$\begin{aligned}\tilde{\phi}'_1 &= \frac{1}{\alpha^2 - \beta^2} \left(\frac{A}{G \sin^2 \theta} + \beta \omega_1 - \tilde{\gamma} \alpha \omega_2 \cos^2 \theta \right), \\ \tilde{\phi}'_2 &= \frac{1}{\alpha^2 - \beta^2} \left(\frac{B}{G \sin^2 \theta} + \beta \omega_2 + \tilde{\gamma} \alpha \omega_1 \sin^2 \theta \right), \\ (\theta')^2 &= \frac{1}{(\alpha^2 - \beta^2)^2} \left[(\alpha^2 + \beta^2) \kappa^2 - \frac{A^2}{G \sin^2 \theta} \right. \\ &\quad \left. - \frac{B^2}{G \cos^2 \theta} - \alpha^2 \omega_1^2 \sin^2 \theta - \alpha^2 \omega_2^2 \cos^2 \theta \right. \\ &\quad \left. + 2\tilde{\gamma} \alpha (\omega_2 A \cos^2 \theta - \omega_1 B \sin^2 \theta) \right].\end{aligned} \quad (2.10)$$

A and B are integration constants and the prime denotes a derivative with respect to y . The other Virasoro constraint provides the following relation between the parameters:

$$A \omega_1 + B \omega_2 + \beta \kappa^2 = 0. \quad (2.11)$$

This system has three conserved quantities—the energy and two angular momenta:

$$\begin{aligned}E &= 2T \frac{\kappa}{\alpha} \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'}, \\ J_1 &= 2 \frac{T}{\alpha} \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} G \sin^2 \theta [\omega_1 + \beta \tilde{\phi}'_1 + \tilde{\gamma} \alpha \cos^2 \theta \tilde{\phi}'_2], \\ J_2 &= 2 \frac{T}{\alpha} \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} G \cos^2 \theta [\omega_2 + \beta \tilde{\phi}'_2 + \tilde{\gamma} \alpha \sin^2 \theta \tilde{\phi}'_1]\end{aligned} \quad (2.12)$$

where the integration is performed over the range of the coordinate θ . In the analysis below, we will find solutions of the above equations and relations between the energy and the angular momenta for some special values of the parameters. These solutions include the giant magnon and the single spike solution on the deformed S^3 .

III. SPIKES AND GIANT MAGNONS ON S^2 AND S^3

A. S^2 case

We can reproduce the known solutions describing a giant magnon [3] and a single spike [30] on the nondeformed S^2 by taking $\tilde{\gamma} = 0$ and $\phi_2 = 0$. This simplifies the equations of motion to

$$\begin{aligned}\tilde{\phi}'_1 &= \frac{1}{\alpha^2 - \beta^2} \left(\frac{A}{\sin^2 \theta} + \beta \omega_1 \right), \\ \theta' &= \frac{\alpha \omega_1}{(\alpha^2 - \beta^2) \sin \theta} \sqrt{(\sin^2 \theta - \sin^2 \theta_1)(\sin^2 \theta_0 - \sin^2 \theta)}\end{aligned} \quad (3.1)$$

where

$$\sin \theta_0 = \frac{\beta \kappa}{\alpha \omega_1}, \quad \sin \theta_1 = \frac{\kappa}{\omega_1}, \quad (3.2)$$

and θ takes values between θ_0 and θ_1 . We will be interested in solutions describing a string wound around the equator

(Fig. 1) and thus require that one of the turning points θ_0 or θ_1 is equal to $\pi/2$. This leads to two possible string solutions which, as we will show, correspond to the giant magnon and the single spike solutions:

- (i) $\frac{\kappa^2}{\omega_1^2} = 1$, the giant magnon solution of [5];
- (ii) $\frac{\kappa^2 \beta^2}{\alpha^2 \omega_1^2} = 1$, the single spike solution of [25].

