Traversable wormholes: Minimum violation of the null energy condition revisited

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It was argued in literature that traversable wormholes can exist with an arbitrarily small violation of null energy conditions. I show that if the amount of exotic material near the wormhole throat tends to zero, either this leads to a horn instead of a wormhole or the throat approaches the horizon in such a way that infinitely large stresses develop on the throat.

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I. INTRODUCTION AND BASIC EQUATIONS

One of the key features of wormholes is that the null energy condition (NEC) should be violated in the vicinity of a throat [1]. The corresponding material has a property unusual for laboratory physics and is called "exotic." Although the inevitability of violation of NEC is well-known in the physics of wormholes, the extent to which it occurs is still being debated in literature. It was stated in [2] that the amount of exotic matter needed to support wormholes can be made as small as one likes (see also [3–6]). Let there be a sequence of traversable wormholes depending on some parameter ε such that in the limit $\varepsilon \to 0$ the amount of exotic material tends to zero. According to [2], one is inclined to think that a configuration with arbitrary small but nonzero ε represents a usual traversable wormhole and, thus, some kind of discontinuity happens that separates the state with $\varepsilon = 0$ from those with $\varepsilon \neq 0$. By itself, this does not necessarily mean something wrong since topological properties of the configuration with $\varepsilon = 0$ and $\varepsilon \neq 0$ are qualitatively different, so one may or may not expect discontinuity. Indeed, a state with small $\varepsilon \neq 0$ connects two asymptotically flat universes whereas the state with $\varepsilon = 0$ does not.

Nonetheless, the work [2] left one key question open. Let me call, for brevity, a wormhole standard if (1) there is no horizon, (2) its geometry remains regular (Kretschmann scalar is finite), and (3) the areal radius r as a function of the proper distance l grows away from the throat in some finite vicinity. It is not quite clear from the results of [2] whether or not the limiting configuration remains the standard wormhole when the amount of exotic material tends to zero. The aim of the present article is to show that the answer to this question is negative. In this sense, if I restrict myself to standard wormholes, an amount of exotic material *cannot* be made arbitrarily small. I also find what metric appears in the limit $\varepsilon = 0$ depending on which of the conditions (1)–(3) is violated.

It is worth noting another approach in which the constraints on the wormhole geometries are derived from quantum inequalities which the null-contracted stress-energy tensor should obey, when averaged over a timelike worldline [7]. Then, it turns out that in the concrete models considered as candidates for arbitrarily small violation of NEC, the typical throat radius cannot be macroscopic. Meanwhile, in my approach I do not constrain the properties of the source (so, classical sources are, in principle, also allowed provided they are compatible with NEC violation) and appeal directly to geometrical consequences which follow from the definitions of wormholes and Einstein equations.

I consider the spherically-symmetric metrics

$$ds^2 = -e^{2\Phi}dt^2 + \frac{dr^2}{V} + r^2 d\omega^2, \qquad d\omega^2 = \sin^2\theta d\phi^2 + d\theta^2, \qquad V = 1 - \frac{b}{r}, \tag{1}$$

where b and Φ are the shape and redshift functions, respectively. I assume that the stress-energy tensor has the diagonal form

$$T_\mu^\nu = \mathrm{diag}(-\rho, p_r, p_t, p_t). \tag{2}$$

Then, it follows from the Einstein equations that

$$\rho = \frac{b'}{8\pi r^2}, \qquad p_r = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right],$$
$$p_t = \frac{1}{8\pi}\left\{\left(1-\frac{b}{r}\right)\left[\Phi'' + \Phi'\left(\Phi' + \frac{1}{r}\right)\right] - \frac{b' - \frac{b}{r}}{2r}\left(\Phi' + \frac{1}{r}\right)\right\}. \tag{3}$$

