

Quantum algebra of N superspace

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We identify the quantum algebra of position and momentum operators for a quantum system bearing an irreducible representation of the super Poincaré algebra in the $N > 1$ and $D = 4$ superspace, both in the case where there are no central charges in the algebra, and when they are present. This algebra is noncommutative for the position operators. We use the properties of superprojectors acting on the superfields to construct explicit position and momentum operators satisfying the algebra. They act on the projected wave functions associated to the various supermultiplets with defined superspin present in the representation. We show that the quantum algebra associated to the massive superparticle appears in our construction and is described by a supermultiplet of superspin 0. This result generalizes the construction for $D = 4$, $N = 1$ reported recently. For the case $N = 2$ with central charges, we present the equivalent results when the central charge and the mass are different. For the κ -symmetric case when these quantities are equal, we discuss the reduction to the physical degrees of freedom of the corresponding superparticle and the construction of the associated quantum algebra.

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I. INTRODUCTION

The adequate formulation of the quantum mechanics for a given classical system has been the subject of much attention ever since the method of canonical quantization was proposed in the late twenties, but there are still unanswered questions in relation to this important problem. In the presence of second class constraints, neither the Dirac procedure nor any of the modern approaches, including the Becchi-Rouet-Stora-Tyutin (BRST) construction, give a general recipe to represent the algebra of observables. It has been proposed [1–3] that any second class constrained system may be extended by introducing new variables to a first class constrained one such that under partial or total gauge fixing reduces to it. But although there are cases in which this approach provides a solution to the construction of the observables of the theory, there are others for which it does not. In particular, for extended systems, it is not known how to realize this approach locally in a systematic way. In this paper we discuss this fundamental aspect of the quantization procedure for superparticles in $D = 4$. We show that properties of the superprojectors which reduce the superfields to the sectors associated to the irreducible supermultiplets determine the algebra of the projected position and momentum operators. For the superspin 0 sector this algebra coincides with the quantum algebra of the massive superparticle.

A noncommutative structure of the geometric space appears in various physical contexts. The Seiberg-Witten [4] limit of constant antisymmetric B field in string theory

leads to a noncommutativity of the related super–Yang–Mills and super–Born–Infeld field theories. Besides, interacting super D-brane theories require noncommutative coordinates expressed as Lie algebra valued coordinates for consistency. Also the quantization of the massive spinning and supersymmetric particles may be represented in terms of a noncommutative superspace.

As we review below, for massive superparticles the classical Dirac algebra of the position operators describe a noncommutative superspace. The Dirac brackets of the X^μ coordinate variables are proportional to the internal angular momentum. For the quantization of this system, a set of operators that is different from the standard multiplicative position operators, which realizes the noncommutative algebra should be found. In Ref. [5] we found indeed the general solution for the quantum algebra of $N = 1$ superparticles in four dimensions in terms of the superprojectors to the chiral and antichiral and tensorial multiplets. In this case the chiral and antichiral $N = 1$ multiplets are associated to the usual $N = 1$ superparticle and we found that the $N = 1$ tensorial spin 1/2 multiplet should be associated to a new superparticle action [5,6].

In this paper we extend our previous work for the $N > 1$ and $D = 4$ superspace both in the case where there are not central charges in the algebra and where they are present. After a discussion of the classical mechanics of the corresponding massive superparticles—which also serves to identify the general form of the quantum algebra in each of the superspaces—we present an explicit solution for the quantum operators using the superprojectors and demonstrate that they satisfy the right algebra.

We notice that the solution of the quantum algebra is of course Lorentz covariant, providing a general approach for the covariant quantization of massive superparticles.

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II. THE CLASSICAL SYSTEM WITHOUT CENTRAL CHARGES

Let us consider first the massive superparticle [5,7] in $D = 4$, N superspace without central charges (in this section we follow the discussion in Ref. [5]). The metric signature is $\eta_{\mu\nu} = \text{diag}\{-1, +1, +1, +1\}$ and the superspace coordinates are $(x^\mu, \theta^{ai}, \bar{\theta}^{\dot{a}}_i)$, where $a = 1, 2$ is a spinor index and $i = 1, \dots, N$ is the number of supersymmetric charges. Naturally $(\theta^{ai})^* = \bar{\theta}^{\dot{a}}_i$. We choose Dirac matrices to be off-diagonal and given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (1)$$

with σ^μ_{ab} the Pauli matrices. The action principle for the massive superparticle is given by [7]

$$S = \frac{1}{2} \int d\tau (e^{-1} \omega^\mu \omega^\nu \eta_{\mu\nu} - em^2), \quad (2)$$

where $\omega^\mu = \dot{x}^\mu - i\dot{\theta}^{ai} \sigma^\mu_{ab} \bar{\theta}^{\dot{b}}_i + i\theta^{ai} \sigma^\mu_{ab} \dot{\bar{\theta}}^{\dot{b}}_i$ is defined for convenience. The generalized momenta are given by

$$\pi_e = 0, \quad (3)$$

$$p_\mu = e^{-1} \omega_\mu, \quad (4)$$

$$\pi_{ai} = -ip_\mu \sigma^\mu_{ab} \bar{\theta}^{\dot{b}}_i, \quad (5)$$

$$\bar{\pi}^{\dot{a}}_i = -ip_\mu \bar{\theta}^{\dot{a}}_i \sigma^\mu_{ab}, \quad (6)$$

and satisfy the canonical Poisson bracket relations [7]