Let us first consider case (i); this will give the giant magnon solution on S^2 . Using the equation of motion for $\tilde{\phi}_1$ and the expressions for the energy and the angular momentum (2.12), one finds

$$E - J_1 = 2T \cos \theta_0, \quad (3.4)$$

$$\Delta \phi_1 = 2 \int_{\theta_0}^{\pi/2} \frac{d\theta}{\theta'} \tilde{\phi}_1' = 2 \arcsin(\cos \theta_0) = \pi - 2\theta_0,$$

and thus we end up with the dispersion relation for the giant magnon on S^2 ,

$$E - J_1 = 2T \cos \theta_0 = \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{\Delta \phi_1}{2}\right). \quad (3.5)$$

The angle difference $\Delta \phi_1$ can be interpreted as the string

$$\tilde{\phi}_1' = \frac{1}{\alpha^2 - \beta^2} \left(\frac{A}{\sin^2 \theta} + \beta \omega_1 \right), \quad \tilde{\phi}_2' = \frac{1}{\alpha^2 - \beta^2} \left(\frac{B}{\cos^2 \theta} + \beta \omega_2 \right),$$

$$\theta' = \frac{1}{\alpha^2 - \beta^2} \sqrt{(\alpha^2 + \beta^2) \kappa^2 - \frac{A^2}{\sin^2 \theta} - \frac{B^2}{\cos^2 \theta} - \alpha^2 \omega_1^2 \sin^2 \theta - \alpha^2 \omega_2^2 \cos^2 \theta}. \quad (3.7)$$

We are interested in string configurations wound around the equator so we should impose one of the turning points of θ to be at $\pi/2$. This condition requires $B = 0$ and one of the following two choices:

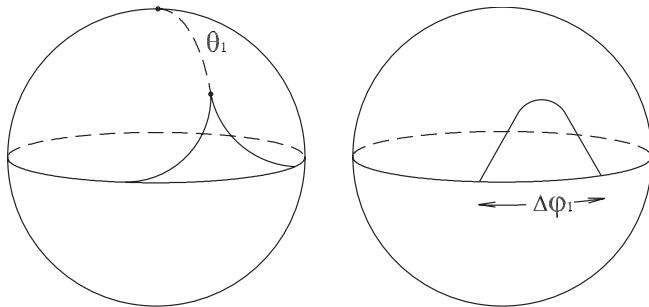


FIG. 1. A single spike (left diagram) and a giant magnon (right diagram). The single spike is infinitely wound around the equator and has height $\frac{\pi}{2} - \theta_1$, whereas the giant magnon has a finite deficit angle $\Delta \phi_1$, which is interpreted as the momentum of the corresponding spin chain excitation.

counterpart of the momentum p of the magnon excitations of the corresponding spin chain. The above dispersion relation matches the one from the spin chain analysis [34].

Now consider case (ii); this corresponds to the single spike solution on S^2 [30]. The equations of motion yield the following relations:

$$J_1 = 2T \cos \theta_1 = \frac{\sqrt{\lambda}}{\pi} \cos \theta_1, \quad (3.6)$$

$$E - T \Delta \phi_1 = \frac{\sqrt{\lambda}}{\pi} \left(\frac{\pi}{2} - \theta_1 \right).$$

This solution corresponds to an infinitely wound string around the equator of S^2 (Fig. 1) with infinite energy and finite angular momentum. The interpretation from the field theory side is not clear but there is a spin chain interpretation of this system [30]. The single spike can be thought of as a perturbation of the “slow strings” of [35].

B. S^3 case

If we take $\tilde{\gamma} = 0$, we end up with the dynamics of rigid strings on nondeformed S^3 . The equations of motion reduce to

- (i) $\frac{\kappa^2}{\omega_1^2} = 1$, giant magnon solution on S^3 ;
- (ii) $\frac{\kappa^2 \beta^2}{\alpha^2 \omega_1^2} = 1$, single spike solution on S^3 .