The $r - r$ component of the conservation law $T^\nu_{\mu;\nu} = 0$ that can be obtained from Einstein Eqs. (3) reads

$$p_r' + \Phi'\rho + \left(\Phi' + \frac{2}{r}\right)p_r - \frac{2p_t}{r} = 0. \tag{4}$$

It is convenient to introduce the quantity

$$\xi \equiv 8\pi r^2(p_r + \rho) \tag{5}$$

and use, along with the coordinate r, also the proper distance

$$l = \int \frac{dr}{\sqrt{V}}. \tag{6}$$

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Then one obtains from (1) and (3) that

$$V = 1 - \frac{r_0}{r} - \frac{2m(r)}{r}, \qquad m(r) = 4\pi \int_{r_0}^{r} \rho \bar{r}^2 d\bar{r}, \quad (7)$$

$$\xi = -V'r + 2\Phi' rV, \quad (8)$$

$$\frac{d^2 r}{dl^2} = \frac{V'(r)}{2}. \quad (9)$$

At the throat $r = r_0$ I must have, by definition, $\frac{dr}{dl} = 0$, so that $b(r_0) = r_0$, and

$$\frac{d^2 r}{dl^2}(r_0) = -\frac{\xi(r_0)}{2r_0}, \quad (10)$$

$$p_t(r_0) = \frac{1 - b'(r_0)}{16\pi r_0}\left[\Phi'(r_0) + \frac{1}{r_0}\right]. \quad (11)$$

I am interested in traversable wormholes, so the horizon is supposed to be absent; $\Phi(r_0)$ is finite.

II. MEASURING THE AMOUNT OF EXOTIC MATTER

Now I must choose the method for measuring the degree of "exoticism." In general, NEC is violated on the throat and/or in some vicinity of it. Therefore, one can, instead of NEC itself, consider the averaged null energy condition obtained by the integration of NEC [8]. This procedure was somewhat changed in [2]. To gain information about the total amount of matter which violates NEC, it was suggested in [2] to consider the volume integral $I \equiv \int_{r_0}^{\infty} dr\xi$. In this connection, I must make a technical remark. The boundary term at infinity was lost in Eq. (12) of [2] which is equal to $b(\infty)$ and, in general, does not vanish contrary to what is stated in [2]. What is more important, the definition of I should be, in my view, modified. In the form in which it was introduced in [2], the integral extends to infinity. Therefore, it may happen that $I > 0$, but there is a region of strong violation of NEC compensated by the contribution of the normal matter. Meanwhile, it looks more natural to be interested in the contribution from the exotic region alone. Therefore, I will consider the somewhat different definition

$$I \equiv \int_{r_0}^{a} dr\xi = 8\pi \int_{r_0}^{a} drr^2(p_r + \rho), \quad (12)$$

where it is assumed that the exotic matter fills the inner region $r_0 \le r < a$, while the outer region $r \ge a$ is occupied by the normal matter with NEC satisfied or by vacuum ("normal region" for brevity). In doing so, I use the same integral measure as in [2] but restrict the integration by the exotic region. The appearance of the factor drr^2 in the integral (12) is motivated by the analogy with the Arnowitt-Deser-Misner mass formula as is explained in [2]. For my system, the quantity $\xi < 0$ for $r_0 \le r < a$ and $\xi \ge 0$ for $r \ge a$. Actually, in concrete examples considered in [2], the value of I does not depend on which of two definitions is used since the outer region lies in vacuum and does not contribute to the integral I.

As $\xi \le 0$ in the integrand, I have $I \le 0$. My goal is to elucidate whether and under which conditions one can obtain $I \to 0$. As I want to preserve regularity I consider everywhere finite ξ. Then $I \to 0$ entails that either (1) $a \to r_0$ (limits of integration shrink) or (2) $\xi \to 0$ (the integrand vanishes) in the whole exotic region [or both (1) and (2) hold]. In other words, I can try to minimize either the size of the region or exoticism or violation of NEC in the relationship between p_r and ρ (i.e., in the equation of state). Let me discuss different cases separately.

