

Towards a wave-extraction method for numerical relativity. II. The quasi-Kinnersley frameAndrea Nerozzi,^{1,2} Christopher Beetle,^{3,4} Marco Bruni,¹ Lior M. Burko,^{3,5} and Denis Pollney⁶¹*Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom*²*Center for Relativity, Department of Physics, University of Texas at Austin, Austin, Texas 78712-1081, USA*³*Department of Physics, University of Utah, Salt Lake City, Utah 84112, USA*⁴*Department of Physics, Florida Atlantic University, Boca Raton, Florida 33431, USA*⁵*Department of Physics and Astronomy, Bates College, Lewiston, Maine 04240, USA*⁶*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, 14476 Golm, Germany*

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The Newman-Penrose formalism may be used in numerical relativity to extract coordinate-invariant information about gravitational radiation emitted in strong-field dynamical scenarios. The main challenge in doing so is to identify a null tetrad appropriately adapted to the simulated geometry such that Newman-Penrose quantities computed relative to it have an invariant physical meaning. In black hole perturbation theory, the Teukolsky formalism uses such adapted tetrads, those which differ only perturbatively from the background Kinnersley tetrad. At late times, numerical simulations of astrophysical processes producing isolated black holes ought to admit descriptions in the Teukolsky formalism. However, adapted tetrads in this context must be identified using only the numerically computed metric, since no background Kerr geometry is known *a priori*. To do this, this paper introduces the notion of a quasi-Kinnersley frame. This frame, when space-time is perturbatively close to Kerr, approximates the background Kinnersley frame. However, it remains calculable much more generally, in space-times nonperturbatively different from Kerr. We give an explicit solution for the tetrad transformation which is required in order to find this frame in a general space-time.

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I. INTRODUCTION

One of the main challenges currently faced by numerical relativity is that of interpreting its results in a physically meaningful way. That is, once a given simulation is complete, one must find ways to quantify invariantly the physical information contained in the gravitational field described by the numerical variables. A new generation of experiments designed to detect and interpret gravitational radiation has lent particular importance to one such problem: that of extracting information about gravitational waves far from a modeled source. A great deal is known about gravitational radiation in various approximation schemes, such as the standard quadrupole formula of linearized gravity and the various approaches (Regge-Wheeler [1], Zerilli [2], Teukolsky [3]) to black hole perturbation theory. However, these theories are well-defined only in the perturbative regime. Each is founded on an assumed knowledge of a specific background metric on space-time which, in a typical simulation of strongly dynamical gravitational fields, is not known *a priori*.

What is needed is a background-independent formalism which does not rely on such *a priori* structures. Rather, one should seek an approach based on quantities which are calculable solely from the physical metric, and which yield information about gravitational radiation in those cases where such radiation is unambiguously present. Since the quantities we imagine here would be defined in terms of the physical metric, they could in principle be calculated at any point of any space-time. In generic situations, however,

their interpretation in terms of gravitational radiation would be lost.

Calculations of the Newman-Penrose Weyl scalar Ψ_4 have been used in numerical studies of gravitational wave forms [4–7]. This technique looks very promising because Weyl scalars are first of all coordinate independent quantities. In addition, once a suitable tetrad is found, extracting Ψ_4 one has immediately the interpretation in terms of the outgoing radiation.

For such an analysis, an appropriate Newman-Penrose tetrad must be found. This paper aims to address the problem of finding the right tetrad to calculate Ψ_4 . That is, given *only* the output of a numerical simulation, we construct a particular null frame. For space-times which truly describe perturbations of a Kerr background, our frame approximates the Kinnersley frame of that background. However, the construction works somewhat more generally, and can be applied to many numerical space-times, including some which differ from Kerr nonperturbatively. Specifically the tetrad we seek belongs to one of the *transverse frames* introduced in [8]. While three such frames exist in algebraically general space-times, only one can approximate the Kinnersley frame when the space-time is a perturbation of Kerr. Here, we show how to calculate this physically interesting *quasi-Kinnersley frame*. This is meant to be one of the two steps required to have the right quantities computed in the Teukolsky formalism, the second one being related to fixing the scalings of the vectors constituting such frames (see [9] for further details), in order to get the right radial falloffs

for the relevant quantities such as Ψ_0 and Ψ_4 . This second step will be the subject of future work. Once this construction is complete, the goal is to deploy the entire Teukolsky formalism of black hole perturbation theory in the weak-field radiation zones of a numerical evolution.

This paper constructs the quasi-Kinnersley frame within the Newman-Penrose formalism. That is, it operates by transforming a given, fiducial null tetrad on space-time to one satisfying the transversality conditions. Because the Teukolsky formalism is built on the Newman-Penrose approach, our results take a particularly clear form in this language. However, many numerical relativity codes do not currently incorporate the infrastructure needed to define and transform Newman-Penrose frames on space-time. Rather, many are based on various $3 + 1$ decompositions of the Einstein equations wherein the quantities of interest describe a spatial geometry evolving in parameter time. This approach is meant to be alternative to the one presented in [9], hereafter paper I, where the quasi-Kinnersley frame is constructed *ab initio*, starting from spatial, rather than space-time, data. Although both papers aim at the same goal, their techniques are rather different. We present them separately to preserve clarity in each. The issue of the quasi-Kinnersley frame is also presented in [10] where this frame is explicitly found, together with the radiation scalar [8], for some specific cases.

The outline of this paper is as follows. Section II establishes notation and gives general definitions, including those of both transverse and quasi-Kinnersley frames. Sections III and IV set up and solve the problem of calculating the three transverse frames in an algebraically general space-time. Section V shows how to select the unique quasi-Kinnersley frame from among those three transverse frames. Section VI will test the construction of the quasi-Kinnersley frame in a simple case. Finally, Appendix B gives closed-form expressions for the Weyl scalars and for the tetrad vectors when the fiducial frame is the principal null frame, while Appendix C discusses the existence and plurality of transverse frames in algebraically special space-times.

II. DEFINITIONS

A. Weyl scalars

In vacuum space-times, curvature is entirely encoded in the Weyl tensor C_{abcd} . The ten independent components of this tensor can be expressed in the five complex Weyl scalars

$$\Psi_0 = C_{pqrs} \ell^p m^q \ell^r m^s \quad (2.1a)$$

$$\Psi_1 = C_{pqrs} \ell^p n^q \ell^r m^s \quad (2.1b)$$

$$\Psi_2 = C_{pqrs} \ell^p m^q \bar{m}^r n^s \quad (2.1c)$$

$$\Psi_3 = C_{pqrs} \ell^p n^q \bar{m}^r n^s \quad (2.1d)$$

$$\Psi_4 = C_{pqrs} \bar{m}^p n^q \bar{m}^r n^s, \quad (2.1e)$$

where ℓ^p, n^p, m^p and $\bar{m}^p$ comprise a null tetrad. The first pair of vectors here are real, while the second pair are both complex and conjugate to one another. The only nonvanishing inner products are $\ell^p n_p = -1$ and $m^p \bar{m}_p = 1$. Relative to this noncoordinate basis, the Weyl scalars are naturally coordinate independent, but they do depend on the particular tetrad choice. The freedom in the tetrad is given by the six-dimensional Lorentz group which, in this context, is conveniently generated by elementary transformations of three types. For an exhaustive presentation of these transformations we refer to Appendix A.