Let us first examine the giant magnon solution, i.e. $\frac{\kappa^2}{\omega_1^2} = 1$. One of the turning points of θ is $\theta_1 = \pi/2$; the other one occurs at $\sin \theta_0 = \frac{\beta \omega_1}{\alpha \sqrt{\omega_1^2 - \omega_2^2}}$. The equation of motion for θ reads

$$\theta' = \frac{\alpha \sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} \frac{\cos \theta}{\sin \theta} \sqrt{\sin^2 \theta - \sin^2 \theta_0}. \quad (3.9)$$

This form of θ' allows one to compute explicitly the integrals for the conserved quantities (2.12) and obtain the following relations,

$$\Delta \phi_1 = \pi - 2\theta_0, \quad E - J_1 = \frac{2T \omega_1}{\sqrt{\omega_1^2 - \omega_1^2}} \cos \theta_0,$$

$$J_2 = \frac{2T \omega_2}{\sqrt{\omega_1^2 - \omega_2^2}} \cos \theta_0, \quad (3.10)$$

where to compute these quantities we have used the following integrals:

$$\begin{aligned} \int_{\xi}^{\pi/2} \frac{\sin\theta \cos\theta}{\sqrt{\sin^2\theta - \sin^2\xi}} d\theta &= \cos\xi, \\ \int_{\xi}^{\pi/2} \frac{\cos\theta}{\sin\theta \sqrt{\sin^2\theta - \sin^2\xi}} d\theta &= \frac{1}{\sin\xi} \left(\xi - \frac{\pi}{2} \right). \end{aligned} \quad (3.11)$$

With the above expressions for the energy and the angular momenta in hand, we find the dispersion relation of the giant magnon with two angular momenta on S^3 ,

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2\theta_0}. \quad (3.12)$$

If we identify the momentum of the spin chain magnon excitation as $p = \pi - 2\theta_0$, we reproduce the known dispersion relation for bound states of spin chain magnons,

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}. \quad (3.13)$$

The other possible choice of the parameters which will describe a string wound around the equator is case (ii). For this choice we have $\kappa^2\beta^2 = \alpha^2\omega_1^2$, and the equation of motion for θ is

$$\theta' = \frac{\alpha\sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} \frac{\cos\theta}{\sin\theta} \sqrt{\sin^2\theta - \sin^2\theta_1}, \quad (3.14)$$

where we have defined $\sin\theta_1 = \frac{\alpha\omega_1}{\beta\sqrt{\omega_1^2 - \omega_2^2}}$. Using this expression it is not hard to find the following relations between the conserved quantities:

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2\theta_1}, \quad E - T\Delta\phi_1 = \frac{\sqrt{\lambda}}{\pi} \left(\frac{\pi}{2} - \theta_1 \right), \quad (3.15)$$

which matches the results for the single spike solution first found in [30] with the Nambu-Goto action. It is interesting to see how the giant magnons and single spike solutions arise as limiting cases of a rigid string rotating around the equator; this implies that they fall into the same class of solutions and suggests that the single spike solutions might be important from the spin chain/gauge theory point of view.

IV. RELATION TO THE SINE-GORDON MODEL

The classical string solutions on S^3 described in the previous section have a relation to the sine-Gordon model. This is important for calculations of the scattering of single spikes and magnons. This relation can be explicitly seen by computing the determinant of the two dimensional world-sheet metric

$$\sqrt{-h} = \alpha^2(\theta'^2 + \sin^2\theta \tilde{\phi}_1'^2 + \cos^2\theta \tilde{\phi}_2'^2) \quad (4.1)$$

where we have used the Virasoro constraints. For the giant magnon solution we have $\kappa^2 = \omega_1^2$ and we can use the integral

$$\int \frac{\sin\theta d\theta}{\cos\theta \sqrt{\sin^2\theta - \sin^2\xi}} = \frac{1}{\cos\xi} \operatorname{arctanh}\left(\sqrt{\frac{\cos(2\xi) - \cos(2\theta)}{1 + \cos(2\xi)}}\right) \quad (4.2)$$

to find the following solution for θ :

$$\cos\theta = \frac{\cos\theta_0}{\cosh(Cy)} \quad \text{where } C = \frac{\alpha\sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} \cos\theta_0 \quad (4.3)$$

and thus

$$\frac{\sqrt{-h}}{(\alpha^2 - \beta^2)C^2} = \frac{1}{\cosh^2(Cy)}. \quad (4.4)$$