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \quad (7)$$

$$\{\theta^{ai}, \pi_{bj}\} = -\delta^a_b \delta^i_j. \quad (8)$$

The total angular momentum is given by

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad (9)$$

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad (10)$$

$$S_{\mu\nu} = -\frac{1}{4}(\theta^{ai} \sigma_{\mu\nu a}^b \pi_{bi} + \bar{\pi}^{\dot{a}}_i \bar{\sigma}_{\mu\nu}^{\dot{a} b} \bar{\theta}^{\dot{b}}_i). \quad (11)$$

For this system, there is one first class constraint $\pi_e = 0$ related to the reparametrization invariance of the action which implies the secondary first class constraint $p^2 + m^2 = 0$. There also appear the constraints

$$d_{ai} \equiv \pi_{ai} + ip_\mu \sigma^\mu_{ab} \bar{\theta}^{\dot{b}}_i = 0, \quad (12)$$

$$\bar{d}^{\dot{a}}_i \equiv \bar{\pi}^{\dot{a}}_i + ip_\mu \bar{\theta}^{\dot{a}}_i \sigma^\mu_{ab} = 0, \quad (13)$$

$$\{d_{ai}, \bar{d}^{\dot{a}}_i\} = -i2\delta^{\dot{a}}_i p_\mu \sigma^\mu_{ab} \equiv C_{ai\dot{b}}^j, \quad (14)$$

which are second class since the matrix $C_{ai\dot{b}}^j$ is nonsingu-

lar if $m \neq 0$. The Dirac brackets are given by

$$\begin{aligned} \{F, G\}_D &= \{F, G\} - \{F, d_{ai}\} \hat{C}^{ai\dot{b}}_j \{\bar{d}^{\dot{b}}_j, G\} \\ &\quad - \{F, \bar{d}^{\dot{a}}_i\} \hat{C}^{ai\dot{b}}_j \{d_{aj}, G\}. \end{aligned} \quad (15)$$

Here $\hat{C}^{ai\dot{b}}_j = \frac{1}{2ip^2} p_\mu \bar{\sigma}^{\mu\dot{a}b} \delta^i_j$ and verifies

$$C_{ai\dot{b}}^j \hat{C}^{bk\dot{c}}_j = \delta_i^k \delta_a^{\dot{c}}. \quad (16)$$

Calculating the Dirac brackets for all coordinates and momenta the result is [7]

$$\{\theta^{ai}, \theta^{bj}\}_D = \{\pi_{ai}, \pi_{bj}\}_D = \{p_\mu, p_\nu\}_D = 0, \quad (17)$$

$$\{\theta^{ai}, \bar{\theta}^{\dot{a}}_j\}_D = \frac{-1}{2ip^2} p_\mu \bar{\sigma}^{\mu\dot{a}a} \delta^i_j, \quad (18)$$

$$\{x^\mu, \theta^{ai}\}_D = \frac{1}{2p^2} p_\nu \bar{\sigma}^{\nu\dot{a}a} \theta^{\dot{b}}_i \sigma^\mu_{b\dot{a}}, \quad (19)$$

$$\{x^\mu, \bar{\theta}^{\dot{a}}_i\}_D = \frac{1}{2p^2} p_\nu \bar{\sigma}^{\nu\dot{a}a} \bar{\theta}^{\dot{b}}_i \sigma^\mu_{ab}, \quad (20)$$

$$\{x^\mu, x^\nu\}_D = \frac{-S^{\mu\nu}}{p^2}. \quad (21)$$

The nontrivial Poisson bracket (21) was first discussed by Pryce [8] and in the context of superparticles by Casalbuoni [7]. As we discuss below it is a most important ingredient in the formulation of the quantum theory. Note that using (12) and (13), the equation for the internal angular momentum $S^{\mu\nu}$ (11) may be written also in the form

$$S^{\mu\nu} = \varepsilon^{\mu\nu\rho\lambda} p_\rho \theta^{ai} \sigma_{\lambda ab} \bar{\theta}^{\dot{b}}_i. \quad (22)$$

III. THE QUANTUM ALGEBRA

In order to construct a quantum theory on the superspace, related to the classical system of the previous section or to any other supersymmetric system of interest formulated on this superspace, one should be able to identify a set of quantum operators satisfying the corresponding algebra. For the superparticle this algebra is identified by applying the quantization rules to the Dirac algebra discussed above. For other systems it should be a suitable covariant generalization. In the functional space of all wave functions defined on the N superspace the standard position operators X^μ , Θ^{ai} , and $\bar{\Theta}^{\dot{a}}_i$ act multiplicatively and together with

$$P_\mu = -i\partial_\mu, \quad (23)$$

$$\Pi_{ai} = -i\partial_{ai}, \quad \bar{\Pi}^{\dot{a}}_i = -i\bar{\partial}^{\dot{a}}_i, \quad (24)$$

$$Q_{ai} = \Pi_{ai} - iP_\mu \sigma^\mu_{a\dot{a}} \bar{\Theta}^{\dot{a}}_i, \quad (25)$$

$$\bar{Q}_{\dot{a}}^i = -\bar{\Pi}_{\dot{a}}^i + iP_\mu \sigma^\mu_{a\dot{a}} \Theta^{ai}, \quad (26)$$

give a representation of the super Poincaré algebra without central charges [9,10]. The covariant derivatives are defined as

$$D_{ai} = \partial_a + i\sigma^\mu_{a\dot{b}} \bar{\Theta}^{\dot{b}}_i \partial_\mu = i\Pi_{ai} - P_\mu \sigma^\mu_{a\dot{b}} \bar{\Theta}^{\dot{b}}_i, \quad (27)$$

$$\bar{D}_{\dot{b}}^i = -\bar{\partial}_{\dot{b}}^i - i\Theta^{ai} \sigma^\mu_{a\dot{b}} \partial_\mu = -i\bar{\Pi}_{\dot{b}}^i + P_\mu \Theta^{ai} \sigma^\mu_{a\dot{b}}. \quad (28)$$

