

What does $\mu - \tau$ symmetry imply about neutrino mixings?

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The requirement of the $\mu - \tau$ symmetry in the neutrino sector that yields the maximal atmospheric neutrino mixing is shown to yield either $\sin\theta_{13} = 0$ (referred to as C1) or $\sin\theta_{12} = 0$ (referred to as C2)), where $\theta_{12(13)}$ stands for the solar (reactor) neutrino mixing angle. We study general properties possessed by approximately $\mu - \tau$ symmetric textures. It is argued that the tiny $\mu - \tau$ symmetry breaking generally leads to $\cos 2\theta_{23} \sim \sin\theta_{13}$ for C1) and $\cos 2\theta_{23} \sim \Delta m_{\odot}^2 / \Delta m_{\text{atm}}^2 (\equiv R)$ for C2), which indicates that the smallness of $\cos 2\theta_{23}$ is a good measure of the $\mu - \tau$ symmetry breaking, where Δm_{atm}^2 ($\Delta m_{\odot}^2$) stands for the square mass difference of atmospheric (solar) neutrinos. We further find that the relation $R \sim \sin^2\theta_{13}$ arises from contributions of $\mathcal{O}(\sin^2\theta_{13})$ in the estimation of the neutrino masses ($m_{1,2,3}$) for C1), and that possible forms of textures are strongly restricted to realize $\sin^2 2\theta_{12} = \mathcal{O}(1)$ for C2). To satisfy $R \sim \sin^2\theta_{13}$ for C1), neutrinos exhibit the inverted mass hierarchy, or the quasi degenerate mass pattern with $|m_{1,2,3}| \sim \mathcal{O}(\sqrt{\Delta m_{\text{atm}}^2})$, and, to realize $\sin^2 2\theta_{12} = \mathcal{O}(1)$ for C2), there should be an additional small parameter η whose size is comparable to that of the $\mu - \tau$ symmetry breaking parameter ε , giving $\tan 2\theta_{12} \sim \varepsilon/\eta$ with $\eta \sim \varepsilon$ to be compatible with the observed large mixing.

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I. INTRODUCTION

Since the confirmation of the atmospheric neutrino oscillations in 1998 [1], neutrino oscillations have been observed in various neutrinos [2] coming from the Sun [3,4], accelerators [5] and reactors [6]. The results of the neutrino oscillations are known to be interpreted in terms of the mixings of three flavor neutrinos, ν_e , ν_μ and ν_τ , which evolve into three massive neutrinos ν_1 , ν_2 and ν_3 during their flights. Observed in experiments are three mixing angles denoted by θ_{12} for $\nu_e - \nu_\mu$, θ_{23} for $\nu_\mu - \nu_\tau$ and θ_{13} for $\nu_e - \nu_\tau$ and two neutrino mass squared differences Δm_{atm}^2 for atmospheric neutrinos and $\Delta m_{\odot}^2$ for solar neutrinos. These masses and mixing angles are currently constrained to satisfy [7]:

$$\begin{aligned} \Delta m_{\odot}^2 &= 7.92(1 \pm 0.09) \times 10^{-5} \text{ eV}^2, \\ \Delta m_{\text{atm}}^2 &= 2.4 \begin{pmatrix} +0.21 \\ 1 \\ -0.26 \end{pmatrix} \times 10^{-3} \text{ eV}^2, \end{aligned} \quad (1)$$

and

$$\begin{aligned} \sin^2\theta_{12} &= 0.314 \begin{pmatrix} +0.18 \\ 1 \\ -0.15 \end{pmatrix}, \\ \sin^2\theta_{23} &= 0.44 \begin{pmatrix} +0.41 \\ 1 \\ -0.22 \end{pmatrix}, \\ \sin^2\theta_{13} &= \begin{pmatrix} +2.3 \\ 0.9 \\ -0.9 \end{pmatrix} \times 10^{-2}, \end{aligned} \quad (2)$$

where $\Delta m_{\odot}^2 = m_2^2 - m_1^2$ (> 0) [8] and $\Delta m_{\text{atm}}^2 = |m_3^2 - (m_1^2 + m_2^2)/2|$ and m_1 , m_2 and m_3 , respectively, stand for the masses of ν_1 , ν_2 and ν_3 . These experimental data have indicated two distinct properties: (i) The atmospheric and solar mixing angles measured as $\sin^2 2\theta$ are $\mathcal{O}(1)$ while the reactor mixing angle θ_{13} is quite small, and (ii) The mass squared differences Δm_{atm}^2 and $\Delta m_{\odot}^2$ exhibit the hierarchy as $\Delta m_{\odot}^2 / \Delta m_{\text{atm}}^2 \ll 1$.