If we define the sine-Gordon field as $\sin^2\Psi_0 = \frac{\sqrt{-h}}{(\alpha^2 - \beta^2)C^2} = \frac{1}{\cosh^2(Cy)}$ it is straightforward to check that Ψ_0 is a solution to the sine-Gordon equation

$$\partial_\tau^2\Psi_0 - \partial_\sigma^2\Psi_0 + \frac{(\alpha^2 - \beta^2)C^2}{2} \sin(2\Psi_0) = 0. \quad (4.5)$$

One can perform an analogous calculation for the single spike solution on S^3 with $\beta^2\kappa^2 = \alpha^2\omega_1^2$. For this case we have a solution for θ of the form

$$\cos\theta = \frac{\cos\theta_1}{\cosh(Dy)} \quad \text{where } D = \frac{\alpha\sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} \cos\theta_1, \quad (4.6)$$

and if we impose $D^2 = \frac{\kappa^2}{\beta^2 - \alpha^2}$ we find

$$\sin^2\Psi_1 = \frac{\sqrt{-h}}{(\beta^2 - \alpha^2)D^2} = \tanh^2(Dy) \quad (4.7)$$

where Ψ_1 solves the sine-Gordon equation

$$\partial_\tau^2\Psi_1 - \partial_\sigma^2\Psi_1 + \frac{(\beta^2 - \alpha^2)D^2}{2} \sin(2\Psi_1) = 0. \quad (4.8)$$

Although we do not explore further the connection between the single spike solutions on S^3 and the sine-Gordon model, it seems interesting to use it and study scattering of single spikes, as this was done for the giant magnons.

V. γ DEFORMATIONS, SPIKY STRINGS, AND GIANT MAGNONS

In this section we will look for string solutions wound around the equator of the γ -deformed S^3 . To ensure this, we again require that one of the turning points of θ occurs at $\theta = \pi/2$. This imposes some constraints on the parameters of the solution. It turns out that the deformation

parameter does not play a role on the constraints, and they are the same as for the undeformed case, namely, $B = 0$ and

$$\kappa^2 = \omega_1^2 \quad \text{or} \quad \beta^2 \kappa^2 = \alpha^2 \omega_1^2. \quad (5.1)$$

As was noted in [24,25],³ the nontrivial deformation of the $\text{AdS}_5 \times S^5$ background imposes certain conditions. It turns out that this solution is required to live on the “three-sphere” and we need to consider a classical string moving on S_γ^3 . As before, we use (2.4) to parametrize the ambient space, i.e. the deformed three-sphere is parametrized by the three angles θ , ϕ_1 , and ϕ_2 .

A. Giant magnons

If we choose $\kappa^2 = \omega_1^2$ (which through the Virasoro constraint implies $A = -\beta\omega_1$) we get the giant magnon solution on S_γ^3 found in [25]. The equations of motion for this case are

$$\begin{aligned} \tilde{\phi}'_1 &= -\frac{\cos^2\theta}{\alpha^2 - \beta^2} \left(\frac{\beta\omega_1}{\sin^2\theta} + \tilde{\gamma}\alpha\omega_2 + \tilde{\gamma}^2\beta\omega_1 \right), \\ \tilde{\phi}'_2 &= \frac{\beta\omega_2 + \tilde{\gamma}\alpha\omega_1\sin^2\theta}{\alpha^2 - \beta^2}, \\ \theta' &= \frac{\alpha\Omega_0}{(\alpha^2 - \beta^2)} \frac{\cos\theta}{\sin\theta} \sqrt{\sin^2\theta - \sin^2\theta_0} \end{aligned} \quad (5.2)$$

where we have defined

$$\sin\theta_0 = \frac{\beta\omega_1}{\alpha\Omega_0} \quad \text{and} \quad \Omega_0 = \sqrt{\omega_1^2 - \left(\omega_2 + \tilde{\gamma} \frac{\beta\omega_1}{\alpha} \right)^2}. \quad (5.3)$$