The total angular momentum is given by

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad (29)$$

$$L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu, \quad (30)$$

$$S_{\mu\nu} = -\frac{1}{4}(\Theta^i \sigma_{\mu\nu} \Pi_i + \bar{\Pi}^i \bar{\sigma}_{\mu\nu} \bar{\Theta}_i).$$

This representation is reducible and does not correspond to the quantization of the superparticle. Let us introduce another set of quantum operators $\hat{X}^\mu$, $\hat{\Theta}^{ai}$, and $\hat{\Theta}^{\dot{a}}_i$ and their corresponding momentum operators acting on the superfields in such a way that the corresponding representation of the super Poincaré algebra is irreducible. We propose that these operators should satisfy the quantum algebra given by

$$[\hat{P}_\mu, \hat{\Theta}^{ai}] = [\hat{P}_\mu, \hat{\Theta}^{\dot{a}}_i] = \{\hat{\Theta}^{ai}, \hat{\Theta}^{bj}\} = \{\hat{\Theta}^{\dot{a}}_i, \hat{\Theta}^{\dot{b}}_j\} = 0, \quad (31)$$

$$[\hat{X}^\mu, \hat{P}^\nu] = i\eta^{\mu\nu}, \quad (32)$$

$$\{\hat{\Theta}^{ai}, \hat{\Theta}^{\dot{a}}_j\} = \frac{-1}{2\hat{P}^2} \hat{P}_\mu \bar{\sigma}^{\mu\dot{a}a} \delta^i_j, \quad (33)$$

$$[\hat{X}^\mu, \hat{\Theta}^{ai}] = \frac{i}{2\hat{P}^2} \hat{P}_\nu \sigma^{\nu\dot{a}a} \hat{\Theta}^{bi} \sigma^\mu_{b\dot{a}}, \quad (34)$$

$$[\hat{X}^\mu, \hat{\Theta}^{\dot{a}}_i] = \frac{i}{2\hat{P}^2} \hat{P}_\nu \bar{\sigma}^{\nu\dot{a}a} \hat{\Theta}^{\dot{b}}_i \sigma^\mu_{a\dot{b}}, \quad (35)$$

$$[\hat{X}^\mu, \hat{X}^\nu] = \frac{-i\hat{S}^{\mu\nu}}{\hat{P}^2}, \quad (36)$$

where $\hat{S}^{\mu\nu}$ is the internal angular momentum operator in the subspace where we are representing the algebra and it reduces to the form (30) (in terms of the new operators) for the representation corresponding to superspin 0 supermultiplets. This algebra indeed generalizes the quantum algebra of the massive superparticle obtained by applying the quantization rules to the classical superparticle described in the previous section. As stated before, to perform the quantization of the system one should exhibit a set of

operators and a space of wave functions for which the quantum algebra above holds. The related problem was solved for $N = 1$ in Ref. [5] using the properties of the operators [11–13] which project superfields to the chiral, antichiral, and tensorial supermultiplets, which carry the irreducible representations of the supersymmetry algebra present in the superfields in this case. Following this approach, let us consider the operators $\mathbb{P}_G$ acting on the superfields which project to the irreducible representations of the supersymmetry in the extended case. As we show below, the projected operators

$$\hat{X}^\mu = X^\mu_G \equiv \mathbb{P}_G X^\mu \mathbb{P}_G, \quad (37)$$

$$\hat{\Theta}^{ai} = \Theta^{ai}_G \equiv \mathbb{P}_G \Theta^{ai} \mathbb{P}_G, \quad (38)$$

$$\hat{\Theta}^{\dot{a}}_i = \bar{\Theta}^{\dot{a}}_{iG} \equiv \mathbb{P}_G \bar{\Theta}^{\dot{a}}_i \mathbb{P}_G, \quad (39)$$

$$\hat{P}_\mu = P^\mu_G \equiv \mathbb{P}_G P^\mu \mathbb{P}_G \quad (40)$$

satisfy the desired algebra with the internal angular momentum given by

$$\hat{S}^{\mu\nu} = S^{\mu\nu}_G \equiv \mathbb{P}_G S^{\mu\nu} \mathbb{P}_G. \quad (41)$$

For the particular case in which $\mathbb{P}_G$ projects to the super-spin zero representation we obtain the quantization of the massive superparticle. The explicit expressions for $\mathbb{P}_G$ in $D = 4$ for arbitrary N were obtained in Ref. [12]. In terms of the covariant derivatives they are of the general form

$$\mathbb{P}_G \propto D^q \bar{D}^{2N} D^p, \quad (42)$$

where $q + p = 2N$. They satisfy

$$\mathbb{P}_G D_{ai} \mathbb{P}_G = \mathbb{P}_G \bar{D}_{\dot{a}}^i \mathbb{P}_G = 0, \quad (43)$$

$$[\mathbb{P}_G, Q_{ai}] = [\mathbb{P}_G, \bar{Q}_{\dot{a}}^i] = [J^{\mu\nu}, \mathbb{P}_G] = [P_\mu, \mathbb{P}_G] = 0. \quad (44)$$

A direct computation of the quantum algebra using the expressions for the projectors as was done for the $N = 1$ case is not practical in this case. Instead we present our general result in the following theorem valid with convenient modifications also for $D > 4$.