III. MINIMIZING THE SIZE OF THE REGION WITH EXOTIC MATERIAL

A. Continuous distribution

1. $\xi(r_0) < 0$

Let me now assume that pressures p_r, p_t and the energy density ρ are continuous functions of r. Therefore, on the border between the exotic and normal regions,

$$\xi(a) = 0. \quad (13)$$

On the throat, it is supposed that NEC is violated as usual [1,8], so $\xi(r_0) < 0$. I am interested in regular configurations. The function $\Phi(r)$ and its first and second derivatives are supposed to be bounded, unless the opposite is stated explicitly. In the limit under discussion $\lim_{a\to r_0} \frac{d\xi}{dr}(a) = \lim_{a\to r_0} \frac{\xi(a)-\xi(r_0)}{a-r_0} \to \infty$. It follows also from (4) that p_r' is finite. Then, the only way to reconcile the finiteness of p_r' with the divergency of ξ' consists in considering infinite ρ'. Let me consider, without a big loss of generality, the density profile of the form

$$\rho = \rho_0 + \Delta\rho \frac{(r - r_0)^n}{(a - r_0)^n} \quad (14)$$

for $r_0 \le r \le a$, and $\rho(r)$ is some smooth function with bound derivatives for $r > a$. It follows from (14) that $\rho(r_0) = \rho_0$, $\rho(a) = \rho_0 + \Delta\rho$. From (13), I have that $\rho(a) = -p_r(a)$. In the limit $a \to r_0$ $\rho'(a) \to \infty$, so I have a jump in ρ. (For $a - r_0$ small but nonzero the distribution of ρ is still continuous.) Because of the continuity of p_r and the finiteness of p_r', in this limit I obtain also that $\rho(a) = -p_r(r_0) + O(\delta)$, $\delta = a - r_0$. On the throat, Eq. (3) entails a well-known equality $p_r(r_0) = -(8\pi r_0^2)^{-1}$ [1,8]. Therefore, for small δ

$$\rho(a) = \frac{1}{8\pi a^2} + O(\delta). \quad (15)$$

How does the geometry look in this limit? To answer this question, consider first the geometry for $r > a$. I assume that for $r \to a$ the corresponding function admits the Taylor expansion $V = V(a) + V'(a)(r - a) + \frac{V''(a)}{2} \times (r - a)^2 + O((r - a)^3)$ with finite coefficients. Then, it follows from (7) and (15) that $V(a) \to 0$ and $V'(a) \to 0$ in the limit $\delta \to 0$. As a result, the metric in the vicinity of $r = a$ looks like

$$ds^2 = -dt^2 \exp[2\Phi(a)] + \frac{2dr^2}{V''(a)(r-a)^2} + a^2 d\omega^2 \quad (16)$$

and represents a horn in the sense that the proper distance between $r = a$ and $r > a$ diverges. In doing so, the quantity $\exp[2\Phi(a)]$ does not vanish since there is no horizon by my assumption. The part of the manifold $r \geq a$ becomes geodesically complete and does not resemble a wormhole.

It is also instructive to trace what happens to the region $r_0 \leq r < a$ in this limit. Again, I assume the Taylor expandability in the vicinity of r_0. Then, it is easy to obtain that for small δ the expansion reads

$$V = \frac{r - r_0}{r} Z, \quad (17)$$

$$Z = -\xi(r_0) - 8\pi r_0^2 \Delta\rho \frac{(r - r_0)^n}{(n+1)\delta^n} + O(\delta^2). \quad (18)$$

Making a substitution $r = r_0 + \delta y$, where $0 \leq y \leq 1$, I obtain that

$$ds^2 = -dt^2 \exp[2\Phi(a)] + \delta \frac{dy^2}{f(y)} + r_0^2 d\omega^2, \quad (19)$$

$f(y) = -\xi(r_0)y - 8\pi r_0^2 \Delta\rho \frac{y^n}{(n+1)}$ does not contain δ. Then, it becomes obvious that the proper distance between the throat and the boundary $r = a$ between the normal and exotic regions is finite and, moreover, tends to zero, so that this part of space shrinks to a disc and, thus, is removed from the manifold.