Using the expressions for the energy and the angular momentum (2.12) and Eqs. (5.2), we find

$$\begin{aligned} E - J_1 &= 2T \frac{\omega_1}{\Omega_0} \cos\theta_0, \\ J_2 &= 2T \left(\frac{\omega_2}{\Omega_0} + \tilde{\gamma} \frac{\beta\omega_1}{\alpha\Omega_0} \right) \cos\theta_0. \end{aligned} \quad (5.4)$$

These expressions lead to the dispersion relation for the giant magnon solution on γ -deformed S^3 [24,25],

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2\theta_0}. \quad (5.5)$$

In order to make a connection with the spin chain description, we should identify $\cos\theta_0 = \sin(\frac{p}{2} - \pi\beta)$, where p is the momentum of the magnon excitation on the spin chain and $\beta = \tilde{\gamma}/\sqrt{\lambda}$. So the prediction for the relevant spin chain dispersion relation is

³Semiclassical string solutions in the Lunin-Maldacena background were considered in [36–40]; see also [41]. For a nice review on the subject and a complete list of references, see [42].

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2} - \pi\beta\right)}; \quad (5.6)$$

this relation is invariant under $p \rightarrow p + 2\pi$ and $\beta \rightarrow \beta + 1$, as is required by the spin chain analysis [36,43]. We note that in the deformed background we find

$$\frac{\Delta\phi_1}{2} = \left(\frac{\pi}{2} - \theta_0 \right) - \tilde{\gamma} \frac{\sqrt{\omega_1^2 - \Omega_0^2}}{\Omega_0} \cos\theta_0, \quad (5.7)$$

and in contrast to the undeformed giant magnon solution, the momentum p of the spin chain magnon is no longer interpreted as the angular difference $\Delta\phi$.

B. Single spikes

Considering the case $\beta^2\kappa^2 = \alpha^2\omega_1^2$, which yields $A = -\frac{\omega_1\alpha^2}{\beta}$, we expect to find a rigid string solution on S_γ^3 which is the analogue of the single spike solution on S^3 found in [30]. The equations of motion are

$$\begin{aligned} \tilde{\phi}'_1 &= \frac{1}{\alpha^2 - \beta^2} \left(\beta\omega_1 - \frac{\alpha^2\omega_1}{\beta\sin^2\theta} - \tilde{\gamma}\alpha\sqrt{\omega_1^2 - \Omega_1^2} \cos^2\theta \right), \\ \tilde{\phi}'_2 &= \frac{1}{\alpha^2 - \beta^2} (\beta\omega_2 + \tilde{\gamma}\alpha\omega_1\sin^2\theta), \\ \theta' &= \frac{\alpha\Omega_1}{(\alpha^2 - \beta^2)} \frac{\cos\theta}{\sin\theta} \sqrt{\sin^2\theta - \sin^2\theta_1} \end{aligned} \quad (5.8)$$

where

$$\sin\theta_1 = \frac{\alpha\omega_1}{\beta\Omega_1}, \quad \Omega_1 = \sqrt{\omega_1^2 - \left(\omega_2 + \tilde{\gamma} \frac{\alpha\omega_1}{\beta} \right)^2}. \quad (5.9)$$

The two angular momenta are

$$J_1 = 2T \frac{\omega_1}{\Omega_1} \cos\theta_1, \quad J_2 = -2T \frac{\sqrt{\omega_1^2 - \Omega_1^2}}{\Omega_1} \cos\theta_1. \quad (5.10)$$

This leads to the relation

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2\theta_1}. \quad (5.11)$$

This looks identical to the corresponding expression in the undeformed case; the dependence on the deformation parameter $\tilde{\gamma}$ is buried in the definition of $\cos\theta_1$. Similarly to the giant magnon solution, if we want to make a connection with the spin chain description we should identify $\cos\theta_1 = \sin(\frac{p}{2} - \pi\beta)$. However, the interpretation of the single spike solutions from the dual spin chain/gauge theory side is not completely clear.