Despite the complicated appearance of some of the transformation laws for the Weyl scalars, some combinations of them are independent of the tetrad. For example, two well-known scalar curvature invariants are defined by

$$I = \frac{1}{16}(C_{pq}{}^{rs} C_{rs}{}^{pq} - i C_{pq}{}^{rs*} C_{rs}{}^{pq}) \quad (2.2a)$$

$$J = \frac{1}{96}(C_{pq}{}^{rs} C_{rs}{}^{mn} C_{mn}{}^{pq} - C_{pq}{}^{rs} C_{rs}{}^{mn*} C_{mn}{}^{pq}), \quad (2.2b)$$

where $*C_{pq}{}^{rs} = \frac{1}{2} \epsilon_{pq}{}^{mn} C_{mn}{}^{rs}$ is the Hodge dual of the Weyl tensor. By definition I and J do not depend on tetrads. However, they can easily be expressed in terms of the Weyl scalars in an arbitrary tetrad:

$$I = \Psi_4 \Psi_0 - 4 \Psi_3 \Psi_1 + 3 \Psi_2^2 \quad (2.3a)$$

$$J = \det \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}. \quad (2.3b)$$

For more details we refer to [11,12].

B. Principal null directions and additional scalar quantities

Every curvature tensor picks out a family of preferred principal null directions; principal null directions are those preferred directions for which Ψ_0 or Ψ_4 are vanishing. More specifically, ℓ is a principal null direction if $\Psi_0 = 0$ while n is a principal null direction if $\Psi_4 = 0$ (see [13,14] for further details). Since these directions are determined invariantly, they are natural structures to consider for the type of tetrad construction we contemplate here. In this section, we review the process of identifying the principal null directions starting from a fiducial tetrad. This process introduces a number of quantities whose definitions will be important below.

The equation to be solved to find the principal directions sets Ψ_4 (Ψ_0) to zero after an n (ℓ) null vector rotation:

$$a^{*4} \Psi_0 + 4a^{*3} \Psi_1 + 6a^{*2} \Psi_2 + 4a^{*} \Psi_3 + \Psi_4 = 0. \quad (2.4)$$

Provided we have not started in a frame where ℓ is already a principal null vector, so $\Psi_0 \neq 0$, we introduce the new reduced variable

$$z = \Psi_0 a^{*} + \Psi_1, \quad (2.5)$$

so that Eq. (2.4) becomes the reduced equation

$$z^4 + 6Hz^2 + 4Gz + K = 0. \quad (2.6)$$

Here, H , G and K are

$$H = \Psi_0\Psi_2 - \Psi_1^2 \quad (2.7a)$$

$$G = \Psi_0^2\Psi_3 - 3\Psi_0\Psi_1\Psi_2 + 2\Psi_1^3 \quad (2.7b)$$

$$K = \Psi_0^2I - 3H^2. \quad (2.7c)$$

They can be related directly to the curvature invariants I and J using

$$\Psi_0^2I = K + 3H^2 \quad (2.8a)$$

$$\Psi_0^3J = HK - H^3 - G^2. \quad (2.8b)$$

Unlike I and J , the new quantities H , G and K take different values in different tetrads. The solution for the principal null directions is then achieved introducing three additional quantities α , β and γ defined by

$$\alpha^2 = 2\Psi_0\lambda_1 - 4H \quad (2.9a)$$

$$\beta^2 = 2\Psi_0\lambda_2 - 4H \quad (2.9b)$$

$$\gamma^2 = 2\Psi_0\lambda_3 - 4H, \quad (2.9c)$$

where the λ variables are the eigenvalues of a specific matrix Q built from the Weyl scalars (see [12] for further details). They are given by

$$\lambda_1 = -\left(P + \frac{I}{3P}\right) \quad (2.10a)$$

$$\lambda_2 = -\left(e^{2\pi i/3}P + e^{4\pi i/3}\frac{I}{3P}\right) \quad (2.10b)$$

$$\lambda_3 = -\left(e^{4\pi i/3}P + e^{2\pi i/3}\frac{I}{3P}\right), \quad (2.10c)$$

where

$$P = [J + \sqrt{J^2 - (I/3)^3}]^{1/3}. \quad (2.11)$$

Equation (2.11) may lead to some ambiguity. It is easy to see that different choices of the branch of the cubic root permute the definitions for the λ_i variables. The breaking of this permutation symmetry is essential to the definition of the quasi-Kinnersley frame [9].

In the end, we find four solutions for Eq. (2.6) which are

$$\begin{aligned} z_1 &= (\alpha + \beta + \gamma)/2 & z_2 &= (\alpha - \beta - \gamma)/2 \\ z_3 &= (-\alpha + \beta - \gamma)/2 & z_4 &= (-\alpha - \beta + \gamma)/2, \end{aligned}$$

and the solutions of Eq. (2.4) are easily derived from them using Eq. (2.5).

The triples of quantities (α, β, γ) and (H, G, K) are both tetrad-dependent. In fact, there is the same amount of information classifying a given tetrad contained in each triple. This assertion follows from the relations

$$\alpha^2 + \beta^2 + \gamma^2 = -12H \quad (2.12a)$$

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = 36H^2 - 4K \quad (2.12b)$$

$$\alpha\beta\gamma = 4G. \quad (2.12c)$$

The calculation described above could equally well be done by rotating a given tetrad to make ℓ , rather than n , a principal null direction. The calculation is essentially the same, but we outline it here to introduce notation. The operative equation to solve is

$$b^4\Psi_4 + 4b^3\Psi_3 + 6b^2\Psi_2 + 4b\Psi_1 + \Psi_0 = 0. \quad (2.13)$$

In this case, assuming $\Psi_4 \neq 0$, we can introduce the reduced variable

$$\hat{z} = \Psi_4 b + \Psi_3, \quad (2.14)$$

to find the reduced equation

$$\hat{z}^4 + 6\hat{H}\hat{z}^2 + 4\hat{G}\hat{z} + \hat{K} = 0, \quad (2.15)$$

where this time $\hat{H}$, $\hat{G}$ and $\hat{K}$ are defined as

$$\hat{H} = \Psi_4\Psi_2 - \Psi_3^2 \quad (2.16a)$$

$$\hat{G} = \Psi_4^2\Psi_1 - 3\Psi_4\Psi_3\Psi_2 + 2\Psi_3^3 \quad (2.16b)$$

$$\hat{K} = \Psi_4^2I - 3\hat{H}^2. \quad (2.16c)$$

The procedure is in this case analogous to the one already presented, and it uses the definition of other variables $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ which are given by

$$\hat{\alpha}^2 = 2\Psi_4\lambda_1 - 4\hat{H} \quad (2.17a)$$

$$\hat{\beta}^2 = 2\Psi_4\lambda_2 - 4\hat{H} \quad (2.17b)$$

$$\hat{\gamma}^2 = 2\Psi_4\lambda_3 - 4\hat{H}. \quad (2.17c)$$

It is worth noticing that hatted variables are obtained from nonhatted ones by simply swapping $\Psi_0 \leftrightarrow \Psi_4$ and $\Psi_1 \leftrightarrow \Psi_3$.