2. *Case* $\xi(r_0) = 0$

Up to now, I assumed that $\xi(r_0) < 0$. Correspondingly, the asymptotic expansion of the metric coefficient began from the linear terms according to (18). The situation changes if $\xi(r_0) = 0$. The exotic region $\xi(r) < 0$ is assumed to occupy the interval $r_0 < r < a$ and $\xi(a) = 0$. Let me try again to find the configuration with the almost zero violation of NEC. As now $\xi(r_0) = 0$, it follows from (8) that $V'(r_0) = 0$. It follows from (7) that this condition means $\rho = \rho_0 = (8\pi r_0^2)^{-1}$. I assume that in the vicinity of the throat I have the asymptotics

$$V = A(r - r_0)^{n+1}, \quad (20)$$

with $A = \text{const}$ and $n > 0$. Instead of a usual minimum of the function $r(l)$ typical of $n = 0$, now I deal with the minimum of the higher order. I want to find a regular traversable wormhole configuration, so the proper distance to the throat must be finite. Thus, $0 < n < 1$. This is similar to the case discussed in Sec. 4.1 of [6].

Then, I obtain from (8) that

$$\xi = -r_0(n+1)A(r - r_0)^n + B(r - r_0)^{n+1} + \ldots \quad (21)$$

The exact value of the coefficient B (as well as the coefficients at the higher degrees in the expansion) is irrelevant in the given context. What is important is the fact that A does not contain the small parameter, so that ξ has the general form $\xi = -A_1 y + A_2 y^m$, $m = \frac{n+1}{n} > 1$, $y = r - r_0$, A_1 and A_2 are finite, and $A_1 > 0$. Then, it becomes clear that the function ξ attains its minimum at some r_1 with finite $r_1 - r_0$. In turn, this means that $a - r_0 > a - r_1$ is also finite and cannot be made arbitrarily small. Thus, although now the proper distance is finite, efforts to achieve the almost zero violation of NEC failed again.

On the other hand, I may impose by brute force the condition $a \to r_0$, allowing a sequence of configurations with more and more small A. As the point $r = a$ is supposed to be a regular point, I may exploit the Taylor expansion $V = V(a) + V'(a)(r - a) + V''(a)\frac{(r-a)^2}{2} + \ldots$. It follows from the condition $\xi(a) = 0$ and Eq. (8) that $V'(a) \sim V(a) \to V(r_0) = 0$. Thus, only the term proportional to $(r - a)^2$ may survive here, so the proper distance to $r = a$ diverges.

The analysis becomes especially simple if $\Phi = 0$. Then, I have from (8) that

$$\xi = -V' r. \quad (22)$$

Then, V' changes the sign at $r = a$, so that $V' < 0$ everywhere in a normal region. It means that $V(r)$ decreases to zero which does not correspond to a wormhole configuration (by assumption, $V = 0$ at the throat r_0 only but $V > 0$ for $r > r_0$).

In the work [5] the trial form of the shape function was chosen so that $V = \exp[\frac{K}{(r-r_0)^n}]$ with $n \geq 1$. Then, it is obvious that V diverges so strongly near the throat that the proper distance $l \to \infty$ and I have a horn instead of a wormhole.