It has been a guiding principle that the presence of hierarchies or of tiny quantities implies a presence of a certain protection symmetry in underlying physics [9]. Candidates of such a symmetry in neutrino physics [10] may include $U(1)_{L'}$ based on the conservation of $L_e - L_\mu - L_\tau$ ($\equiv L'$) [11,12], and a $\mu - \tau$ symmetry based on the invariance of flavor neutrino mass terms under the interchange of ν_μ and ν_τ characterized by Z_2 [13–17], where $L_{e,\mu,\tau}$, respectively, represent the e -, μ -, and τ -number. These symmetries show that $\Delta m_{\odot}^2 / \Delta m_{\text{atm}}^2 = 0$, $\sin^2 2\theta_{12} = 1$ and $\sin\theta_{13} = 0$ for $U(1)_{L'}$ and $\sin^2 2\theta_{23} = 1$ and $\sin\theta_{13} = 0$ for the $\mu - \tau$ symmetry. Since the charged leptons clearly violate these symmetries, the effect from the charged leptons yields deviations from these values and we expect that their contributions finally give compatible results with Eqs. (1) and (2). However, the charged leptons and neutrinos are the $SU(2)_L$ -doublets and, for example, the $\mu - \tau$ symmetry respected by neutrinos should be respected by the charged leptons. This fact apparently disfavors the requirement of the $\mu - \tau$ symmetry. To have $\mu - \tau$ symmetric mass terms, we must introduce several Higgs scalars with

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the different Z_2 parity, where their vev's can provide charged lepton masses [15,17,18] in such a way that the charged leptons acquire almost diagonal masses, which badly break the $\mu - \tau$ symmetry. The price to pay is to have flavor-changing neutral current interactions due to the direct exchanges of these Higgs scalars. Effects from the interactions become sizable for quarks when the $\mu - \tau$ symmetry is applied to grand unified models [19] and should be suppressed. If neutrinos are Majorana particles, which are different from charged leptons of the Dirac type, it is expected that this difference may readily supply approximately $\mu - \tau$ symmetric neutrino flavor structure.

In this article, we discuss details of physical results from the requirement of the $\mu - \tau$ symmetry in neutrino mixings without CP phases. The influence from CP phases will be discussed in a subsequent article [20]. Throughout this article, we assume that the effects from the charged leptons are fully contained in our discussions as $\mu - \tau$ symmetry breaking effects. We calculate eigenvectors associated with a given flavor neutrino mass matrix M_ν , which determines the Pontecorvo-Maki-Nakagawa-Sakata unitary matrix U_{PMNS} [21] that converts the flavor neutrinos $\nu_{e,\mu,\tau}$ into the massive neutrinos $\nu_{1,2,3}$: $\nu_f = (U_{\text{PMNS}})_{fi} \nu_i$, where $f = e, \mu, \tau$ and $i = 1, 2, 3$. The genuine use of the $\mu - \tau$ symmetry indicates two categories of $\mu - \tau$ symmetric textures, which give either $\sin\theta_{13} = 0$, or $\sin\theta_{12} = 0$ in U_{PMNS} depending on the order of the eigenvalues. After including a $\mu - \tau$ symmetry breaking effect characterized by a parameter ε , yielding either $\sin\theta_{13} \sim \varepsilon$, or $\sin\theta_{12} \sim \varepsilon$, we find general constraints on flavor neutrino masses that yield $\sin^3\theta_{13} \ll 1$ and $\Delta m_\odot^2/\Delta m_\text{atm}^2 \ll 1$. Furthermore, to obtain $\sin^2 2\theta_{12} = \mathcal{O}(1)$ from $\sin\theta_{12} \sim \varepsilon$ gives a severe constraint on sizes of the flavor neutrino masses. The $\mu - \tau$ symmetry breaking results in a correlation of $\cos\theta_{23}$ to $\sin\theta_{13}$, or to $\Delta m_\odot^2/\Delta m_\text{atm}^2$ [22]. If $m_1 + m_2 \sim 0$ giving $\Delta m_\odot^2 \sim 0$, we show that the relation of $\Delta m_\odot^2/\Delta m_\text{atm}^2 \sim \sin^2\theta_{13}$ arises in textures leading to $\sin\theta_{13} = 0$ in the $\mu - \tau$ symmetric limit.

In the next section, we explain how $\sin\theta_{13} = 0$ and $\sin\theta_{12} = 0$ are obtained in the $\mu - \tau$ symmetric limit. The useful formula are shown to calculate neutrino masses and mixing angles, where two categories depend on the signs of $\sin\theta_{23}$ for a given M_ν . In Sec. III, we derive various constraints to realize $\sin^2\theta_{13} \ll 1$ and $\sin^2 2\theta_{12} = \mathcal{O}(1)$. In Sec. IV, applying these constraints to textures, we find general relations among $\cos 2\theta_{23}$, $\sin^2\theta_{13}$, and $\Delta m_\odot^2/\Delta m_\text{atm}^2$, which do not depend on details of textures.¹ The final section, Sec. V, is devoted to summary, and discussions.

II. $\mu - \tau$ SYMMETRIC TEXTURE

Let us define a neutrino mass matrix M_ν parametrized by²

$$M_\nu = \begin{pmatrix} M_{ee} & M_{e\mu} & M_{e\tau} \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ M_{e\tau} & M_{\mu\tau} & M_{\tau\tau} \end{pmatrix}. \quad (3)$$

The $\mu - \tau$ symmetry is based on the invariance of the flavor neutrino mass terms in the lagrangian under the interchange of $\nu_\mu \leftrightarrow \nu_\tau$ or $\nu_\mu \leftrightarrow -\nu_\tau$. As a result, we obtain $M_{e\tau} = M_{e\mu}$ and $M_{\mu\mu} = M_{\tau\tau}$ for $\nu_\mu \leftrightarrow \nu_\tau$ or $M_{e\tau} = -M_{e\mu}$ and $M_{\mu\mu} = M_{\tau\tau}$ for $\nu_\mu \leftrightarrow -\nu_\tau$. We use the sign factor $\sigma = \pm 1$ to have $M_{e\tau} = -\sigma M_{e\mu}$ for the $\mu - \tau$ symmetric part under the interchange of $\nu_\mu \leftrightarrow -\sigma\nu_\tau$. We divide M_ν into the $\mu - \tau$ symmetric part M_{sym} and its breaking part M_b [24–26] expressed in terms of $M_{e\mu}^{(\pm)} = (M_{e\mu} \pm (-\sigma M_{e\tau}))/2$ and $M_{\mu\mu}^{(\pm)} = (M_{\mu\mu} \pm M_{\tau\tau})/2$:

$$M_\nu = M_{\text{sym}} + M_b \quad (4)$$

with

$$M_{\text{sym}} = \begin{pmatrix} M_{ee} & M_{e\mu}^{(+)} & -\sigma M_{e\mu}^{(+)} \\ M_{e\mu}^{(+)} & M_{\mu\mu}^{(+)} & M_{\mu\tau} \\ -\sigma M_{e\mu}^{(+)} & M_{\mu\tau} & M_{\mu\mu}^{(+)} \end{pmatrix}, \quad (5)$$

$$M_b = \begin{pmatrix} 0 & M_{e\mu}^{(-)} & \sigma M_{e\mu}^{(-)} \\ M_{e\mu}^{(-)} & M_{\mu\mu}^{(-)} & 0 \\ \sigma M_{e\mu}^{(-)} & 0 & -M_{\mu\mu}^{(-)} \end{pmatrix},$$

where obvious relations of $M_{e\mu} = M_{e\mu}^{(+)} + M_{e\mu}^{(-)}$, $M_{e\tau} = -\sigma(M_{e\mu}^{(+)} - M_{e\mu}^{(-)})$, $M_{\mu\mu} = M_{\mu\mu}^{(+)} + M_{\mu\mu}^{(-)}$ and $M_{\tau\tau} = M_{\mu\mu}^{(+)} - M_{\mu\mu}^{(-)}$ are used. The lagrangian for M_{sym} : $-\mathcal{L}_{\text{mass}} = \psi^T M_{\text{sym}} \psi/2$ with $\psi = (\nu_e, \nu_\mu, \nu_\tau)^T$ turns out to be invariant under the exchange of $\nu_\mu \leftrightarrow -\sigma\nu_\tau$.

It is not difficult to find three eigenvalues of the $\mu - \tau$ symmetric M_{sym} . After a little calculus, we obtain three eigenvalues $\lambda_\pm$ and λ

$$\lambda_\pm = M_{\mu\mu}^{(+)} - \sigma M_{\mu\tau} + M_{e\mu}^{(+)} x_\pm, \quad \lambda = M_{\mu\mu}^{(+)} + \sigma M_{\mu\tau}, \quad (6)$$

where

¹Specific forms of textures that respect our constraints will be presented elsewhere to make testable predictions [23].

²It is understood that the charged leptons and neutrinos are rotated, if necessary, to give diagonal charged-current interactions and to define the flavor neutrinos of ν_e , ν_μ and ν_τ .

$$x_{\pm} = \frac{M_{ee} - M_{\mu\mu}^{(+)} + \sigma M_{\mu\tau} \pm \sqrt{(M_{ee} - M_{\mu\mu}^{(+)} + \sigma M_{\mu\tau})^2 + 8M_{e\mu}^{(+2)}}}{2M_{e\mu}^{(+)}}. \quad (7)$$

The ordering of $|\lambda_{\pm}|$ and $|\lambda|$ determines masses of $\nu_{1,2,3}$. For example, if $|\lambda_-| < |\lambda_+| < |\lambda|$, the neutrino masses are given by $m_1 = \lambda_-$, $m_2 = \lambda_+$ and $m_3 = \lambda$ as the normal mass hierarchy and by $m_1 = \lambda_+$, $m_2 = \lambda$ and $m_3 = \lambda_-$ as the inverted mass hierarchy. The quasi degenerate mass pattern further requires $|m_i - m_j| \ll |m_{1,2,3}|$ ($i, j = 1, 2, 3$). These two examples show the typical cases, where λ is assigned to ν_3 or to the others. We will see that, if λ is assigned to the mass of ν_3 , $\sin\theta_{13} = 0$ is derived, while if λ is assigned to the mass of ν_2 , $\sin\theta_{12} = 0$ is derived.³ The eigenvectors are also calculated to be

$$|\lambda_- \rangle = n_- \begin{pmatrix} -x_- \\ -1 \\ \sigma \end{pmatrix}, \quad |\lambda_+ \rangle = n_+ \begin{pmatrix} x_+ \\ 1 \\ -\sigma \end{pmatrix}, \quad |\lambda \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sigma \\ 1 \end{pmatrix}, \quad (8)$$

respectively, for λ_- , λ_+ and λ , where $n_{\pm} = \sqrt{2 + x_{\pm}^2}$. These eigenvectors are determined up to an arbitrary relative phase as long as corresponding eigenvalues remain intact. The orthogonally condition is obviously satisfied because of $x_+ x_- = -2$. These eigenvectors form the PMNS unitary matrix expressed by

$$U_{\text{PMNS}}^{(0)} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}, \quad (9)$$

where $c_{ij} \equiv \cos\vartheta_{ij}$ and $s_{ij} \equiv \sin\vartheta_{ij}$.