C. Null tetrads and null frames

Hereafter, we will adopt a terminology that clearly discriminates null frames and null tetrads, as follows:

- (i) A *null tetrad* is a specific set of two real null vectors ℓ and n and two complex conjugates null vectors m and $\bar{m}$.
- (ii) A *null frame* is a class of null tetrads connected by a spin/boost (type III) transformation.

D. Transverse frames

Although we are not interested in calculating principal null directions the definitions given in Sec. II B will help us provide a rigorous definition of transverse frame for a general Petrov type I space-time.

Following [8] we first define a transverse frame as

Definition 1.—A transverse frame is a frame in which $\Psi_1 = \Psi_3 = 0$.

We want to stress here the point that Def. I really identifies a frame, i.e. a class of tetrads, as it is invariant under a spin/boost (type III) transformation.

A useful geometrical property of transverse frames is given by the following proposition:

Proposition I.—A transverse frame for a Petrov type I space-time is a frame which sees principal null directions in pairs, each pair being, in the stereographic sphere, at the same angle θ and at angles ϕ_1 and ϕ_2 such that $\phi_2 - \phi_1 = \pi$.

Let us note at this point that it is clear from Eq. (2.4) that it becomes a biquadratic if and only if the frame is transverse. Proposition 1 can then be proved as follows: let us assume that we are in transverse frame and want to compute the principal null directions. Then Eq. (2.4) becomes a biquadratic and therefore if $(a^*)_1$ is a solution, then $(a^*)_2 = -(a^*)_1$ will be another solution. Using stereographic coordinates, i.e. writing the general solution for Eq. (2.4) as

$$a^* = \cot\left(\frac{\theta}{2}\right)e^{i\phi}, \quad (2.18)$$

we see that this property corresponds to seeing the two principal null directions at the same angle θ and at angles ϕ_1 and ϕ_2 such that $\phi_2 - \phi_1 = \pi$.

To prove the equivalence of Def. 1 and Prop. 1 in the other direction let us suppose that we are in a frame in which our parameters to get the principal null directions have the property described in Prop. 1, i.e. we can write them down in the following way:

$$\begin{aligned} a_1^* &= \cot\left(\frac{\theta_1}{2}\right)e^{i\phi_1} & a_2^* &= \cot\left(\frac{\theta_1}{2}\right)e^{i\phi_1+i\pi} \\ a_3^* &= \cot\left(\frac{\theta_2}{2}\right)e^{i\phi_2} & a_4^* &= \cot\left(\frac{\theta_2}{2}\right)e^{i\phi_2+i\pi}. \end{aligned} \quad (2.19)$$

Using these values to build up the polynomial defined in Eq. (2.4) we would end up with the term in a^* and a^{*3} missing, this corresponding to having $\Psi_1 = \Psi_3 = 0$ in the frame we are in.

We will hereafter refer to the property introduced in Prop. 1 as *seeing principal null directions in conjugate pairs*, in order to distinguish it from the normal principal null directions in pairs which define a Petrov type D space-time. Proposition 1 will be our starting point to define, in the next section, the quasi-Kinnersley frame.

E. The quasi-Kinnersley frame

The Kinnersley frame [15] is defined for a Petrov type D space-time. Its definition states that

Definition 2.—A Kinnersley frame for a type D space-time is a frame where the two real tetrad null vectors coincide with the two repeated principal null directions of the Weyl tensor.

In his original article, Kinnersley makes a second step with an additional condition that sets the spin coefficient ϵ to zero. This corresponds to fixing the additional degrees of

freedom coming from a spin/boost transformation, i.e. to identifying a particular tetrad out of the Kinnersley frame. In this paper we will not consider this second step, which deserves further study, and focus our attention to finding a particular frame, i.e. a particular class of null tetrads, which converges to the Kinnersley frame when the space-time approaches a type D (see also paper I for further details).

In a type D space-time the following relations hold:

$$S = 1 \quad G = 0 \quad K = 9H^2, \quad (2.20)$$

where S is the speciality index defined in [16]

$$S = \frac{27J^2}{I^3}. \quad (2.21)$$

We know that the Kinnersley frame has the additional property that all the scalars are vanishing except Ψ_2 , i.e. it is also a canonical frame [17] for Petrov type D. We would like here to find that particular frame which converges to the Kinnersley frame when $S \rightarrow 1$. We will dub this *quasi-Kinnersley frame* for a Petrov type I space-time. Our definition is then

Definition 3.—A quasi-Kinnersley frame, for a Petrov type I space-time, is a frame which converges to the Kinnersley frame when $S \rightarrow 1$.

Let us consider a transverse frame as defined in Prop. 1, such that it sees the principal null directions in conjugate pairs. The difference between the angles ϕ of each pair of null directions must remain fixed to π , even in the limit $S \rightarrow 1$. On the other hand, we know that for $S \rightarrow 1$ the two principal null directions will eventually converge. The only way we can see, from our transverse frame, the two parameters coinciding asymptotically, but keeping the difference in ϕ , is that their absolute value must tend to zero. Hence, if asymptotically our parameters for finding the principal null directions tend to zero, this means that our ℓ vector is converging to the principal null directions, i.e. we are in a quasi-Kinnersley frame.

