

Universal criterion for black hole stability

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It is shown that a nonrotating macroscopic black hole with very large horizon area can remain in stable thermal equilibrium with Hawking radiation provided *its mass, as a function of horizon area, exceeds its microcanonical entropy, i.e., its entropy when isolated, without thermal radiation or accretion, and having a constant horizon area* (in appropriate units). The analysis does not use properties of specific classical spacetimes, but depends only on the plausible assumption that the mass is a function of the horizon area for large areas.

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Black holes whose spacetimes are flat at infinity, like the Schwarzschild spacetime, exhibit a thermal instability: they either radiate or accrete thermally without limit. In the former case they may evaporate away completely [1,2] with possible loss of information. In the latter case they may turn into supermassive black holes the likes of which have recently been claimed to have been observed. The thermal instability of asymptotically flat black holes has been variously attributed to a negative heat capacity or a superexponential density of states leading to a diverging canonical partition function. In contrast, black hole spacetimes which asymptotically have constant negative curvature (anti-de Sitter or AdS) have special properties enabling them to coexist in stable thermal equilibrium with a bath of their own radiation [3]. Are there other black holes which may be in stable equilibrium with radiation? It is clear that semiclassical approaches employing explicitly classical background metrics will have limited applicability in unravelling issues like this, since black hole thermodynamics has origins most definitely grounded in the quantum nature of spacetime [4].

A background-independent canonical quantization of general relativity, known as Loop Quantum Gravity (LQG) [5] has yielded a fairly complete understanding of the entropy of *isolated* black holes [6–8] which are neither radiating nor accreting thermally, and are therefore of fixed horizon area $\mathcal{A}$. For nonrotating isolated black holes with large $\mathcal{A} \gg l_{\text{Planck}}^2$ (l_{Planck} is the Planck length $(G\hbar/c^3)^{1/2}$) the microcanonical entropy $S_{MC}(\mathcal{A})$ is given by an infinite series in inverse powers of horizon area [7,8],

$$S_{MC}(\mathcal{A}) = S_{BH} - \frac{\xi}{2} \log S_{BH} + \text{const.} + O(S_{BH}^{-1}), \quad (1)$$

where, $S_{BH} \equiv \mathcal{A}/4l_{\text{Planck}}^2$ is the Bekenstein-Hawking entropy, $\xi = 3$, if the residual gauge invariance on a spatial slice of the horizon (leftover from local Lorentz invariance) is $SU(2)$, and $\xi = 1$ if the gauge group is $U(1)$. The subleading terms in Eq. (1) are all finite and unambigu-

ously calculable. The only ambiguity involves the Barbero-Immirzi parameter γ which is fixed by fitting the leading area term to S_{BH} . The $\log(\text{area})$ corrections are clearly independent of this ambiguity. In obtaining this result, crucial use has been made of the fact that in LQG, geometrical quantities like area and volume are represented by self-adjoint operators acting on the (kinematical) Hilbert space, which can be shown to have a discrete spectrum [9]. For very large areas (in units of Planck area), the area spectrum can be shown to be characterized by an integer, $\mathcal{A}_n \sim l_{\text{Planck}}^2 n$ with $n \gg 1$.

The microcanonical approach is however inadequate from a physical perspective since a black hole does necessarily radiate and accrete thermally. To handle this dynamical situation, Loop Quantum Gravity *per se* is not very useful, because of long-standing difficulties involving the quantum Hamiltonian constraint [5]. In absence of a direct quantization of a dynamical horizon, the present authors have adopted an indirect heuristic approach based on the standard canonical ensemble of equilibrium statistical mechanics. This has been applied to nonrotating black holes assumed to be in contact with their radiation bath [10–12]. The horizon is assumed to be an inner boundary of spacetime. With this assumption, and the fact that quantum states corresponding to bulk three dimensional space (on a spatial slice) are annihilated by the quantum Hamiltonian operator, the partition function essentially reduces to computing the state sum over the horizon states. This computation is performed in the saddle point approximation [10,11], including the Gaussian fluctuations around the saddle point (identified here with the classical mass M), and one obtains

$$Z_{hor} \simeq \exp \left\{ S_{MC}(M) - \beta M - \log \left| \frac{dE}{dx} \right|_{E=M} \right\} \times \left[\frac{\pi}{-S''_{MC}(M)} \right]^{1/2}. \quad (2)$$

In this equation, x is the continuum variable which has replaced the area quantum number n introduced earlier for large areas. The quantity $dE/dx|_M = M'(\mathcal{A}) \cdot \text{const.}$ since $d\mathcal{A}/dx = \text{const.}$ from the area spectrum for large

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areas (with prime indicating a derivative with respect to the argument). Notice that we have made the tacit assumption that the black hole mass is a function of the area. This is not really an assumption for many classical general relativistic black holes for asymptotically large areas. Such a functional dependence is plausible even in LQG given that the bulk Hamiltonian can be related to the volume operator [13] in LQG. We also note that the quantity in square brackets in eQ. (2) is the contribution of Gaussian fluctuation around the saddle point at $E = M$.