For the relation between E and $\Delta\phi_1$ we find

$$E - T\Delta\phi_1 = \frac{\sqrt{\lambda}}{\pi} \left(\frac{\pi}{2} - \theta_1 \right) - \tilde{\gamma} \frac{\sqrt{\lambda}}{\pi} \frac{\sqrt{\omega_1^2 - \Omega_1^2}}{\Omega_1} \cos\theta_1. \quad (5.12)$$

As should be expected in the limit $\tilde{\gamma} \rightarrow 0$, this expression reduces to the one for the single spike solution on undeformed S^3 . Similarly to the deformed giant magnon solution, we see that there is a nontrivial $\tilde{\gamma}$ dependence in the angle difference $\Delta\phi_1$ which prevents us from interpreting $E - T\Delta\phi_1$ as the momentum of the spin chain excitation.

VI. COMMENTS AND CONCLUSIONS

In this short paper we investigated the multispin string solutions representing infinitely wound strings which develop a single spike in the Lunin-Maldacena background [31]. First we gave a derivation of the solutions in conformal gauge using the Polyakov sigma model action on S_γ^3 . When reduced to the undeformed S^2 and S^3 , the solutions actually reproduce the single spike, or giant magnon, solutions of [3,30]. Although the results are the same, the derivation is performed with the sigma model action which allows us to make connections to more general cases, for instance, those based on the reduction to the Neumann-Rosochatius system [21] or to solutions on deformed backgrounds. The new solution we found is a single spike solution on the deformed S^3 of the Lunin-Maldacena background; it has two finite angular momenta and infinite energy and winding angle.

In contrast to the magnon solutions, in the single spike case the role of the angle difference $\Delta\phi_1$ and the role of the spin J_1 interchange. The angle difference becomes infinite while all the spins J_i remain finite. Nevertheless, the difference $E - \Delta\phi_1$ is finite. Formally one can interpret the angle $\pi - 2\theta_1$ as the momentum of the corresponding spin chain excitation making the expressions very similar to the ones for the giant magnon. However, in this case one should interpret J_1 and not E as the energy of the excitation,

$$E - T\Delta\phi = \frac{\sqrt{\lambda}}{\pi} \left(\frac{\pi}{2} - \theta_1 \right).$$

Then one can loosely think of this case as arising from an interchange of the world-sheet coordinates σ and τ . This is supported also by the way the single spike solution reduces to the sine-Gordon model. In the case of dual spikes in the AdS part of geometry, the interpretation is somehow more transparent, but we did not consider that case here.⁴

The new single spike solution we obtained in the gamma deformed geometry shares some of the properties of the deformed giant magnons. The angle difference $\Delta\phi_1$ is

shifted by a $\tilde{\gamma}$ -dependent term. For the deformed magnon solution, it is given by

$$\frac{\Delta\phi}{2} = \left(\frac{\pi}{2} - \theta_0 \right) - \tilde{\gamma} \frac{\sqrt{\omega_1^2 - \Omega_0^2}}{\Omega_0} \cos\theta_0$$

where

$$\Omega_0 = \sqrt{\omega_1^2 - \left(\omega_2 + \tilde{\gamma} \frac{\beta\omega_1}{\alpha} \right)^2},$$

and for the deformed single spike, we find

$$E - T\Delta\phi = \frac{\sqrt{\lambda}}{\pi} \left(\frac{\pi}{2} - \theta_1 \right) - \tilde{\gamma} \frac{\sqrt{\lambda}}{\pi} \frac{\sqrt{\omega_1^2 - \Omega_1^2}}{\Omega_1} \cos\theta_1$$

where

$$\Omega_1 = \sqrt{\omega_1^2 - \left(\omega_2 + \tilde{\gamma} \frac{\alpha\omega_1}{\beta} \right)^2}.$$