Theorem—Let $\mathbb{P}_G$ be a projector operator that satisfy (43) and (44). Then the set of operators (37)–(39) satisfy relations (31)–(36) and $S^{\mu\nu}_G$ may be expressed in the form

$$S^{\mu\nu}_G = \tilde{S}^{\mu\nu}_G + W^{\mu\nu}_G \quad (45)$$

with

$$\tilde{S}^{\mu\nu}_G = -\frac{1}{4}(\Theta_G \sigma^{\mu\nu} \Pi_G + \bar{\Pi}_G \bar{\sigma}^{\mu\nu} \bar{\Theta}_G), \quad (46)$$

$$W^{\mu\nu}_G = \frac{P_\alpha}{4P^2} \varepsilon^{\mu\nu\alpha\lambda} \mathbb{P}_G \bar{D}^i \bar{\sigma}_\lambda D_i \mathbb{P}_G. \quad (47)$$

Proof—To obtain this result we consider all the operators

corresponding to the generators of the super Poincaré algebra projected as Eqs. (37)–(40). Since the projectors commute with the generators the projected generators also satisfy the super Poincaré algebra. In particular, let us take the supercharges Q_{ai}^G and $\bar{Q}_{G\dot{a}}^i$. The first step is to note that Eqs. (43) are equivalent to

$$\Pi_{ai}^G = -iP_\mu \sigma^\mu_{a\dot{a}} \bar{\Theta}_G^{\dot{a}i}, \quad (48)$$

$$\bar{\Pi}^G_{\dot{a}i} = -iP_\mu \sigma^\mu_{a\dot{a}} \Theta_G^{ai}. \quad (49)$$

Here and in what follows we use $[P_\mu, \mathbb{P}_G] = 0$ and we write P_μ instead of P_μ^G . We have then

$$Q_{ai}^G = -2iP_\mu \sigma^\mu_{a\dot{b}} \bar{\Theta}_G^{\dot{b}i}, \quad (50)$$

$$\bar{Q}_{G\dot{a}}^i = 2iP_\mu \sigma^\mu_{a\dot{a}} \Theta_G^{ai}, \quad (51)$$

and so we may write Θ^{ai} and $\bar{\Theta}^{\dot{b}i}$ in terms of the charges

$$\Theta_G^{ai} = \frac{i}{4P^2} P_\mu \bar{\sigma}^{\mu\dot{b}a} \bar{Q}_{G\dot{b}}^i, \quad (52)$$

$$\bar{\Theta}_G^{\dot{b}i} = \frac{-i}{4P^2} P_\mu \bar{\sigma}^{\mu\dot{b}a} Q_{ai}^G. \quad (53)$$

Using the supersymmetry algebra for Q_{ai}^G and $\bar{Q}_{G\dot{b}}^i$, one readily shows that relations (31) to (33) hold. To see that relations (34) and (35) also hold we need only to note that

$$[X_G^\mu, Q_{ai}^G] = -\sigma^\mu_{a\dot{a}} \bar{\Theta}_G^{\dot{a}i}, \quad (54)$$

$$[X_G^\mu, \bar{Q}_{G\dot{a}}^i] = \sigma^\mu_{a\dot{a}} \Theta_G^{ai}. \quad (55)$$

Finally, we show the noncommutativity of X_G^ν . We first proof Eq. (45). Writing Π_{ai} and Θ^{ai} in terms of the covariant derivatives and the super charges

$$\Pi_{ai} = \frac{1}{2}(Q_{ai} - iD_{ai}), \quad (56)$$

$$\Theta^{ai} = \frac{-1}{2P^2} P_\nu \bar{\sigma}^{\nu\dot{b}a} (\bar{D}_{\dot{b}}^i - i\bar{Q}_{\dot{b}}^i), \quad (57)$$

and using $\mathbb{P}_G D_{ai} \mathbb{P}_G = 0$ implies also that $\mathbb{P}_G Q_{bj} D_{ai} \mathbb{P}_G = 0$, and we see that

$$\begin{aligned} S_G^{\mu\nu} &= \mathbb{P}_G S^{\mu\nu} \mathbb{P}_G = \frac{-1}{4} \mathbb{P}_G (\Theta^i \sigma_{\mu\nu} \Pi_i + \bar{\Pi}^i \bar{\sigma}_{\mu\nu} \bar{\Theta}_i) \mathbb{P}_G \\ &= \frac{P_\alpha}{8P^2} \mathbb{P}_G ((\bar{D}^i - i\bar{Q}^i) \bar{\sigma}^\alpha \sigma_{\mu\nu} (Q_i - iD_i) + \text{c.c.}) \mathbb{P}_G \\ &= \frac{-iP_\alpha}{8P^2} \mathbb{P}_G (\bar{D}^i \bar{\sigma}^\alpha \sigma_{\mu\nu} D_i) \mathbb{P}_G \\ &\quad + \frac{-iP_\alpha}{8P^2} \mathbb{P}_G (\bar{Q}^i \bar{\sigma}^\alpha \sigma_{\mu\nu} Q_i) \mathbb{P}_G + \text{c.c.} \end{aligned} \quad (58)$$

The first term is Eq. (47). For the second term we use Eq. (52) and the relation $Q^G = 2\Pi^G$ to obtain the right side of Eq. (46). This proves Eq. (45) for the projected angular momentum operator. We note that from Eq. (45), Pryce

condition [8] $P_\mu S^{\mu\nu} = 0$ may be readily deduced. Using this and the algebra of $J^{\mu\nu}$ one obtains

$$[J^{\mu\nu}, X_G^\lambda] = -iX_G^\mu \eta^{\nu\lambda} + iX_G^\nu \eta^{\mu\lambda}, \quad (59)$$

$$\begin{aligned} [S_G^{\mu\nu}, X_G^\lambda] &= -iX_G^\mu \eta^{\nu\lambda} + iX_G^\nu \eta^{\mu\lambda} \\ &\quad - [X_G^\mu P^\nu - X_G^\nu P^\mu, X_G^\lambda], \end{aligned} \quad (60)$$

$$[S_G^{\mu\nu}, X_G^\lambda] = -[X_G^\mu, X_G^\lambda] P^\nu + [X_G^\nu, X_G^\lambda] P^\mu, \quad (61)$$

$$0 = [P_\mu S_G^{\mu\nu}, X_G^\lambda] = -iS^{\lambda\nu} + P_\mu [S_G^{\mu\nu}, X_G^\lambda], \quad (62)$$

$$iS^{\lambda\nu} = -P_\mu [X_G^\mu, X_G^\lambda] P^\nu + P_\mu [X_G^\nu, X_G^\lambda] P^\mu. \quad (63)$$