Following this idea, we can conclude that a well-motivated strategy to find a quasi-Kinnersley frame is to look for a transverse frame. This conclusion is however not enough. By saying that a transverse frame sees principal null directions in conjugate pairs, we are not specifying *which* directions it sees in conjugate pairs. Figures 1 and 2 explain better this concept. Let us suppose that our Petrov type I space-time converges to a type D one, such that the principal null directions z_1 and z_2 will converge, and the same for z_3 and z_4 . In Fig. 1 we have constructed a transverse frame whose ℓ null vector sees z_1 and z_2 as conjugate pair (which, in the graph, is indicated by putting ℓ in the middle of the two principal null directions it sees in pairs); consequently its n vector will see z_3 and z_4 as conjugate pair, although this is not shown in the figure. It turns out that this is the quasi-Kinnersley frame as z_1 and z_2 will converge and, in particular, they will converge to ℓ . A

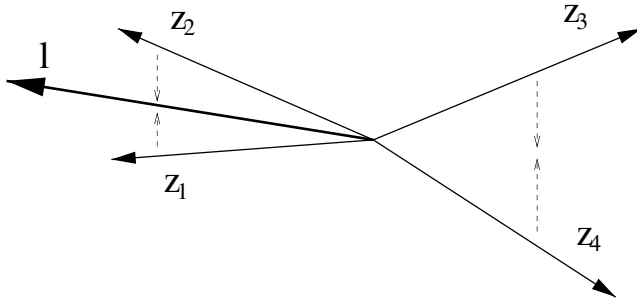


FIG. 1. A transverse frame which is also a quasi-Kinnersley frame: the ℓ vector of the transverse frame sees the two principal null directions z_1 and z_2 as conjugate pair. As the space-time approaches a Petrov type D one, z_1 and z_2 will converge and, in particular, they converge to ℓ .

counterexample is shown in Fig. 2; here l sees z_2 and z_3 as conjugate pair, such that, when z_1 and z_2 will converge, they will not converge to ℓ . This is telling us that we need an additional condition that the quasi-Kinnersley frame has to satisfy among all the transverse frames.

As mentioned earlier in the original Kinnersley paper the additional condition that all the scalars are vanishing except Ψ_2 holds. In a general transverse frame we know from Def. 1 that $\Psi_1 = \Psi_3 = 0$, so we want to impose the additional condition that $\Psi_0, \Psi_4 \rightarrow 0$ when $S \rightarrow 1$ in a quasi-Kinnersley frame. By introducing the radiation scalar $\xi = \Psi_0 \Psi_4$ (notice that ξ is to be evaluated in a transverse frame; see [8]) we end up with the following proposition:

Proposition 2.—In a Petrov type I space-time, a transverse frame where the radiation scalar $\xi \rightarrow 0$ for $S \rightarrow 1$ is a quasi-Kinnersley frame.

From the definitions given in this section and in Sec. II D and discussion above, it is clear that a good strategy to find a quasi-Kinnersley frame is to search it among transverse frames, although a transverse frame in general will not be a quasi-Kinnersley frame.

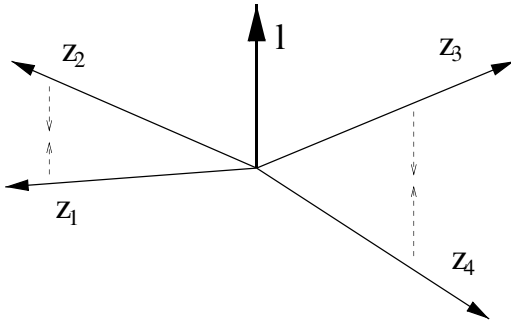


FIG. 2. A transverse frame which is not a quasi-Kinnersley frame: the ℓ vector of the frame sees the two principal null directions z_2 and z_3 as conjugate pair. When z_1 and z_2 converge, they do not converge to ℓ .

F. The linear theory

Teukolsky [3] studied a perturbed Kerr black hole space-time in the Newman-Penrose formalism, choosing the Kinnersley frame for the background metric, where for a Kerr black hole the only nonvanishing scalar is Ψ_2 . Having chosen this frame, the equations governing the dynamics of all the scalars simplify considerably, thus leading to separate evolution equations (Teukolsky equation) for Ψ_0 and Ψ_4 . It turns out that, within the linearized framework, i.e. considering infinitesimal transformations of the original Kinnersley background, the values of Ψ_0 and Ψ_4 are invariant under gauge or tetrad transformations, so that they can be given pure physical interpretation of ingoing or outgoing gravitational radiation, while Ψ_1 and Ψ_3 can be easily set to zero, thus being related to gauge degrees of freedom. Ψ_2 is instead related to the background metric. An analogous interpretation for the scalars, not restricted to linear theory, is given in [18]: here Ψ_0 and Ψ_4 are shown to be associated with transverse gravitational fields (although not necessarily representing gravitational radiation), Ψ_1 and Ψ_3 to longitudinal ones, while Ψ_2 is related to the Coulombian part of the gravitational field.

It is evident that if we choose the quasi-Kinnersley frame in our numerical simulations, and we fix the particular tetrad in this frame which shows the correct radial falloffs, we will be able to interpret, in the linear regime, Ψ_4 as the outgoing wave contribution. Moreover, the determination of whether we are or not in the linearized regime can be easily achieved using the speciality index defined in Eq. (2.21) as well described in [16].

III. THE TRANSVERSE FRAME

In the previous section we defined a transverse frame for a Petrov type I space-time. Here we want to describe the general problem of finding a transverse frame, as well as determining how many transverse frames we expect. We start from a general Petrov type I space-time having all the five Weyl scalars nonvanishing; we then perform an n null rotation (type I) with parameter a and an ℓ null rotation (type II) with parameter b , and set to zero the final values of Ψ_3 and Ψ_1 , ending up with a system of two equations to be solved for parameters a_* and b

$$\Psi_3 + 3a^* \Psi_2 + 3a^{*2} \Psi_1 + a^{*3} \Psi_0 + b(\Psi_4 + 4a^* \Psi_3 + 6a^{*2} \Psi_2 + 4a^{*3} \Psi_1 + a^{*4} \Psi_0) = 0 \quad (3.1)$$

$$\begin{aligned} &\Psi_1 + a^* \Psi_0 + 3b(\Psi_2 + 2a^* \Psi_1 + a^{*2} \Psi_0) \\ &+ 3b^2(\Psi_3 + 3a^* \Psi_2 + 3a^{*2} \Psi_1 + a^{*3} \Psi_0) + b^3(\Psi_4 \\ &+ 4a^* \Psi_3 + 6a^{*2} \Psi_2 + 4a^{*3} \Psi_1 + a^{*4} \Psi_0) = 0. \end{aligned} \quad (3.2)$$

If we derive b from Eq. (3.1), we get

$$b = -\frac{\Psi_3 + 3a^* \Psi_2 + 3a^{*2} \Psi_1 + a^{*3} \Psi_0}{\Psi_4 + 4a^* \Psi_3 + 6a^{*2} \Psi_2 + 4a^{*3} \Psi_1 + a^{*4} \Psi_0}. \quad (3.3)$$

This expression for b is well-posed. We might be wondering if the denominator of Eq. (3.3) can be vanishing. It turns out that it cannot, as this would mean that the n vector after the n null rotation (type I) with parameter a^* coincides with one principal null direction. From the definitions and propositions given in Sec. II D it is clear that the ℓ and n vectors of a transverse frame do not coincide with the principal null directions for a Petrov type I space-time.