We first assume that M_{ν} gives $|\lambda_-| < |\lambda_+| < |\lambda|$ in the normal mass hierarchy and, thus, assign λ to m_3 . As a result, U_{PMNS} can be described by

$$\left(\frac{1}{\sqrt{2 + (x_-)^2}} \begin{pmatrix} -x_- \\ -1 \\ \sigma \end{pmatrix} \frac{1}{\sqrt{2 + (x_+)^2}} \begin{pmatrix} x_+ \\ 1 \\ -\sigma \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sigma \\ 1 \end{pmatrix} \right). \quad (10)$$

By comparing it with $U_{\text{PMNS}}^{(0)}$, we obtain that

$$\begin{aligned} \tan 2\theta_{12} &= \frac{2\sqrt{2}M_{e\mu}^{(+)}}{M_{\mu\mu}^{(+)} - \sigma M_{\mu\tau} - M_{ee}}, \\ \sin 2\theta_{23} &= \sigma, \\ \sin\theta_{13} &= 0. \end{aligned} \quad (11)$$

This prediction leads to the statement that the $\mu - \tau$ symmetry guarantees the appearance of the maximal atmospheric neutrino mixing, and of the vanishing θ_{13} . The $\mu - \tau$ symmetric mass matrix with $|\lambda_-| < |\lambda_+| < |\lambda|$ itself can give consistent mixing angles with the experimental data provided that an appropriate magnitude of θ_{12} is produced. The similar conclusion can be obtained for $|\lambda| < |\lambda_-| < |\lambda_+|$ in the inverted mass hierarchy, where m_3 is still assigned to λ .

Starting with the same M_{ν} , we next assume $|\lambda_+| < |\lambda| < |\lambda_-|$ in the normal mass hierarchy, and assign λ to

³This kind of consideration has been done in Ref. [27], where it was mainly applied to the analysis on the CKM unitary matrix [28]. See also Ref. [24], where the possible choice of $\sin\theta_{12} = 0$ was phrased as the requirement of $m_1 = m_2$, whose consequence was not fully discussed.

m_2 . In this mass-ordering, we construct U_{PMNS} to be:

$$\left(\frac{1}{\sqrt{2 + (x_+)^2}} \begin{pmatrix} x_+ \\ -1 \\ \sigma \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \sigma \end{pmatrix} \frac{1}{\sqrt{2 + (x_-)^2}} \begin{pmatrix} \sigma x_- \\ -\sigma \\ 1 \end{pmatrix} \right), \quad (12)$$

from which we obtain that

$$\begin{aligned} \sin\theta_{12} &= 0, \quad \sin 2\theta_{23} = -\sigma, \\ \tan 2\theta_{13} &= -\frac{2\sqrt{2}\sigma M_{e\mu}^{(+)}}{M_{\mu\mu}^{(+)} - \sigma M_{\mu\tau} - M_{ee}}, \end{aligned} \quad (13)$$

Therefore, we have $\sin\theta_{12} = 0$ instead of $\sin\theta_{13} = 0$. The similar conclusion can be obtained for $|\lambda_-| < |\lambda_+| < |\lambda|$ in the inverted mass hierarchy, where m_2 is still assigned to λ .

There is a general formula [29] that can treat both cases in a unified way. The mixing angles and masses are given by

$$\begin{aligned} \tan 2\theta_{12} &= \frac{2X}{\lambda_2 - \lambda_1}, \\ (M_{\tau\tau} - M_{\mu\mu}) \sin 2\theta_{23} - 2M_{\mu\tau} \cos 2\theta_{23} &= 2s_{13}X, \\ \tan 2\theta_{13} &= \frac{2Y}{\lambda_3 - M_{ee}}, \end{aligned} \quad (14)$$

and

$$\begin{aligned} m_1 &= c_{12}^2 \lambda_1 + s_{12}^2 \lambda_2 - 2c_{12}s_{12}X, \\ m_2 &= s_{12}^2 \lambda_1 + c_{12}^2 \lambda_2 + 2c_{12}s_{12}X, \\ m_3 &= c_{13}^2 \lambda_3 + 2c_{13}s_{13}Y + s_{13}^2 M_{ee}, \end{aligned} \quad (15)$$

where

$$\begin{aligned}
X &= \frac{c_{23}M_{e\mu} - s_{23}M_{e\tau}}{c_{13}}, \\
Y &= s_{23}M_{e\mu} + c_{23}M_{e\tau}, \\
\lambda_1 &= c_{13}^2 M_{ee} - 2c_{13}s_{13}Y + s_{13}^2 \lambda_3, \\
\lambda_2 &= c_{23}^2 M_{\mu\mu} + s_{23}^2 M_{\tau\tau} - 2s_{23}c_{23}M_{\mu\tau}, \\
\lambda_3 &= s_{23}^2 M_{\mu\mu} + c_{23}^2 M_{\tau\tau} + 2s_{23}c_{23}M_{\mu\tau}.
\end{aligned} \tag{16}$$

This formula reproduces the obtained results Eq. (11) for $c_{23} = \sigma s_{23} = 1/\sqrt{2}$ and Eq. (13) for $c_{23} = -\sigma s_{23} =$

$1/\sqrt{2}$ because M_{sym} is specified by $M_{e\mu} = M_{e\mu}^{(+)}$, $M_{e\tau} = -\sigma M_{e\mu}^{(+)}$ and $M_{\mu\mu} = M_{\tau\tau} = M_{\mu\mu}^{(+)}$.