Note that the Jacobi identity implies that P_μ commutes with $[X_G^\alpha, X_G^\beta]$. In the last equation multiplying by P_ν and doing the sum it is shown that $P_\nu [X_G^\nu, X^\lambda] = 0$. From here the result

$$[X_G^\mu, X_G^\nu] = -iS_G^{\mu\nu}/P^2 \quad (64)$$

is obtained.

IV. THE PAULI-LUBANSKI VECTOR AND $W^{\mu\nu}$

Let us see now that the $W^{\mu\nu}$ term of the internal angular momentum operator [see Eq. (45)] is closely related to the super Pauli-Lubanski vector and fix the superspin of the representation. Defining the Pauli-Lubanski vector in the representation space projected by $\mathbb{P}_G$ in the usual form

$$W_\mu^G = \mathbb{P}_G W_\mu \mathbb{P}_G = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} P_G^\nu J_G^{\alpha\beta} + \frac{1}{8} \bar{Q}_G^i \bar{\sigma}_\mu Q_i^G, \quad (65)$$

where $J_G^{\mu\nu} = L_G^{\mu\nu} + S_G^{\mu\nu}$ and

$$S_G^{\mu\nu} = \tilde{S}_G^{\mu\nu} + W_G^{\mu\nu}, \quad (66)$$

$$W_G^{\mu\nu} = \frac{-P_\alpha}{4P^2} \varepsilon^{\mu\nu\alpha\lambda} \mathbb{P}_G D_i \sigma_\lambda \bar{D}^i \mathbb{P}_G, \quad (67)$$

one can show that

$$\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} P_G^\nu (L_G^{\mu\alpha} + \tilde{S}_G^{\mu\alpha}) + \frac{1}{8} \bar{Q}_G^i \bar{\sigma}_\mu Q_i^G = 0. \quad (68)$$

Then the only nonvanishing contribution to the super Pauli-Lubanski four vector comes from the $W^{\mu\nu}$ term, and one can write

$$W_\mu^G = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} P_G^\nu W_G^{\alpha\beta} \quad (69)$$

or equivalently

$$W_G^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} P_{G\alpha} W_\beta^G. \quad (70)$$

This shows directly that in the quantum theory of the superparticle for which one should have $W_G^{\mu\nu} = 0$, W_μ^G also vanishes and so the corresponding super multiplet has superspin 0. Supermultiplets with higher values of the superspin appear in the other cases.

V. CENTRAL CHARGES AND SUPERPROJECTORS

Let us now turn to the construction of the quantum algebra when there are central charges in the supersymmetry algebra. In what follows we consider the $N = 2$ superspace. Other cases may be discussed along the same lines.

The supersymmetry algebra is given by

$$\{Q_{ai}, Q_{bj}\} = 2i\varepsilon_{ab}\varepsilon_{ij}Z, \quad \{Q_{ai}, \bar{Q}_{\dot{a}}^j\} = 2\delta_i^j\sigma^\mu_{a\dot{a}}P_\mu. \quad (71)$$

Superspace coordinates are now $(x^\mu, z, \bar{z}, \theta^{ai}, \bar{\theta}^{\dot{a}}_i)$ and the algebra is represented on the superfields by the operators

$$Z = -i\frac{\partial}{\partial z}, \quad \bar{Z} = -i\frac{\partial}{\partial \bar{z}}, \quad (72)$$

$$\Pi_{ai} = -i\partial_{ai}, \quad \bar{\Pi}_{\dot{a}}^i = -i\bar{\partial}_{\dot{a}}^i, \quad (73)$$

$$Q_{ai} = \Pi_{ai} - iP_\mu\sigma^\mu_{a\dot{a}}\bar{\Theta}^{\dot{a}}_i - \varepsilon_{ab}\varepsilon_{ij}Z\Theta^{bj}, \quad (74)$$

$$\bar{Q}_{\dot{a}}^i = -\bar{\Pi}_{\dot{a}}^i + i\Theta^{ai}P_\mu\sigma^\mu_{a\dot{a}} - \varepsilon_{ab}\varepsilon_{ij}\bar{Z}\bar{\Theta}^{\dot{b}}_j. \quad (75)$$

As a guide for the construction of the quantum algebra let us consider the dynamics of the superparticle in this superspace. The action for this system was proposed some time ago by Azcarraga and Lukierski [14]. There are three simple invariants that can be build in terms of the coordinates, $w^\mu w_\mu$ as before, $(\varepsilon_{ab}\varepsilon_{ij}\theta^{ai}\bar{\theta}^{bj} + \dot{z})$ and its complex conjugate. Therefore a general action invariant under the supersymmetry algebra is

$$S = \int \left(\frac{1}{2}(e^{-1}w^\mu w_\mu - em^2) + \ell(\theta^{ai}\dot{\theta}_{ai} + \dot{z}) + \ell^*(\bar{z} - \bar{\theta}^{\dot{a}}_i\dot{\bar{\theta}}^{\dot{a}i}) \right) d\tau. \quad (76)$$