B. Jump of ξ

The above arguments do not work directly if I allow the jump of the quantity $\xi(r)$. Let this quantity change from $\xi(a - 0) < 0$ in the exotic region to $\xi(a + 0) > 0$ in the normal one. Then, for a close to r_0 it follows from (8) that V' changes by a jump from positive to negative values at $r = a$, while the term proportional to V is negligible. However, if the function $V(r)$ is small and negative near $a \to r_0$ and has a negative finite derivative $V' = -\frac{\xi}{r} + O(V)$ for $r > a$, it means that the function V changes sign at some $b = a + O(V)$, in obvious contradiction with the properties of wormhole metrics. If, instead, ξ changes by jump from $\xi(a - 0) < 0$ to $\xi(a + 0) = 0$, the derivative $V'(a) \to 0$ and I have a horn.

IV. MINIMIZING EXOTICISM

Let $\xi(r) \to 0$ inside some region including $r = r_0$. If $\xi = 0$ (or, equivalently, $p_r = -\rho$) it follows from the Einstein Eqs. (3) that only two possibilities exist.

A. Horizons instead of traversable wormholes

First case: $\exp(2\Phi) = 1 - \frac{b}{r}$. In this case at the supposed throat $r = r_0$ where $b(r_0) = r_0$ the g_{00} component of the metric tensor in (1) vanishes as well, in contradiction with the assumption about the absence of the horizon. Therefore, this case should be rejected.

B. Hornlike configurations

Second case: $b = r$. Then, formally, $g_{11} = \infty$. Actually, this only means that the coordinate r is degenerate and cannot be used. If, instead, I use l, one can see that $\frac{dr}{dl} = 0$, so $r = r_0 = \text{const}$ inside the corresponding region. Thus, instead of the wormhole I have a horn. In principle, one can take a finite piece of this horn and glue it to the metric with $\frac{dr}{dl} > 0$ on the right side and $\frac{dr}{dl} < 0$ on the left side. In doing so, one can obtain so-called "null wormholes" (N-wormholes) [9]. Although inside the region where $p_r + \rho = 0$ NEC is satisfied on the verge, gluing to the normal matter on the borders needs the presence of exotic material and I return to the situation under discussion with all corresponding difficulties.

One can also consider the situation when $\xi(r) \to 0$ when $r \to r_0$ but does not vanish identically in some region. Then, r is also not identically constant. In the vicinity of r_0 I still have a horn (semi-infinite throat) with small derivatives of r. Indeed, it follows directly from (8) that in this limit $\frac{d^2r}{dl^2}(r_0) \to 0$. In a similar way, one can show, assuming the analyticity of $b(r)$, that all higher derivatives $\frac{d^nr}{dl^n} \to 0$. Actually, it means that r_0 is at infinite proper distance from any $r > r_0$. As a result, the spacetime with r varying from r_0 to the right infinity is geodesically complete. Therefore, my configuration cannot be considered as a wormhole.

As an example of such a configuration, I can point to the exact solutions in dilaton gravity [10]:

$$ds^2 = -dt^2 + \left(1 - \frac{r_0}{r}\right)^{-2} dr^2 + r^2 d\omega^2. \tag{23}$$

In this example $\xi = -2\frac{r_0}{r}(1 - \frac{r_0}{r}) \to 0$ when $r \to r_0$; $r - r_0 \simeq \exp(-\frac{l}{r_0})$ in this limit. Meanwhile, r is not identically constant and, moreover, $r \to \infty$ at asymptotically flat infinity. The spacetime with $r_0 \leq r < \infty$ is geodesically complete and no wormhole arises.

One can also consider the case when $\xi(r_0) \neq 0$ but is small. When, for nonzero $\xi < 0$ a wormhole configuration will be indeed possible. However, in the limit $\xi(r_0) \to 0$ it will be approaching the hornlike one closer and closer, with $\frac{d^2r}{dl^2}(r_0)$ becoming smaller and smaller, so in the limit $\xi(r_0) = 0$ I return to the situation described above: either it will be a horn everywhere ($r = \text{const}$) or asymptotically for $r \to r_0$. Anyway, the limit of the sequence of such a configuration does not represent a wormhole.