Substituting Eq. (3.3) into Eq. (3.2), we obtain the following sixth order equation for the parameter a^* :

$$\begin{aligned} \mathcal{P}_1 a^{*6} + \mathcal{P}_2 a^{*5} + \mathcal{P}_3 a^{*4} + \mathcal{P}_4 a^{*3} \\ + \mathcal{P}_5 a^{*2} + \mathcal{P}_6 a^* + \mathcal{P}_7 = 0, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} \mathcal{P}_1 &= -\Psi_3 \Psi_0^2 - 2\Psi_1^3 + 3\Psi_2 \Psi_1 \Psi_0 \\ \mathcal{P}_2 &= -2\Psi_3 \Psi_1 \Psi_0 - \Psi_0^2 \Psi_4 + 9\Psi_2^2 \Psi_0 - 6\Psi_2 \Psi_1^2 \\ \mathcal{P}_3 &= -5\Psi_1 \Psi_4 \Psi_0 - 10\Psi_3 \Psi_1^2 + 15\Psi_3 \Psi_2 \Psi_0 \\ \mathcal{P}_4 &= -10\Psi_4 \Psi_1^2 + 10\Psi_3^2 \Psi_0 \\ \mathcal{P}_5 &= 5\Psi_3 \Psi_0 \Psi_4 + 10\Psi_1 \Psi_3^2 - 15\Psi_1 \Psi_2 \Psi_4 \\ \mathcal{P}_6 &= 2\Psi_3 \Psi_1 \Psi_4 + \Psi_4^2 \Psi_0 - 9\Psi_2^2 \Psi_4 + 6\Psi_2 \Psi_3^2 \\ \mathcal{P}_7 &= \Psi_1 \Psi_4^2 + 2\Psi_3^3 - 3\Psi_2 \Psi_3 \Psi_4. \end{aligned}$$

Equation (3.4) is of course very difficult to solve analytically and we might turn to numerical methods to find solutions.

It is worth pointing out here that we could be misled to the conclusion that we have six transverse frames, as the equation is of sixth order. This turns out to be wrong, due to a degeneracy of the transverse frame if we exchange the ℓ and n vectors: the nonvanishing scalars would be exchanged as follows:

$$\Psi_0 \rightarrow \Psi_4^* \quad \Psi_2 \rightarrow \Psi_2 \quad \Psi_4 \rightarrow \Psi_0^*,$$

more precisely, we would obtain a simple exchange $\Psi_0 \leftrightarrow \Psi_4$ without complex conjugation if we exchanged accordingly m and $\bar{m}$, thus preserving the tetrad orientation. This is exactly the exchange operation introduced in [9]. Although the frame we would get after such exchange would result as a different solution of Eq. (3.4), it is actually the same from the physical point of view, as we have just swapped the outgoing and ingoing contribution on the scalars Ψ_0 and Ψ_4 . We will name hereafter this property as the $\ell \leftrightarrow n$ degeneracy.

We conclude then that it is possible to find three transverse frames for a Petrov type I space-time, up to spin/boost transformations. This result is in agreement with what was found in [8].

Another comment to be done on Eq. (3.4) is that its solutions are all we really need, as once a is obtained, the parameter b can be easily derived from Eq. (3.3). For this reason we will no longer mention the parameter b from

now on, and we will restrict our attention to finding the solutions for a .

IV. FINDING THE TRANSVERSE FRAMES

We will now derive the general solution for the parameter a which leads to the three transverse frames. Our goal is to solve Eq. (3.4). It can be shown easily that this equation corresponds to setting to zero the quantity $\hat{G}$ (2.16b) after the n null rotation (type I) with parameter a , i.e.

$$\hat{G}^a = \frac{\hat{\alpha}^a \hat{\beta}^a \hat{\gamma}^a}{4} = 0, \quad (4.1)$$

where the index a tells us that these are the quantities in the frame which we get *after* the n null rotation. The equivalence of Eq. (4.1) with Eq. (3.4) is evident if, in the substitution of Eq. (3.3) into Eq. (3.2), one does not explicitly expand the Weyl scalars in terms of a^* after the first n null rotation.

Equation (4.1) expresses in a much more evident way the presence of three transverse frames. Moreover it gives us a straightforward way to factorize Eq. (3.4), as each of the three transverse frames can be defined as follows:

$$\text{I: } \hat{\alpha}^a = 0 \quad (4.2a)$$

$$\text{II: } \hat{\beta}^a = 0 \quad (4.2b)$$

$$\text{III: } \hat{\gamma}^a = 0. \quad (4.2c)$$

This conclusion allows us to reduce the degree of the polynomial originally defined in Eq. (3.4). Let us now focus our attention on just one transverse frame (frame I) which verifies the condition $\hat{\alpha}^a = 0$. For the sake of simplicity, as we have defined $\hat{\alpha}^2$ in Eq. (2.17a), we will study the completely equivalent condition $(\hat{\alpha}^a)^2 = 0$. If we write this condition in terms of the variables in the original frame [using Eqs. (A2)] we get

$$\mathcal{Q}_1 z^4 + \mathcal{Q}_2 z^3 + \mathcal{Q}_3 z^2 + \mathcal{Q}_4 z + \mathcal{Q}_5 = 0, \quad (4.3)$$

where

$$\begin{aligned} \mathcal{Q}_1 &= \Psi_0 \lambda_1 - 2H & \mathcal{Q}_2 &= -4G \\ \mathcal{Q}_3 &= 6\Psi_0 \lambda_1 H + 6H^2 - 2K & \mathcal{Q}_4 &= 4G(H + \Psi_0 \lambda_1) \\ \mathcal{Q}_5 &= -2KH + 2G^2 + \Psi_0 \lambda_1 K, \end{aligned}$$

and z is the reduced variable defined in Eq. (2.5). Equation (4.3) is already a good achievement as we passed from a sixth order equation to a fourth order one. But still this is not enough. As mentioned previously we are actually studying the condition $(\hat{\alpha}^a)^2 = 0$ so we want to be able to calculate the square root of this polynomial and reduce it to a second order equation.

Using Eqs. (2.7), (2.8), and (2.9) it is possible to do that, the second order polynomial being

$$z^2 - \left(\frac{2G}{\Psi_0 \lambda_1 - 2H} \right) z - (H + \Psi_0 \lambda_1) = 0, \quad (4.4)$$

whose solutions are

$$z_{1,2} = \frac{G \pm \sqrt{G^2 + (\Psi_0 \lambda_1 - 2H)^2 (H + \Psi_0 \lambda_1)}}{\Psi_0 \lambda_1 - 2H}. \quad (4.5)$$

The $\pm$ in Eq. (4.5) is related to the $\ell \leftrightarrow n$ degeneracy. We can reexpress Eq. (4.5) in a more elegant and suitable form, as a function of the α , β and γ variables. Moreover, the same procedure can be applied to Eqs. (4.2b) and (4.2c) in order to find the parameter to get to the other two transverse frames. The final result is

$$(z)_I = \frac{1}{2\alpha} [\beta\gamma \pm \sqrt{(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2)}] \quad (4.6a)$$

$$(z)_{II} = \frac{1}{2\beta} [\alpha\gamma \pm \sqrt{(\beta^2 - \gamma^2)(\beta^2 - \alpha^2)}] \quad (4.6b)$$

$$(z)_{III} = \frac{1}{2\gamma} [\alpha\beta \pm \sqrt{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}]. \quad (4.6c)$$

The initial parameter a^* can be easily found using Eq. (2.5).