Here ℓ is an arbitrary parameter with mass units. Generalized momenta are given by

$$p_\mu = e^{-1}\omega_\mu, \quad \pi_z = \ell, \quad \pi_{\bar{z}} = \ell^*, \quad (77)$$

$$\begin{aligned} \pi_{ai} &= -ip_\mu\sigma^\mu_{a\dot{b}}\bar{\theta}^{\dot{b}}_i - \pi_z\theta_{ai}, \\ \bar{\pi}_{\dot{a}}^i &= -ip_\mu\sigma^\mu_{a\dot{a}}\theta^{ai} - \bar{\theta}_{\dot{a}}^i\pi_{\bar{z}}. \end{aligned} \quad (78)$$

The constraints are now

$$\pi_e = 0 \rightarrow p^2 + m^2 = 0, \quad (79)$$

$$d_{ai} = \pi_{ai} + ip_\mu\sigma^\mu_{a\dot{b}}\bar{\theta}^{\dot{b}}_i + \pi_z\theta_{ai}, \quad (80)$$

$$\bar{d}_{\dot{a}}^i = \bar{\pi}_{\dot{a}}^i + ip_\mu\sigma^\mu_{a\dot{a}}\theta^{ai} - \bar{\theta}_{\dot{a}}^i\pi_{\bar{z}}. \quad (81)$$

And satisfy:

$$\{d_{ai}, d_{bj}\} = -2\varepsilon_{ij}\varepsilon_{ab}\pi_z, \quad \{\bar{d}_{\dot{a}}^i, \bar{d}_{\dot{b}}^j\} = 2\varepsilon^{ij}\varepsilon_{\dot{a}\dot{b}}\pi_{\bar{z}}, \quad (82)$$

$$\{d_{ai}, \bar{d}_{\dot{a}}^j\} = -2i\delta_i^j p_\mu\sigma^\mu_{a\dot{a}}. \quad (83)$$

The nature of the constraints depends on the value of ℓ . We consider first the case $|\ell| \neq m$. Then, all the constraints are second class. Introducing $\hat{C}$ the inverse of the constraints matrix defined by (82) and (83) we have

$$\hat{C}^{abij} = \frac{1}{2\Delta}\varepsilon^{ab}\varepsilon^{ij}\pi_{\bar{z}}, \quad (84)$$

$$\hat{C}^{a\dot{a}i}_{\dot{j}} = \frac{1}{2i\Delta}p_\mu\bar{\sigma}^{\mu\dot{a}a}\delta^i_j, \quad (85)$$

$$\Delta = p^2 + \pi_z\pi_{\bar{z}}. \quad (86)$$

Dirac brackets are then obtained as

$$\{\theta^{ai}, \theta^{bj}\}_D = -\hat{C}^{abij}, \quad \{\bar{\theta}^{\dot{a}}_i, \bar{\theta}^{\dot{b}}_j\}_D = -\hat{C}^{\dot{a}\dot{b}}_{ij}, \quad (87)$$

$$\{\theta^{ai}, x^\mu\}_D = -\hat{C}^{acik}i\sigma^\mu_{c\dot{a}}\bar{\theta}^{\dot{a}}_k - \hat{C}^{a\dot{a}i}_{\dot{k}}i\sigma^\mu_{b\dot{a}}\theta^{bk}, \quad (88)$$

$$\{x^\mu, x^\nu\}_D = -\frac{S^{\mu\nu}}{p^2}, \quad (89)$$

where $S^{\mu\nu}$ is given by Eq. (11) which in this case is not equal to Eq. (22).

Quantization of this system may be done on the space of superfields using the projectors approach. Straightforward quantization of the classical algebra leads to the following quantum algebra:

$$\{\hat{\Theta}^{ai}, \hat{\Theta}^{bj}\} = \frac{i}{2\Delta}\varepsilon^{ab}\varepsilon^{ij}\hat{\Pi}_{\bar{z}}, \quad (90)$$

$$\{\hat{\Theta}^{\dot{a}}_i, \hat{\Theta}^{\dot{b}}_j\} = \frac{-i}{2\Delta}\varepsilon^{\dot{a}\dot{b}}\varepsilon_{ij}\hat{\Pi}_z, \quad (91)$$

$$\begin{aligned} [\hat{\Theta}^{ai}, \hat{X}^\mu] &= \frac{1}{2\Delta}\varepsilon^{ab}\varepsilon^{ij}\ell^*\sigma^\mu_{b\dot{b}}\hat{\Theta}^{\dot{b}}_j \\ &+ \frac{i}{2\Delta}P_\mu P_\nu \bar{\sigma}^{\nu\dot{a}a}\sigma^\mu_{b\dot{a}}\hat{\Theta}^{\dot{b}}_b, \end{aligned} \quad (92)$$

$$[\hat{X}^\mu, \hat{X}^\nu] = \frac{-i\hat{S}^{\mu\nu}}{p^2}. \quad (93)$$

To find a representation of this quantum algebra we use the superprojectors method discussed above. The superprojector operator $\mathbb{P}_G$ should commute with all the generators of the super Poincaré algebra and satisfies

$$\mathbb{P}_G D_{ai} \mathbb{P}_G = \mathbb{P}_G \bar{D}_{\dot{a}}^i \mathbb{P}_G = 0, \quad (94)$$

$$D_{ai} = i\Pi_{ai} - P_\mu\sigma^\mu_{a\dot{a}}\bar{\Theta}^{\dot{a}}_i + i\varepsilon_{ab}\varepsilon_{ij}Z\Theta^{bj}, \quad (95)$$

$$\bar{D}_{\dot{a}}^i = -i\bar{\Pi}_{\dot{a}}^i + \Theta^{ai}P_\mu\sigma^\mu_{a\dot{a}} + i\varepsilon_{ab}\varepsilon_{ij}\bar{Z}\bar{\Theta}^{\dot{b}}_j. \quad (96)$$