C. Combined case

One can also try to combine both factors and consider decreasing $\xi(r_0)$ along with decreasing $a - r_0$. Actually, a model of this kind was considered in [4], where it was claimed that the violation of NEC can be arbitrarily small, but it was not discussed what happens to the wormhole metric in this limit. In my notation, this model can be described as follows: Let $\Phi = 0$, $b = 1 - k - \epsilon(r)$ for $r_0 \leq r \leq a$ where $\epsilon'(r_0) = 0$, $k \to 1$, $a \to r_0$, and $\epsilon(r)$ have the order $k - 1$ in this limit. Then, one can obtain from [4] or directly from (3) that $\rho + p_r = -\frac{1-k}{8\pi r_0^2} < 0$ inside the interval $[r_0, a]$ where it can be made arbitrarily small by taking sufficiently small $1 - k$. However, in this limit the proper distance between any $r > a$ and $r = a$ behaves like $l(a, r) = \frac{r-a}{\sqrt{1-k}}$ and diverges in this limit. As a result, I again obtain a geodesically complete spacetime with a horn instead of a wormhole. It was already mentioned in [7] that in this model the proper distance between the throat and the border at a behaves similarly like $l(r_0, a) = \frac{r_0 - a}{\sqrt{1-k}}$, so that only fine-tuning between parameters may warrant the finite $l(r_0, a)$ in the limit under discussion. I would like to point out that even such a fine-tuning does not save the matter since $l(a, r)$ diverge even if $l(r_0, a)$ remains finite. Shortcomings of this model as well as of some other models also flaring outward too slowly (see [7] for their detailed criticism) were repaired in [11] where minimizing the size of the exotic region was discussed without the requirement of making this region arbitrarily small. Instead, some balance between this size and restrictions due to quantum inequalities [7,12] was suggested. This issue, however, is beyond the scope of the present paper.

V. LIMITING CONFIGURATIONS WITH A HORIZON

My goal is to try to find wormhole configurations with the minimum violation of NEC. Previous sections showed that, typically, it is the flare-out condition which is violated in the limit under consideration, so a wormhole becomes more and more extended and approaches the horn. It is seen from (8) that the first term is negative near the throat while the second one is positive and small, provided $\Phi' > 0$ is finite. Roughly speaking, what I did in previous sections is attempt to diminish the first negative term by diminishing V'. This attempt resulted in the appearance of horns instead of throats since small V' entails large proper distances to the border between regions. Let me try another method to achieve positive ξ in the small vicinity of the throat: instead of diminishing the first negative term I can try to increase the second positive one. I assume that $V \sim r - r_0$ near the throat. If I still take Φ'' bounded near the throat, the factor V will make the second term in (8) negligible. Instead, I should take $\Phi \sim \ln(r - r_0)$. As I want to have the throat, I need Φ to be finite on the throat. The natural choice is

$$\exp(\Phi) = \varepsilon + \lambda f(r), \qquad r_0 \leq r \leq a, \qquad \lambda > 0, \quad (24)$$

which is glued smoothly to the patch with $r > a$. If $\varepsilon = 0$ from the very beginning, I obtain the horizon. Then, the requirement of regularity along with the asymptotics $V \sim r - r_0$ leads to the condition $f \sim Af_0$, $f_0 = \sqrt{1 - \frac{r_0}{r}}$ (by rescaling λ I always may achieve $A = 1$). For simplicity and without big loss of generality, I restrict myself to the case when $f = f_0$, $V = 1 - \frac{r_0}{r}$ exactly and $\exp(\Phi) = \sqrt{V}$ for $r > a$. This is just the example "Specialization 2" from [2]. The properties of this model were also discussed in [7] where it was shown that quantum inequalities [7] constrain it in such a way that make it unrealistic. However, this does not exclude, in principle, exploiting some unusual classical source for this geometry. In my context, I am interested in inner properties of the system irrespective of how it is created and do not appeal to any numerical estimates.