V. THE QUASI-KINNERSLEY FRAME

Now that we have obtained the solutions for all the transverse frames in a Petrov type I space-time, we wish to check if it is possible to determine which one of them is the quasi-Kinnersley frame we are looking for. As stated in Sec. II E this frame must satisfy the additional condition that $\xi \rightarrow 0$ when $S \rightarrow 1$.

Our starting point is Eqs. (2.10). We need to calculate their limit when $S \rightarrow 1$. Using Eq. (2.11) we know that $P \rightarrow J^{1/3}$. In order to substitute this value into Eqs. (2.10) we need to express it in function of I , using $I^3 \rightarrow 27J^2$. We face here again the problem of branch choosing to take the root of a complex number; let us for the moment fix one branch and have $J^{1/3} \rightarrow (I/3)^{1/2}$. Using this expression we get that

$$\lambda_1 \rightarrow -2\sqrt{I/3} \quad (5.1a)$$

$$\lambda_2 \rightarrow \sqrt{I/3} \quad (5.1b)$$

$$\lambda_3 \rightarrow \sqrt{I/3}. \quad (5.1c)$$

Equations (5.1) help us remove the ambiguity of choosing the right branches. No matter what branches we choose in taking roots of complex numbers, we will end up having three λ variables, one of which will have a greater absolute value, precisely twice as much than the other two, in zones of the space-time close to type D. Once identified that particular λ variable, we will name it λ_1 . The remaining freedom in naming λ_2 and λ_3 is not relevant to identify the quasi-Kinnersley frame.

Using now the properties of a transverse frame given in Def. 1 and Eqs. (2.9) and (4.2), it can be shown that the values of Ψ_2 in the three transverse frames are given by

$$(\Psi_2)_I = \lambda_1/2 \quad (5.2a)$$

$$(\Psi_2)_{II} = \lambda_2/2 \quad (5.2b)$$

$$(\Psi_2)_{III} = \lambda_3/2. \quad (5.2c)$$

Moreover, using Eq. (2.3a), it is possible to show that the radiation scalar ξ has the following value in the three transverse frames:

$$(\xi)_I = (\lambda_2 - \lambda_3)^2/4 \quad (5.3a)$$

$$(\xi)_{II} = (\lambda_1 - \lambda_3)^2/4 \quad (5.3b)$$

$$(\xi)_{III} = (\lambda_1 - \lambda_2)^2/4. \quad (5.3c)$$

Hence, using Eqs. (5.1), we conclude that the asymptotic values for ξ in the three transverse frames when $S \rightarrow 1$ are

$$(\xi)_I \rightarrow 0 \quad (5.4a)$$

$$(\xi)_{II} \rightarrow 3I/4 \quad (5.4b)$$

$$(\xi)_{III} \rightarrow 3I/4, \quad (5.4c)$$

this leads to our conclusion that *transverse frame I is the quasi-Kinnersley frame*, as it is the only one that matches all the criteria given in Prop. 2.

It is worthwhile at this point to compare the definition of the quasi-Kinnersley frame advanced in this paper with that contained in the companion paper [9]. In particular, the present definition operates very simply by identifying that eigenvalue of the Weyl tensor with the largest modulus. The companion paper uses a somewhat more general definition, deriving a nonperturbative formula for the relevant eigenvalue which holds throughout the disk $|S - 1| < 1$, and then identifying the quasi-Kinnersley frame as the eigenvector with that particular eigenvalue. In the limit $S \rightarrow 1$, both of these definitions are entirely equivalent. However, it is initially not at all clear to what extent they remain equivalent when S differs from unity by a finite amount. That is, although the eigenvalues themselves are degenerate only at the critical points where $S = 0$ or $S = 1$, this guarantees nothing about their moduli. There could be points within the region $|S - 1| < 1$ where two eigenvalues differ only by a phase. Thus, we are led to ask what the largest neighborhood of unity in the S plane is in which the quasi-Kinnersley frame, as defined in the companion paper, is actually associated with the eigenvalue of largest modulus. The answer is quite unexpected: it is the entire disk $|S - 1| < 1$.

Combining Eqs. (2.10) and (2.11), one can identify the three possible eigenvalues of the Weyl tensor with the three branches of a simple function of S (times a prefactor involving I , J and $\sqrt{S}$ which is the same for all three branches). This is done explicitly in the companion paper. At $S = 1$, the quasi-Kinnersley frame is associated with

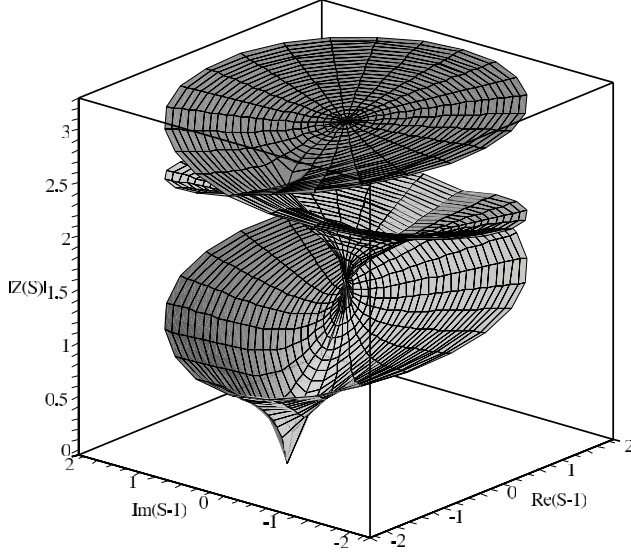


FIG. 3. A representation of a function giving the three eigenvalues of the Weyl tensor as a function of S in the region $|S - 1| < 2$. The front lateral axis is the real part of $S - 1$, while the other lateral axis is its imaginary part. The vertical axis is the modulus of the eigenvalue, and the function itself is clearly triple-valued at most points. The figure demonstrates explicitly that the moduli of the eigenvalues do not equal one another except on the branch lines of the underlying complex function, where of course the eigenvalues themselves are equal.

the branch of largest modulus. Moreover, using these explicit formulas, one can plot the moduli of this branch alongside those of the other two in a finite neighborhood of unity. This is done in Fig. 3 throughout the region $|S - 1| < 2$. The topmost sheet of this surface is clearly associated with the quasi-Kinnersley frame at $S = 1$, the center of the polar coordinates used to generate the figure. Notably, this sheet does not intersect the other sheets, which give the moduli of the other two eigenvalues, except where S is real and nonpositive. Thus, within the region $|S - 1| < 1$ of primary interest, the eigenvalue of largest modulus is always associated with the quasi-Kinnersley frame, as defined in the companion paper. Since outside of this region one encounters subtleties in the branch structure of this complex function which make even the definition of the companion paper somewhat problematic, we can conclude that the two definitions advanced in these papers are effectively equivalent. This observation will simplify considerably the practical problem of identifying the quasi-Kinnersley frame. One need only find the largest eigenvalue of the Weyl tensor.