Using the same techniques as for the case without central charges we can show that using the operators

$$\hat{X}^\mu = \mathbb{P}_G X^\mu \mathbb{P}_G, \quad (97)$$

$$\hat{\Theta}^\mu = \mathbb{P}_G \Theta^\mu \mathbb{P}_G, \quad (98)$$

$$\hat{\bar{\Theta}}^\mu = \mathbb{P}_G \bar{\Theta}^\mu \mathbb{P}_G \quad (99)$$

realize the quantum algebra (89). The first step is to note that using (74), (75), (95), and (96), Θ^{ai} and $\bar{\Theta}^{\dot{a}}_i$ can be written in terms of the supercharges and the covariant derivatives. In a matrix form:

$$\begin{pmatrix} \Theta^{ai} \\ \bar{\Theta}^{\dot{a}}_i \end{pmatrix} = \begin{pmatrix} \frac{-1}{2\Delta} \varepsilon^{ab} \varepsilon^{ij} \bar{Z} & \frac{1}{2i\Delta} P_\mu \bar{\sigma}^{\mu\dot{b}a} \delta^i_j \\ \frac{1}{2i\Delta} P_\mu \bar{\sigma}^{\mu\dot{a}a} \delta^i_j & \frac{1}{2\Delta} \varepsilon^{\dot{a}\dot{b}} \varepsilon_{ij} Z \end{pmatrix} \times \begin{pmatrix} Q_{bj} + iD_{bj} \\ -\bar{Q}_b^j - i\bar{D}_b^j \end{pmatrix}. \quad (100)$$

This implies, in particular, that

$$\hat{\Theta}^{ai} = \frac{-1}{2\Delta} \varepsilon^{ab} \varepsilon^{ij} \bar{Z} \hat{Q}_{bj} - \frac{1}{2i\Delta} P_\mu \bar{\sigma}^{\mu\dot{a}a} \hat{\bar{Q}}^{\dot{a}}_i, \quad (101)$$

$$\hat{\bar{\Theta}}^{\dot{a}}_i = \frac{1}{2\Delta} P_\mu \bar{\sigma}^{\mu\dot{a}a} \hat{Q}_{ai} - \frac{1}{2\Delta} \varepsilon^{\dot{a}\dot{b}} \varepsilon_{ij} Z \hat{\bar{Q}}^{\dot{b}}_j. \quad (102)$$

With these formulas at hand, it is easy to check all commutators in Eq. (87)–(89) except the Pryce relation. To prove this note that $\hat{S}^{\mu\nu}$ may be rewritten exactly in the same form as in the case without central charges

$$\begin{aligned} \hat{S}^{\mu\nu} &= \frac{-1}{4} (\hat{\Theta}^i \sigma_{\mu\nu} \hat{\Pi}_i + \hat{\bar{\Pi}}^i \bar{\sigma}_{\mu\nu} \hat{\bar{\Theta}}_i) \\ &\quad - \frac{-iP_\alpha}{8P^2} \mathbb{P}_G \bar{D}^i \bar{\sigma}_{\mu\nu} D_i \mathbb{P}_G. \end{aligned} \quad (103)$$

Now the arguments that follow Eq. (59) remain unchanged, and we conclude again

$$[\hat{X}^\mu, \hat{X}^\mu] = \frac{-i\hat{S}^{\mu\nu}}{P^2}. \quad (104)$$

VI. THE κ SYMMETRIC CASE

The set of constraints imply

$$\ell \bar{\pi}_{\dot{a}}^i - i p_\mu \sigma^\mu_{\dot{a}a} \pi^{ai} - (p^2 + |\ell|^2) \bar{\theta}_{\dot{a}}^i = 0. \quad (105)$$

If $|\ell| = m$ this constraints are first class and indeed the system is κ -symmetric. Define now,

$$C_a^i = \ell^* \pi_{ai} + i p_\mu \sigma^\mu_{a\dot{a}} \bar{\pi}^{\dot{a}}_i, \quad (106)$$

$$\bar{C}_{\dot{a}}^i = \ell \bar{\pi}_{\dot{a}}^i - p_\mu \sigma^\mu_{\dot{a}a} \pi^{ai}. \quad (107)$$

Following Ref. [14] we have a set of independent first class constraints,

$$\phi = p^2 + m^2 = 0, \quad \pi_z - \ell = 0, \quad \pi_{\bar{z}} - \ell^* = 0, \quad (108)$$

$$C_a^2 = 0, \quad \bar{C}_{\dot{a}}^2 = 0 \quad (109)$$

together with a set of second class constraints

$$d_{a1}, \quad \bar{d}_{\dot{a}}^1. \quad (110)$$

To make the transition to the quantum theory we can split the second class constraints into a set of first class constraints and some gauge fixing conditions, or we may rely on Dirac's procedure and use super projectors. Of course, both mechanisms should give the same answers. The first displays more clearly the particle content of the theory, while the second allows us to represent directly position operators as self-adjoint operators in a Hilbert space.