It is readily found that the predictions of masses from Eq. (15) are identical to the three eigenvalues in each mass-ordering. For instance, in the case of $\sin\theta_{12} = 0$, we find that $m_1 = \lambda_1$, which becomes

$$m_1 = M_{\mu\mu} - \sigma M_{\mu\tau} + \frac{1}{1 - t_{13}^2} (M_{ee} - M_{\mu\mu} + \sigma M_{\mu\tau}). \tag{17}$$

By using

$$t_{13} = \sigma \frac{\sqrt{(M_{ee} - M_{\mu\mu} + \sigma M_{\mu\tau})^2 + 8M_{e\mu}^2} - (M_{ee} - M_{\mu\mu} + \sigma M_{\mu\tau})}{2\sqrt{2}M_{e\mu}}, \tag{18}$$

calculated from $\tan 2\theta_{13}$ in Eq. (14), we reach

$$\begin{aligned}
m_1 &= M_{\mu\mu} - \sigma M_{\mu\tau} + \frac{1}{2} (M_{ee} - M_{\mu\mu} + \sigma M_{\mu\tau} \\
&\quad + \sqrt{(M_{ee} - M_{\mu\mu} + \sigma M_{\mu\tau})^2 + 8M_{e\mu}^2}).
\end{aligned} \tag{19}$$

Similarly, we find that

$$\begin{aligned}
m_3 &= M_{\mu\mu} - \sigma M_{\mu\tau} + \frac{1}{2} (M_{ee} - M_{\mu\mu} + \sigma M_{\mu\tau} \\
&\quad - \sqrt{(M_{ee} - M_{\mu\mu} + \sigma M_{\mu\tau})^2 + 8M_{e\mu}^2}).
\end{aligned} \tag{20}$$

These results, respectively, coincide with λ_+ and λ_- , which is the case of Eq. (12).

III. APPROXIMATELY $\mu - \tau$ SYMMETRIC TEXTURE

To discuss how the $\mu - \tau$ symmetry breaking term M_b gives consistent predictions with the observed masses and mixings, we rely upon the formula provided by Eqs. (14) and (15). Since the experimentally allowed value of $\sin^2\theta_{13} = \mathcal{O}(10^{-2})$ can describe the hierarchical ratio of $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$, we retain terms of $\mathcal{O}(\sin^2\theta_{13})$ in our calculations. The $\mu - \tau$ symmetry breaking effect is characterized by the parameter ε , which control $M_{e\mu}^{(-)}$ and $M_{\mu\mu}^{(-)}$. Our mass matrix, then, takes the following form:

$$M_\nu = \begin{pmatrix} a & b & -\sigma b \\ b & d & e \\ -\sigma b & e & d \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & b' & \sigma b' \\ b' & d' & 0 \\ \sigma b' & 0 & -d' \end{pmatrix}. \tag{21}$$

This texture is almost the same as the one discussed in Ref. [24]. However, constraints on the flavor masses are not well clarified, and the case corresponding to $\sin\theta_{12} = 0$ is not discussed. These two subjects are examined in detail by focusing on the flavor structure of M_ν . We discuss how different flavor structure yielding the same mass pattern results in different predictions. The $\mu - \tau$ symmetry breaking generally induces the deviation of the atmospheric neutrino mixing from the maximal one as indicated by Eq. (14) for θ_{23} because of $M_{\mu\mu} \neq M_{\tau\tau}$ and $s_{13} \neq 0$. This deviation is parametrized by Δ :

$$c_{23} = \frac{1 + \Delta}{\sqrt{2(1 + \Delta^2)}}, \quad s_{23} = \pm \sigma \frac{1 - \Delta}{\sqrt{2(1 + \Delta^2)}}, \tag{22}$$

giving $\sin 2\theta_{23} = \pm \sigma(1 - \Delta^2)/(1 + \Delta^2)$ and $\cos 2\theta_{23} = 2\Delta/(1 + \Delta^2)$. The plus (minus) sign in front of σ for s_{23} specifies textures with $\sin\theta_{13} \rightarrow 0$ ($\sin\theta_{12} \rightarrow 0$) as $\varepsilon \rightarrow 0$.

The masses and mixing angles are given by the following equations:

C1) with $\sin\theta_{13} \rightarrow 0$ as $\varepsilon \rightarrow 0$:

$$\begin{aligned}
m_1 &\approx \frac{a + d - \sigma e - (d + \sigma e - a)t_{13}^2 + 2(\sigma e\Delta + \varepsilon d')\Delta}{2} - \frac{X}{\sin 2\theta_{12}}, \\
m_2 &\approx \frac{a + d - \sigma e - (d + \sigma e - a)t_{13}^2 + 2(\sigma e\Delta + \varepsilon d')\Delta}{2} + \frac{X}{\sin 2\theta_{12}}, \\
m_3 &\approx d + \sigma e + (d + \sigma e - a)t_{13}^2 - 2(\sigma e\Delta + \varepsilon d')\Delta,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}\tan 2\theta_{12} &\approx \frac{2X}{d - \sigma e - a + (d + \sigma e - a)t_{13}^2 + 2(\sigma e\Delta + \varepsilon d')\Delta}, \\ \tan 2\theta_{13} &\approx \frac{2Y}{d + \sigma e - a - 2(\sigma e\Delta + \varepsilon d')\Delta}, \quad \cos 2\theta_{23} \approx 2\Delta, \quad \sin 2\theta_{23} \approx \sigma,\end{aligned}\quad (24)$$

with

$$X \approx \sqrt{2}\left(b\left(1 + \frac{t_{13}^2 - \Delta^2}{2}\right) + \varepsilon b'\Delta\right), \quad Y \approx \sqrt{2}\sigma(\varepsilon b' - b\Delta), \quad \Delta \approx -\frac{\sigma\varepsilon d' + \sqrt{2}s_{13}b}{2e}, \quad (25)$$