In both cases Ω_i determines the turning points of the string θ_0 and θ_1 ,

$$\begin{aligned} \sin\theta_0 &= \frac{\beta\omega_1}{\alpha\Omega_0}, & \text{magnon case;} \\ \sin\theta_1 &= \frac{\beta\omega_1}{\alpha\Omega_1}, & \text{single spike case.} \end{aligned}$$

The relation between the angular momenta of the deformed single spike solution looks the same as the one for the undeformed case,

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \cos^2\theta_1};$$

however, there is implicit dependence on the deformation parameter $\tilde{\gamma}$ in the definition of θ_1 .

The study of the magnon and single spike solutions suggests several interesting directions for further studies. First of all the interpretation of the single spike solutions in terms of mapping to a spin chain is still not quite clear, especially on the field theory side. There could be a relation between the single spike solutions and the “slow strings” of [35] and a possible interpretation of the single spikes as an excitation of the “antiferromagnetic” state of the spin chain. Studying the scattering of single spikes and comparison with the magnon case can shed light on the interpretation of these solutions. The single spike solutions could be generalized to spikes on S^5 as well as to string solutions with dynamics on the full $\text{AdS}_5 \times S^5$ geometry. It would be interesting to also find spiky solutions which are not rigid but are both rotating and pulsating. We hope to report on some of these issues in the near future.

⁴For discussion on this point see [29].

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APPENDIX: SINGLE SPIKES FROM THE NEUMANN-ROSOCHATIUS INTEGRABLE SYSTEM

In this appendix we will derive single spike solutions in S^3 using reduction to the Neumann-Rosochatius integrable system [21]. The reduction to the Neumann-Rosochatius system can be done using the following generalized ansatz for the five-sphere coordinates:

$$X_a = x_a(y)e^{i\omega_a\tau}, \quad a = 1, 2, 3; \quad y = \alpha\sigma + \beta\tau, \quad (\text{A1})$$

where the complex quantities x_a satisfy the periodicity condition

$$x_a(y + 2\pi\alpha) = x_a(y), \quad x_a = r_a(y)e^{i\mu_a(y)} \quad (r_a \text{ real}). \quad (\text{A2})$$

Substituting in the Polyakov action one can find

$$\mathcal{L} = \sum_a \left[(\alpha^2 - \beta^2)r_a'^2 + (\alpha^2 - \beta^2)r_a^2 \left(\mu_a' - \frac{\beta\omega_a}{\alpha^2 - \beta^2} \right)^2 - \frac{\alpha^2}{\alpha^2 - \beta^2} \omega_a^2 r_a^2 \right] + \Lambda \left(\sum_a r_a^2 - 1 \right). \quad (\text{A3})$$

One can find μ_a from the equation

$$\mu_a' = \frac{1}{\alpha^2 - \beta^2} \left[\frac{C_a}{r_a^2} + \beta\omega_a \right] \quad (\text{A4})$$

where C_a are integration constants. The Lagrangian reduces to

$$\mathcal{L} = \sum_a \left[(\alpha^2 - \beta^2)r_a'^2 - \frac{C_a}{(\alpha^2 - \beta^2)r_a^2} - \frac{\alpha^2}{\alpha^2 - \beta^2} \omega_a^2 r_a^2 \right] + \Lambda \left(\sum_a r_a^2 - 1 \right). \quad (\text{A5})$$

The conserved quantities are given by the expressions

$$E = T \frac{\kappa}{\alpha} \int dy, \quad (\text{A6})$$

$$J_a = T \int dy \left(\frac{\beta}{\alpha} \frac{C_a}{\alpha^2 - \beta^2} + \frac{\alpha}{\alpha^2 - \beta^2} \omega_a r_a^2 \right), \quad (\text{A7})$$

$$\left(\frac{\alpha^2 + \beta \sum_a C_a / \omega_a}{\alpha^2 - \beta^2} \right) \frac{E}{\kappa} = \sum_a \frac{J_a}{\omega_a} \quad (\text{A8})$$

where the last relation comes from the constraints.