To follow the first approach we should specify an admissible set of first class constraints. For example, we can take $\bar{d}_{\dot{a}}^2$, $\bar{C}_{\dot{a}}^2$, C_a^2 . The quantum mechanics of this system should then be formulated in terms of a superfield $V(x^\mu, \theta^{ai}, \bar{\theta}^{\dot{a}}_i, z)$ that satisfy the equations:

$$(P^2 + m^2)V = 0, \quad \bar{D}_{\dot{a}}^1 V = 0, \quad (111)$$

$$-i\partial_z V = \ell V, \quad i\partial_{\bar{z}} V = \ell^* V,$$

$$\ell \bar{d}_{\dot{a}}^2 V = \sigma^\mu_{\dot{a}a} \partial_\mu \partial^{a2} V, \quad (112)$$

$$\ell^* \partial_{ai} V = -\sigma^\mu_{a\dot{a}} \partial_\mu \bar{d}_{\dot{a}}^1 V. \quad (113)$$

Let us consider now the method of superprojectors. Dirac's procedure for this system has been carried out with some detail in Ref. [14], so we need only to quote the results. First note that the Dirac bracket of θ^{a2} and $\bar{\theta}^{\dot{a}}_2$ with anything remains unchanged because they commute with the constraints (d_{a2} and $\bar{d}_{\dot{a}}^2$). The algebra of the π 's can always be deduced from the rest so the independent degrees of freedom are encoded in the variables $(x^\mu, p_\nu, \theta^{a1}, \bar{\theta}^{\dot{a}}_1)$, and the algebra of these variables is exactly the same as the $N = 1$ case and is given by Eqs. (17)–(21) with $i, j = 1$. Also, the $S^{\mu\nu}$ piece is given by (22) but only keeps the $i = 1$ term. Quantization now follows exactly the same rules as the $N = 1$ case, but the projectors are constructed from the following covariant derivatives:

$$D_{ai} = \partial_{ai} + i\sigma^\mu_{a\dot{b}} \bar{\Theta}^{\dot{b}}_i \partial_\mu + m\varepsilon_{ab} \varepsilon_{ij} \theta^{aj}, \quad (114)$$

$$\bar{D}_{\dot{b}}^i = -\bar{\partial}_{\dot{b}}^i - i\Theta^{ai} \sigma^\mu_{a\dot{b}} \partial_\mu + m\varepsilon_{\dot{a}\dot{b}} \varepsilon^{ij} \bar{\theta}^{\dot{a}}_j. \quad (115)$$

VII. QUANTIZATION OF MASSLESS SUPERPARTICLES

The discussion of the previous section suggests a way to treat another interesting problem, that of the quantization of massless superparticles in $D = 4, 6, 8$. It is known that for massless particles it is difficult to define position operators. Here this is reflected in the fact that for $m = 0$ the constraints (12) and (13) are a mixture of first and second

class constraints. Also, it is not easy to disentangle the first and second class constraints in a Lorentz covariant and irreducible way or to find a Lorentz covariant gauge fixing condition. Instead we note that the set of first and second class constraints

$$p^2, \quad d_a, \quad \bar{d}_{\dot{a}} \quad (116)$$

is equivalent to a system with only first class constraints

$$p^2, \bar{d}_{\dot{a}}, \phi^a \equiv S^{\mu ab} P_{\mu} d_b. \quad (117)$$

That is the set (116) is obtained from the set (117) upon gauge fixing. To show this, we need to see that if d_a is any field satisfying $p_{\mu} S^{\mu ab} d_b = 0$ then there is a gauge transformation generated by $\bar{d}_{\dot{a}}$ that sets $d_a = 0$. Now the solution of $p_{\mu} S^{\mu ab} d_b = 0$ is $d_b = S^{\mu}_{b\dot{a}} \xi^{\dot{a}}$, so the only thing we have to do is make a gauge transformation generated by $\xi^{\dot{a}} \bar{d}_{\dot{a}}$. Now quantization is straightforward. The superfield satisfies

$$\bar{D}_{\dot{a}} V = 0, \quad (118)$$

$$S^{\mu ab} D_b \partial_{\mu} V = 0. \quad (119)$$

This procedure also works nicely for the $D = 10$, IIB superparticle. In that case we have two Majorana-Weyl spinor θ^{ai} of the same chirality that can be recast in a single θ^a , $\bar{\theta}^{\dot{a}}$ Weyl, but not Majorana fermion. In this language $D = 10$, IIB superspace can be treated as before and the equations above may be considered as the quantization of the IIB superparticle which is related to the linearized IIB supergravity.

VIII. CONCLUSION

In this paper we have generalized the construction of the quantum algebra of the $N = 1$, $D = 4$ superspace without central charges for an arbitrary number N of supersymmetric charges. This was done using the properties of the

superprojectors which project to the irreducible supersymmetric multiplets. Here again the most remarkable aspect of the algebra is related to the nontrivial commutator of the position operators, which is proportional to the internal angular momentum operator. As another interesting point, we show that the quantum internal angular momentum splits in the general case in two parts, one with the form suggested by the quantization of the superparticle and a second term which modifies the expression of the super Pauli-Lubanski vector. Our construction gives, in particular, a complete covariant solution of the quantization of the massive superparticle in terms of the superspin 0 supermultiplet, but includes also as for $N = 1$, the quantum algebra of other different systems with higher values of the superspin. The identification of the corresponding pseudoclassical actions may give us some clues on the physics of ten dimensional superspace.

We also applied the same methods to discuss the quantum algebras associated to the $N = 2$, $D = 4$ superspace with central charges and their relation with the corresponding superparticle. For the κ -symmetric case, when the central charge and the mass are equal, we discuss the reduction to the physical degrees of freedom of the superparticle by a reinterpretation of some of the second class constraints as gauge fixing conditions. We also mention how this approach may also be applied to massless particles and the $D = 10$, IIB superparticle.

Our methods could be generalized to other dimensions, in particular $D = 10$ [15], where they may provide a new approach for the covariant quantization of supersymmetric systems.

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