where we see the result of $\sin\theta_{13} \rightarrow 0$ as $\varepsilon \rightarrow 0$. C2) with $\sin\theta_{12} \rightarrow 0$ as $\varepsilon \rightarrow 0$:

$$\begin{aligned}m_1 &\approx \frac{a + d + \sigma e - (d - \sigma e - a)t_{13}^2 - 2(\sigma e\Delta - \varepsilon d')\Delta}{2} - \frac{X}{\sin 2\theta_{12}}, \\ m_2 &\approx \frac{a + d + \sigma e - (d - \sigma e - a)t_{13}^2 - 2(\sigma e\Delta - \varepsilon d')\Delta}{2} + \frac{X}{\sin 2\theta_{12}}, \\ m_3 &\approx d - \sigma e + (d - \sigma e - a)t_{13}^2 + 2(\sigma e\Delta - \varepsilon d')\Delta,\end{aligned}\quad (26)$$

and

$$\begin{aligned}\tan 2\theta_{12} &\approx \frac{2X}{d + \sigma e - a + 2(d - \sigma e - a)t_{13}^2 - 2(\sigma e\Delta - \varepsilon d')\Delta}, \\ \tan 2\theta_{13} &\approx \frac{2Y}{d - \sigma e - a + 2(\sigma e\Delta - \varepsilon d')\Delta} \quad \cos 2\theta_{23} \approx 2\Delta, \quad \sin 2\theta_{23} \approx -\sigma,\end{aligned}\quad (27)$$

with

$$\begin{aligned}X &\approx \sqrt{2}(\varepsilon b' + b\Delta), \\ Y &\approx -\sqrt{2}\sigma\left(b\left(1 - \frac{\Delta^2}{2}\right) - \varepsilon b'\Delta\right), \\ \Delta &\approx \frac{\sigma d' - \sqrt{2}s_{13}b'}{2e + \sqrt{2}s_{13}b}\varepsilon,\end{aligned}\quad (28)$$

where $\sin\theta_{12} \rightarrow 0$ as $\varepsilon \rightarrow 0$. It should be stressed again that the smallness of $\sin^2\theta_{13}$ is not guaranteed by the $\mu - \tau$ symmetry because Y is mainly proportional to b , namely, to $M_{e\mu}^{(+)}$. To obtain its smallness needs an additional requirement.

It appears that, roughly speaking, masses given in C2) are almost the same as those in C1) by the change of $\sigma \rightarrow -\sigma$ for e as $M_{\mu\tau}$. This correspondence occurs only if the contributions from X are suppressed in C1) because the suppression factor ε as shown in Eq. (28) is always accompanied by those in C2). However, X needs not be suppressed in textures for C1), which cannot lead to textures in C2) by the change of $\sigma \rightarrow -\sigma$.

IV. GENERAL RESULTS

In this section, we discuss general features present in C1) and C2) as a consequence of the tiny $\mu - \tau$ symmetry breaking, whose effects on the neutrino masses and mixings are evaluated in the previous section. First of all, the relation between $\Delta m_{\odot}^2$ and X can be expressed as

$$\Delta m_{\odot}^2 = \frac{2\sqrt{2}(m_1 + m_2)X}{\sin 2\theta_{12}}. \quad (29)$$

The condition of $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2 \ll 1$ requires that either $m_1 + m_2$, or X is suppressed. These two options are used in C1) and C2) as follows:

C1) The suppression of X is not a natural consequence, we may have $m_1 + m_2 \approx 0$ [30]. If $m_1 + m_2 \approx 0$, X needs not be suppressed. The requirement of $m_1 + m_2 \approx 0$ can be fulfilled if $a + d - \sigma e = 0$, more precisely, $|a + d - \sigma e| \ll \varepsilon^2$, is satisfied. The sum of $m_1 + m_2$ turns out to be $\mathcal{O}(\sin^2\theta_{13})$, which arises from the terms of t_{13}^2 , ε and Δ in Eq. (23). As a result, we obtain that

$$\frac{\Delta m_{\odot}^2}{\Delta m_{\text{atm}}^2} \sim \sin^2\theta_{13}, \quad (30)$$

which is one of the main results found in this article. In the case of $m_1 + m_2 \neq 0$, X should be suppressed and $b \approx 0$ is required because the b' -term in X is more suppressed by the factor ε^2 .

C2) The suppression of X is a consequence of the tiny $\mu - \tau$ symmetry breaking that leads to $X \propto \varepsilon$. However, the phenomenological requirement of $\sin^2\theta_{13} \ll 1$, thereby, of the relative smallness of Y , must be satisfied. From $Y \approx -\sqrt{2}\sigma(b - \varepsilon b'\Delta)$, we may have $Y \approx -\sqrt{2}\sigma b$. The suppression of Y can be achieved by the smallness of b . The similar situation to Eq. (30) also arises in C2), but $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$ receives an extra suppression due to

ε present in X , which makes $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$ phenomenologically unacceptable.