VI. A SIMPLE CASE

Let us suppose that we are already in a transverse frame and we want to get the parameters that take us to the other two frames. In order to simplify the calculations, let us also fix the particular tetrad in the transverse frame for which $\Psi_0 = \Psi_4$.

Equation (3.4) simplifies enormously if we set $\Psi_1 = \Psi_3 = 0$ and $\Psi_0 = \Psi_4$ in our initial tetrad, and becomes

$$a^{*5} - a^* = 0, \quad (6.1)$$

here, the solution $a^* = 0$ indicates that we are already in a transverse tetrad, while the corresponding tetrad which we would get by the $\ell \leftrightarrow n$ degeneracy cannot be obtained with a type I rotation (equivalently it could be obtained using a parameter $a^* = \infty$), this explaining the one order lowering of the polynomial.

The other relevant solutions are

$$a^* = 1, i, -1, -i. \quad (6.2)$$

Such a solution allows us to derive another simple geometrical explanation to the presence of three transverse frames for a Petrov type I space-time, more directly linked to what an appropriately chosen observer would measure. Once we have the solution for a^* , using Eq. (3.3) we can find the corresponding values for the b parameter related to the ℓ null rotation (type II), the result being

$$b = -1/2, i/2, 1/2, -i/2. \quad (6.3)$$

Now let us suppose that the tetrad we define in the first transverse frame is built from a timelike vector u and three spacelike vectors e_1, e_2 and e_3 in the usual way

$$\ell_1^p = \frac{1}{\sqrt{2}}(u^p + e_3^p) \quad (6.4a)$$

$$n_1^p = \frac{1}{\sqrt{2}}(u^p - e_3^p) \quad (6.4b)$$

$$m_1^p = \frac{1}{\sqrt{2}}(e_1^p + ie_2^p). \quad (6.4c)$$

If we use the parameters $a^* = 1$ and $b = -1/2$ to get to the second transverse frame, we obtain the following expression for the new tetrad vectors:

$$\ell_{II}^p = \frac{1}{\sqrt{2}}(u^p - e_1^p) \quad (6.5a)$$

$$n_{II}^p = \frac{1}{\sqrt{2}}(u^p + e_1^p) \quad (6.5b)$$

$$m_{II}^p = \frac{1}{\sqrt{2}}(e_2^p + ie_3^p), \quad (6.5c)$$

where we have also used a type III rotation to readjust the normalization constants. Analogously, using $a^* = i$ and $b = i/2$ we can get to the third transverse frame, whose tetrad vectors are

$$\ell_{\text{III}}^p = \frac{1}{\sqrt{2}}(u^p + e_2^p) \quad (6.6a)$$

$$n_{\text{III}}^p = \frac{1}{\sqrt{2}}(u^p - e_2^p) \quad (6.6b)$$

$$m_{\text{III}}^p = \frac{1}{\sqrt{2}}(e_1^p - ie_3^p). \quad (6.6c)$$

Equations (6.4), (6.5), and (6.6) show that the presence of three transverse frames corresponds to the freedom an observer has in choosing one of the three spacelike vectors in order to construct the two real null vectors ℓ and n . The remaining two spacelike vectors are then used to construct the complex null vector m .

Following Szekeres's gravitational compass [18] approach, the electric Weyl tensor represents the only direct curvature contribution to the Jacobi (or, in particular, the geodesic deviation) equation, and for any Petrov type I field and any transverse frame can be expressed as [12]

$$E^{pq} = \text{Re}(\Psi_2)e_{\text{C}}^{pq} - \frac{1}{2}\text{Re}(\Psi_0 + \Psi_4)e_{\text{T}+}^{pq} + \frac{1}{2}\text{Im}(\Psi_0 - \Psi_4)e_{\text{T}\times}^{pq}, \quad (6.7)$$

where in a frame as (6.4)

$$e_{\text{C}}^{pq} = e_1^p e_1^q + e_2^p e_2^q - 2e_3^p e_3^q \quad e_{\text{T}\times}^{pq} = e_1^p e_2^q + e_2^p e_1^q \\ e_{\text{T}+}^{pq} = e_1^p e_1^q - e_2^p e_2^q,$$

respectively represent a Coulombian and two transverse basis tensors. It is actually this expression for E^{pq} than justifies in general (and not just in a perturbative context) the “transverse frame” terminology: for a generic tetrad with $\Psi_1 \neq 0$ or $\Psi_3 \neq 0$ there would also be longitudinal contributions to (6.7) [18]. For type D space-times, observers using a canonical null tetrad where only $\Psi_2 \neq 0$ (and associated orthonormal one) do not measure any transverse contribution. On the other hand, in a type I space-time any observer associated with a transverse frame would measure transverse contributions stresses to his/her gravitational compass, even when no gravitational radiation is present, as it is clear, for example, from an analysis of the Kasner [19] and stationary axisymmetric rotating neutron stars space-times [20]. In these cases, however, the observer would unambiguously exclude the presence of gravitational radiation by observing a zero superenergy flux (see e.g. [21]).

VII. CONCLUSIONS

In this paper we have illustrated a method to explicitly construct this quasi-Kinnersley frame within the Newman-Penrose formalism [22]. First we have provided the definition of the quasi-Kinnersley frame for a general Petrov type I space-time. This definition allowed us to write down

the basic equations that this particular frame has to satisfy, and, eventually, to solve them. Using this solution it is possible to rotate our arbitrary initial null tetrad to the quasi-Kinnersley frame. In this way we have completely fixed the 4 degrees of freedom coming from n (type I) and ℓ (type II) vector rotations, remaining with the 2 degrees of freedom coming from spin/boost (type III) transformations, which deserve further study. Finally, in the appendices, we highlighted further details on finding the transverse frames in the general case and for algebraically special space-times.

While using the Newman-Penrose formalism [22] to construct the quasi-Kinnersley frame is certainly well suited for codes using a characteristic formulation [23], most numerical relativity is formulated using the 3 + 1 decomposition of Einstein equations. In this context it is therefore important to construct the quasi-Kinnersley frame directly from the spatial geometry. This approach to the construction of the quasi-Kinnersley frame is complementary to the one presented here, and is presented in paper I. Both approaches identify a quasi-Kinnersley frame as one of the three transverse frames present in a Petrov type I space-time. The problem of understanding which transverse frame is the quasi-Kinnersley frame is faced in both approaches and different solutions are presented. In Sec. V we have shown that these solutions are completely equivalent not only in a perturbative regime, but in the entire disk $|S - 1| < 1$.