It is interesting that this now leads to the following canonical entropy for nonrotating black holes [11]

$$S_{\text{can}} = S_{MC}(\mathcal{A}) - \frac{1}{2} \log \Delta, \quad (3)$$

where

$$\Delta \equiv [\mathcal{A}'(x)]^2 \left[S'_{MC}(\mathcal{A}) \frac{M''(\mathcal{A})}{M'(\mathcal{A})} - S''_{MC}(\mathcal{A}) \right]. \quad (4)$$

Thus, the canonical entropy is expressed in terms of the microcanonical entropy for an average large horizon area, and the mass which is also a function of the area. Clearly, stable equilibrium ensues so long as $\Delta > 0$.

Additional support for this condition can be gleaned by considering the thermal capacity of the system, using the standard relation

$$C(\mathcal{A}) \equiv \frac{dM}{dT} = \frac{M'(\mathcal{A})}{T'(\mathcal{A})}, \quad (5)$$

with T being derived from the microcanonical entropy $S_{MC}(\mathcal{A})$, and hence a function of $\mathcal{A}$. One obtains for the heat capacity the relation

$$C(\mathcal{A}) = \left[\frac{M'(\mathcal{A})}{T(\mathcal{A}) \mathcal{A}'(x)} \right]^2 \Delta^{-1}, \quad (6)$$

so that $C > 0$ if only if $\Delta > 0$. Since the positivity of the heat capacity is certainly a necessary condition for stable thermal equilibrium, it is gratifying that an identical criterion emerges for Δ as found from the canonical entropy (3).

Using now eQ. (4) for the expression for Δ , the criterion for thermal stability of nonrotating macroscopic black holes is then easily seen to be

$$M(\mathcal{A}) > S_{MC}(\mathcal{A}) \quad (7)$$

as already mentioned in the summary. We have been using units in which $G = \hbar = c = k_B = 1$. If we revert back to units where these constants are not set to unity, the lower bound eQ. (7) can be re-expressed as

$$M(\mathcal{A}) > \left(\frac{\hbar c}{G k_B^2} \right)^{1/2} S_{MC}(\mathcal{A}). \quad (8)$$

We remind the reader that in contrast to semiclassical approaches based on specific properties of classical metrics, our approach incorporates crucially the microcanonical entropy generated by quantum spacetime fluctuations that leave the horizon area constant. Apart from the plausible assumption of the black hole mass being dependent only on the horizon area, no other assumption has been made to arrive at the result. Even so, it subsumes most results based on the semiclassical approach. It also supercedes our earlier assay [11] based on an assumption of a power law functional dependence of the mass on the area.

As a byproduct of the above analysis, the canonical entropy for stable black holes can be expressed in terms of the Bekenstein-Hawking entropy S_{BH} as

$$S_{\text{can}} = S_{BH} - \frac{1}{2}(\xi - 1) \log S_{BH} - \frac{1}{2} \times \log \left[\frac{S'_{MC}(\mathcal{A}) M''(\mathcal{A})}{S''_{MC}(\mathcal{A}) M'(\mathcal{A})} \right]. \quad (9)$$

For any smooth $M(\mathcal{A})$, one can truncate its power series expansion in $\mathcal{A}$ at some large order and show that the quantity in square brackets in Eq. (9) does not contribute to the $\log(\text{area})$ term, so that

$$S_{\text{can}} = S_{BH} - \frac{1}{2}(\xi - 1) \log S_{BH} + \text{const.} + O(S_{BH}^{-1}). \quad (10)$$

The interplay between constant area quantum spacetime fluctuations and thermal fluctuations is obvious in the coefficient of the $\log(\text{area})$ term where the contribution due to each appears with a specific sign. It is not surprising that the thermal fluctuation contribution increases the canonical entropy. The cancellation occurring for horizons on which a residual $U(1)$ subgroup of $SU(2)$ survives, because of additional gauge fixing by the boundary conditions describing an isolated horizon [5], may indicate a possible nonrenormalization theorem, although no special symmetry like supersymmetry has been employed anywhere above. It is thus generic for all nonrotating black holes, including those with electric or dilatonic charge. One would expect the result to hold also for rotating black holes, as well, although the details of the microcanonical entropy for such black holes have not yet been worked out.

While this letter restricts attention to thermal fluctuations of area due to energy fluctuations alone, the stability criterion (7) holds when in addition thermal fluctuations of electric charge are incorporated within a grand canonical ensemble. The result for the grand canonical entropy is however somewhat different from that given above (10) when charge fluctuations are included [14].

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