The reduction to the S^3 case implies $r_3 = \mu_3 = 0$ (also $C_3 = 0$). In addition, the condition that one of the turning points of the string is at $\theta = \pi/2$ imposes $C_2 = 0$, $\beta = -\frac{\omega_1 C_1}{\kappa^2}$ (we set $\alpha = 1$ and then $|\beta|$ becomes a “group velocity”). The equation for κ has two solutions: $\kappa = \omega_1$ corresponding to a magnon type solution, and $\kappa = C_1$ corresponding to a single spike solution.

The conserved quantities are [see Eqs. (3.7)–(3.10) in [21]]

$$\begin{aligned} \hat{\mu} &= \frac{C_1}{(1 - \beta^2)} \int \frac{dy}{u} + \frac{\beta\omega_1}{(1 - \beta^2)} \int dy, \\ E &= \kappa T \int dy, \\ J_1 &= \frac{C_1\beta}{(1 - \beta^2)} \int dy + \frac{\omega_1}{(1 - \beta^2)} T \int u dy, \\ J_2 &= \frac{\omega_2}{(1 - \beta^2)} T \int (1 - u) dy \end{aligned} \quad (\text{A9})$$

where $\hat{\mu}_1$ corresponds to our ϕ_1 and $u = r_1^2 = \sin^2\theta$.

To find finite results (which is the case for $E - J_1$ in the magnon case) we consider

$$E - T\hat{\mu}_1 = \frac{2C_1T}{\sqrt{\Delta\omega^2}} \arccos\sqrt{u} = \frac{\sqrt{\lambda}}{\pi} \bar{\theta} \quad (\text{A10})$$

where

$$\bar{\theta} = \frac{\pi}{2} - \theta_0. \quad (\text{A11})$$

For the angular momenta we get

$$J_1 = \frac{2T\omega_1}{\sqrt{\Delta\omega^2}} \cos\theta_0 = \frac{2T\omega_1}{\sqrt{\Delta\omega^2}} \sin\bar{\theta}. \quad (\text{A12})$$

Analogously,

$$J_2 = -\frac{2T\omega_2}{\sqrt{\Delta\omega^2}} \cos\theta_0 = -\frac{2T\omega_2}{\sqrt{\Delta\omega^2}} \sin\bar{\theta}. \quad (\text{A13})$$

Defining $\sin\gamma$ as in [30],

$$\sin\gamma = \frac{\omega_2}{\omega_1}, \quad \sin\theta_0 = \frac{C_1}{\sqrt{\Delta\omega^2}}, \quad (\text{A14})$$

we find

$$J_1 = 2T \frac{1}{\sin\gamma} \sin\bar{\theta}, \quad J_2 = -2T \frac{\sin\gamma}{\cos\gamma} \sin\bar{\theta}. \quad (\text{A15})$$

These give the result for dispersion relations, Eqs. (6.24)

and (6.25) of [30],

$$E - T\hat{\mu}_1 = \frac{\sqrt{\lambda}}{\pi} \bar{\theta}, \quad (\text{A16})$$

$$J_1^2 = J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \bar{\theta}. \quad (\text{A17})$$

As a result we rederived the results for single spikes in S^3 with two spins. We see that setting $J_2 = 0$ corresponds to the single spike solution with one angular momentum.

Using ellipsoidal coordinates as in [21], one can find for the motion in S^5 the results

$$E - T\Delta\hat{\mu}_1 = \frac{\sqrt{\lambda}}{\pi} (\bar{\theta}_2 + \bar{\theta}_3),$$

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \bar{\theta}_2} + \sqrt{J_3^2 + \frac{\lambda}{\pi^2} \sin^2 \bar{\theta}_3}. \quad (\text{A18})$$

The details of this derivation will be given elsewhere [44].

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