In [2], the authors mainly discussed the behavior of bulk density and pressure; meanwhile now I will show that the crucial role is played by surface stresses. They appear on the boundary between two different regions "+" (right) and "−" (left) [13] with components S_0^0 and $S_2^2 = S_3^3$, where

$$8\pi S_0^0 = \frac{2}{r}\left[\left(\frac{dr}{dl}\right)_+ - \left(\frac{dr}{dl}\right)_-\right], \quad (25)$$

$$8\pi S_2^2 = \frac{1}{r}\left[\left(\frac{dr}{dl}\right)_+ - \left(\frac{dr}{dl}\right)_-\right] + \left(\frac{d\Phi}{dl}\right)_+ - \left(\frac{d\Phi}{dl}\right)_-. \quad (26)$$

As V is supposed to be continuous, $S_0^0 = 0$ everywhere. On the border $r = a$

$$8\pi S_2^2(a) = \frac{r_0 \varepsilon}{2\varepsilon_s^2 a^2}, \qquad \varepsilon_s \equiv \sqrt{1 - \frac{r_0}{a}}, \quad (27)$$

where I follow the notation of [2]. I emphasize that for the metric under discussion, there are also the surface stresses on the throat itself where two branches of $r(l)$ with opposite signs of $\frac{dr}{dl}$ meet. Simple calculations give

$$8\pi S_2^2(r_0) = \frac{\lambda}{r_0 \varepsilon}. \quad (28)$$

Now I will show that the stresses (27) and (28) cannot be made finite simultaneously in the limit $a \to r_0$. Indeed, in this limit the quantity $\varepsilon_s \to 0$. If ε is fixed, $S_2^2(a) \to \infty$, which hardly can be accepted physically. One may try to repair this shortcoming and, instead, take simultaneously the limits $\varepsilon_s \to 0$, $\varepsilon \to 0$ in such a way that $\frac{\varepsilon}{\varepsilon_s^2}$ remains finite or even zero. However, in the limit $\varepsilon \to 0$, $S_2^2(r_0) \to \infty$ independently of the relationship between ε and ε_s.

VI. SUMMARY

It turned out that the statement of the kind "there are wormholes with arbitrarily weak violation of NEC" should be taken with great care. I have shown that the construction of traversable wormholes with arbitrarily small amounts of exotic material is achieved by expense of the violation of some properties inherent to standard traversable wormholes. I considered several ways to achieve the arbitrarily small amount of exotic material for wormhole geometry which, actually, can be divided into two types. The first one consists in diminishing V' (then the density approaches the critical value $\rho_0 = (8\pi r_0^2)^{-1}$). Then, typically, instead of a wormhole I obtain a horn in this limit. It does not mean that such configurations have no physical sense. By contrary, in some situations it is the tubelike configurations which are placed to the forefront (for example, in the context of the problem of avoidance singularities [9], in investigating properties of phantom matter [14], etc.). However, in any case, they represent a special class of wormhole configurations which differ from the standard ones. The second type represents wormholes on the threshold of formation of a horizon. Then, infinitely large surface stresses appear on the throat and/or the boundary between exotic and normal regions, so the limit is singular and, in this sense, unphysical.

As the quantitative measure of exoticism I exploit the value of I according to the definition (12) used in [2]. If, instead of I I would consider the similar integral J over the proper distance l, I would obtain that $|J| > |I|$ since $V < 1$. Therefore, as $|I|$ cannot be made arbitrarily small without violation of conditions (1)–(3) indicated in the introduction, the same applies to J. In this sense, using J as the quantifier of exoticism only strengthens the conclusion that the degree of exoticism cannot be made arbitrarily small for standard wormholes.

One can think that if an advanced civilization wants to construct a standard wormhole, it should not make lame excuses about the "small amount" of exotic material and must do an inevitable job of collecting and arranging this material in sufficient supply.

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