To obtain Eq. (30), we can show even in our general discussions that the mass orderings of neutrinos are not arbitrary. Since $m_2 \approx -m_1 \approx \sqrt{2}X/\sin 2\theta_{12}$ and $m_3 \approx d + e\sigma$, the relation can be obtained for the quasi degenerate mass pattern since both $|d + e\sigma|$ and $|X/\sin 2\theta_{12}|$ are not suppressed and give $|m_{1,2,3}| \sim \sqrt{\Delta m_{\text{atm}}^2}$, or for the inverted mass hierarchy if $d + e\sigma \sim 0$, giving $|m_3| \sim \sin^2\theta_{13}|m_{1,2}|$ [23].

It is instructive to note that the smallness of b can be ascribed to the approximate conservation of the electron number L_e . Namely, a has $L_e = 2$, b, b' have $L_e = 1$ and d, e, d' have $L_e = 0$ [31,32]. In the case that the conservation of L_e is perturbatively violated by an interaction of $|\Delta L_e| = 1$ with an appropriate small parameter η [31], it is not absurd to expect $a \propto \eta^2$, $b \propto \eta$ and $d, e \propto \eta^0$, which explain the required suppression of b . This mechanism is only possible for the normal mass hierarchy. It is because the condition of $\tan 2\theta_{12} = \mathcal{O}(1)$ requires either $d - \sigma e \sim 0$, or $d + \sigma e \sim 0$ for $a \sim 0$, which gives $m_{1,2} \sim 0$. If this is the case, we obtain that

- C1) $\sin^2\theta_{13} \ll 1$ due to the approximate $\mu - \tau$ symmetry and $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2 \ll 1$ due to the approximate L_e -conservation (as long as $m_1 + m_2 \neq 0$).
- C2) $\sin^2\theta_{13} \ll 1$ due to the approximate L_e -conservation and $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2 \ll 1$ due to the approximate $\mu - \tau$ symmetry.

Therefore, any underlying dynamics equipped with these two symmetries can describe the gross feature of the neutrino oscillations as the normal mass hierarchy.

There is a severe constraint on the flavor neutrino masses in C2) in order to satisfy $\sin^2 2\theta_{12} \sim \mathcal{O}(1)$. Because X receives the $\mu - \tau$ symmetry breaking, X is suppressed. If $d + \sigma e - a \approx 0$, we have $\tan 2\theta_{12} \gg 1$, leading to the almost maximal mixing, since the corrections to $d + \sigma e - a$ in the denominator of $\tan 2\theta_{12}$ in Eq. (27) are $\mathcal{O}(\varepsilon^2)$. We must obtain that $|d + \sigma e - a| \propto |\varepsilon|$, leading to

$$\tan 2\theta_{12} \approx \frac{2X}{d + \sigma e - a}. \quad (31)$$

As a result, the denominator cancels ε in the numerator to yield $\tan 2\theta_{12} \sim \mathcal{O}(1)$. Therefore, any textures realized in C2) must satisfy that

$$|d + \sigma e - a| \propto |\varepsilon|, \quad (32)$$

to match with $\sin^2 2\theta_{12} \sim \mathcal{O}(1)$. It means that the magnitude of $d + \sigma e - a$ should be adjusted so as to become as small as that of ε .

Our formula further shows a general relation between $\cos 2\theta_{23}$ and other small quantities [22] arising from the effect of the tiny $\mu - \tau$ symmetry breaking. Such a relation comes from the formula of Eq. (14) for θ_{23} and is readily found that

- C1) because $M_{\mu\mu}^{(-)}$ and s_{13} receive the $\mu - \tau$ symmetry breaking effect, their sizes are proportional to ε , leading to

$$\cos 2\theta_{23} \approx \frac{\sigma M_{\mu\mu}^{(-)} - s_{13}X}{M_{\mu\tau}} \propto \varepsilon \sim \sin \theta_{13}, \quad (33)$$

and

- C2) because X receives the $\mu - \tau$ symmetry breaking effect while s_{13} is required to be phenomenologically suppressed, the product of Xs_{13} is doubly suppressed and can be a vanishingly small quantity, leading to

$$\cos 2\theta_{23} \sim \frac{\sigma M_{\mu\mu}^{(-)}}{M_{\mu\tau}} \propto \varepsilon \sim \frac{\Delta m_{\odot}^2}{\Delta m_{\text{atm}}^2}, \quad (34)$$

where ε can be related to $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$ because $X \propto \varepsilon$ in Eq. (29).

The faithful parameter measuring the size of the $\mu - \tau$ symmetry breaking effect is $\cos 2\theta_{23}$.

From Eqs. (25) for C1) and (28) for C2), we can, respectively, obtain that

$$\begin{aligned} s_{13} &\approx \frac{2eb' + \sigma bd'}{\sqrt{2}[\sigma e(d + \sigma e - a) - b^2]} \varepsilon, \\ \cos 2\theta_{23} &\approx -\frac{(d + \sigma e - a)d' + 2bb'}{\sigma e(d + \sigma e - a) - b^2} \varepsilon, \end{aligned} \quad (35)$$

for $d + \sigma e - a \neq 0$, where s_{13} and $\cos 2\theta_{23}$ satisfy Eq. (33), and

$$\begin{aligned} s_{13} &\approx -\frac{\sqrt{2}\sigma b}{d - \sigma e - a}, \\ \cos 2\theta_{23} &\approx \frac{(d - \sigma e - a)d' + 2bb'}{\sigma e(d - \sigma e - a) - b^2} \varepsilon, \end{aligned} \quad (36)$$

for $d - \sigma e - a \neq 0$, which coincides with $\cos 2\theta_{23}$ of Eq. (34) if the phenomenological requirement of $\sin^2\theta_{13} \ll 1$ is translated into the smallness of b .

V. SUMMARY AND DISCUSSIONS

We have clarified the effects from the $\mu - \tau$ symmetry breaking in neutrino mass textures. It is of great significance to recognize that the ordering of the eigenvalues for a given neutrino mass matrix conceptually yields completely different results. If the texture is $\mu - \tau$ symmetric, its diagonalization gives either $\sin\theta_{13} = 0$ for C1) or $\sin\theta_{12} = 0$ for C2). Of course, the case of $\sin\theta_{13} = 0$ is a usually claimed result if the $\mu - \tau$ symmetry is present. However, the case with $\sin\theta_{12} = 0$ is equally possible to arise and the $\mu - \tau$ symmetry breaking points to the suppression of $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$. Practically, including the $\mu - \tau$ symmetry breaking effect of ε , we can choose the case of $\sin\theta_{12} = 0$ as a phenomenologically acceptable one once a fine-tuning is invoked to yield $\sin^2 2\theta_{12} = \mathcal{O}(1)$ provided that another requirement of $\sin^2\theta_{13} \ll 1$ is fulfilled. This observation

indicates that the C2) case necessarily involves two small quantities, which is ε , and another quantity η that keeps $\sin^2\theta_{13} \ll 1$. However, it would be curious to have $\sin\theta_{12} = 0$ in the symmetric limit. It is only possible if the zero-th order contribution is as small as ε . Since the ε -term is placed on the numerator of $\tan 2\theta_{12}$, the necessary condition to have $\sin^2 2\theta_{12} = \mathcal{O}(1)$ is to require $|d + \sigma e - a| \sim \eta$ with $|\eta| \sim |\varepsilon|$ in the denominator.

The relations among $\cos 2\theta_{23}$, $\sin\theta_{13}$ and $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$ are shown to indicate the general property for any texture as described in Eqs. (33) and (34). We have found the relations:

$$\cos 2\theta_{23} \sim \sin\theta_{13}, \quad \cos 2\theta_{23} \sim \frac{\Delta m_{\odot}^2}{\Delta m_{\text{atm}}^2}, \quad (37)$$

respectively, for C1) and C2). It is clear that there is no relationship between $\sin\theta_{13}$ and $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$ as long as the results of the $\mu - \tau$ symmetry breaking are concerned. To have any correlation between $\sin^2\theta_{13}$ and $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$, we need some specific relations among the flavor masses, which currently arise from other phenomenological requirements.

We have also argued that the suppression of $\Delta m_{\odot}^2$, which is estimated to be $\Delta m_{\odot}^2 = 2\sqrt{2}(m_1 + m_2)X/\sin 2\theta_{12}$, requires either

- (i) $m_1 + m_2 \approx 0$, or
- (ii) $X \approx 0$, where $\tan 2\theta_{12}$ is proportional to X .

To satisfy $X \approx 0$ is a consequence of the approximate $\mu - \tau$ symmetry breaking in C2). On the other hand, the C1) case requires $m_1 + m_2 \approx 0$ including the suppression of both m_1 and m_2 such as in the normal mass hierarchy. If we demand that $a + d - \sigma e = 0$, we obtain $m_1 + m_2 \propto \sin^2\theta_{13}$ from Eq. (23), leading to

$$\frac{\Delta m_{\odot}^2}{\Delta m_{\text{atm}}^2} \sim \sin^2\theta_{13}, \quad (38)$$

as in Eq. (30), which arises from contributions of $\mathcal{O}(\sin^2\theta_{13})$. It is also argued that this relation is only possible for the inverted mass hierarchy, and the quasi

degenerate mass pattern with $|m_{1,2,3}| \sim \sqrt{\Delta m_{\text{atm}}^2}$. The similar relation also exists for C2), but it may not be phenomenologically acceptable because of the further suppression of $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$ due to X .

Additional necessary suppression is required in the C2) case to meet $\sin^2\theta_{13} \ll 1$, which may be due to the approximate L_e conservation. The presence of this conservation also helpful to have $\Delta m_{\odot}^2 \ll 1$ for C1) with $m_1 + m_2 \neq 0$. We expect the following scenarios to emerge. First, any underlying dynamics equipped with the approximate $\mu - \tau$ symmetry, and the approximate L_e -conservation explains $\sin^2\theta_{13} \ll 1$ as well as $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2 \ll 1$. Furthermore, in C2), $\sin^2 2\theta_{12} = \mathcal{O}(1)$ should be realized and can be obtained if the dynamics ensures that $|d + \sigma e - a| \propto |\varepsilon|$. Next, especially in C1), the dynamics only equipped with the approximate $\mu - \tau$ symmetry can describe $\sin^2\theta_{13} \ll 1$ and $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2 \ll 1$ if it ensures that $a + d - \sigma e = 0$.

In conclusion, we have demonstrated how the argument based on the $\mu - \tau$ symmetry is powerful not only for the classification of textures but also for the discovery of general correlations among $\sin\theta_{13}$, $\cos 2\theta_{23}$ and $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2$. The origin of $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2 \ll 1$ can be ascribed to the approximate L_e conservation in the normal mass hierarchy, and to the relationship of $\Delta m_{\odot}^2/\Delta m_{\text{atm}}^2 \sim \sin^2\theta_{13}$ in the inverted mass hierarchy, and the quasi degenerate mass pattern. More precise estimation of these relations is possible if we construct explicit textures leading to the normal and inverted mass hierarchies, and to the quasi degenerate mass pattern.

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