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APPENDIX A: TETRAD TRANSFORMATIONS

The six parameters of a Lorentz transformation acting on a null tetrad are conveniently expressed in three complex parameters. These parameters yield frame rotations of three types:

- (i) n vector null rotations (type I) leave ℓ unchanged, while the other vectors are transformed as follows:

$$\begin{aligned} \ell &\rightarrow \ell & n &\rightarrow n + a^*m + a\bar{m} + aa^*\ell \\ m &\rightarrow m + a\ell & \bar{m} &\rightarrow \bar{m} + a^*\ell \end{aligned} \quad (A1)$$

where a is a complex parameter and a^* is its complex conjugate. The effect of this transformation on the Weyl scalars is

$$\Psi_0 \rightarrow \Psi_0 \quad (\text{A2a})$$

$$\Psi_1 \rightarrow \Psi_1 + a^* \Psi_0 \quad (\text{A2b})$$

$$\Psi_2 \rightarrow \Psi_2 + 2a^* \Psi_1 + a^{*2} \Psi_0 \quad (\text{A2c})$$

$$\Psi_3 \rightarrow \Psi_3 + 3a^* \Psi_2 + 3a^{*2} \Psi_1 + a^{*3} \Psi_0 \quad (\text{A2d})$$

$$\Psi_4 \rightarrow \Psi_4 + 4a^* \Psi_3 + 6a^{*2} \Psi_2 + 4a^{*3} \Psi_1 + a^{*4} \Psi_0. \quad (\text{A2e})$$

- (ii) ℓ vector null rotations (type II) change the tetrad vectors in the following way:

$$\begin{aligned} \ell &\rightarrow \ell + b^* m + b \bar{m} + b b^* n & n &\rightarrow n \\ m &\rightarrow m + b n & \bar{m} &\rightarrow \bar{m} + b^* n \end{aligned} \quad (\text{A3})$$

where b is a second complex scalar quantity. The Weyl scalars transform as

$$\Psi_0 \rightarrow \Psi_0 + 4b \Psi_1 + 6b^2 \Psi_2 + 4b^3 \Psi_3 + b^4 \Psi_4 \quad (\text{A4a})$$

$$\Psi_1 \rightarrow \Psi_1 + 3b \Psi_2 + 3b^2 \Psi_3 + b^3 \Psi_4 \quad (\text{A4b})$$

$$\Psi_2 \rightarrow \Psi_2 + 2b \Psi_3 + b^2 \Psi_4 \quad (\text{A4c})$$

$$\Psi_3 \rightarrow \Psi_3 + b \Psi_4 \quad (\text{A4d})$$

$$\Psi_4 \rightarrow \Psi_4. \quad (\text{A4e})$$

- (iii) *Spin/boost transformations (type III)* rescale the vectors ℓ and n , and rotate m and $\bar{m}$ in their complex plane:

$$\begin{aligned} \ell &\rightarrow A^{-1} \ell & n &\rightarrow A n \\ m &\rightarrow e^{i\theta} m & \bar{m} &\rightarrow e^{-i\theta} \bar{m} \end{aligned} \quad (\text{A5})$$

where A and θ are two real scalars. Weyl scalars are modified according to

$$\Psi_0 \rightarrow A^{-2} e^{2i\theta} \Psi_0 \quad (\text{A6a})$$

$$\Psi_1 \rightarrow A^{-1} e^{i\theta} \Psi_1 \quad (\text{A6b})$$

$$\Psi_2 \rightarrow \Psi_2 \quad (\text{A6c})$$

$$\Psi_3 \rightarrow A e^{-i\theta} \Psi_3 \quad (\text{A6d})$$

$$\Psi_4 \rightarrow A^2 e^{-2i\theta} \Psi_4. \quad (\text{A6e})$$

APPENDIX B: MORE COMMENTS ON FINDING THE TRANSVERSE FRAMES

As pointed out in Sec. III, the six transverse frames initially found are $\ell \leftrightarrow n$ degenerate, so that only three independent equivalency classes of transverse frames remain. In this appendix we look in greater detail into the properties of the frames under an exchange operation $\ell \leftrightarrow n$. To facilitate the discussion, we assume here without loss of generality that our algebraically general space-time is written in a principal null frame, for which $\Psi_0 = 0$ and

$\Psi_4 = 0$, which can always be done [11]. This situation is of little interest for numerical relativity applications, and is undertaken in this appendix for illustrating some of the mathematical properties of transverse frames. In this appendix we recapitulate the construction of the transverse frames under the assumption that the initial frame is the principal null one. This assumption allows us to write explicitly in closed form the real null vectors of the transverse frame. Also, we consider the properties of the transverse frame under the exchange operation $\ell \leftrightarrow n$.

1. Finding the transverse frames

We assume an algebraically general space-time in the principal null frame. We then perform two successive null rotations. The first is a class I rotation (which keeps ℓ fixed) with parameter a , followed by a class II rotation (which keeps n fixed) with parameter b . (See Appendix A for details.) In what follows we denote the Weyl scalars of the principal null frame by Ψ_i ($i = 0 \dots 4$), Ψ'_i are the Weyl scalars in the frame obtained after the first null rotation, and Ψ''_i in the frame obtained after the second null rotation. By Def. 1, we are looking for rotations such that both Ψ''_1 and Ψ''_3 are zero simultaneously.

We next use Eq. (3.3) for the particular case $\Psi_0 = 0 = \Psi_4$ (for the principal null frame), which simplifies to

$$b = -\frac{1}{2a^*} \frac{\Psi_3 + 3a^* \Psi_2 + 3a^{*2} \Psi_1}{2\Psi_3 + 3a^* \Psi_2 + 2a^{*2} \Psi_1}, \quad (\text{B1})$$

to make $\Psi''_3 = 0$. Demanding next that Ψ''_1 too is zero, Eq. (3.4) simplifies to

$$\begin{aligned} \Psi_1^3 a^{*6} + 3\Psi_1^2 \Psi_2 a^{*5} + 5\Psi_1^2 \Psi_3 a^{*4} - 5\Psi_1 \Psi_3^2 a^{*2} \\ - 3\Psi_2 \Psi_3^2 a^* - \Psi_3^3 = 0. \end{aligned} \quad (\text{B2})$$

The polynomial on the left-hand side of Eq. (B2) can be easily factored as

$$(\Psi_1 a^{*2} - \Psi_3)^2 (\Psi_1 a^{*2} + x_1)^2 (\Psi_1 a^{*2} + x_2)^2 = 0, \quad (\text{B3})$$

where

$$x_1 = \Psi_3 + \frac{a^*}{2} (3\Psi_2 - \sqrt{9\Psi_2^2 - 16\Psi_1 \Psi_3}) \quad (\text{B4})$$

and

$$x_2 = \Psi_3 + \frac{a^*}{2} (3\Psi_2 + \sqrt{9\Psi_2^2 - 16\Psi_1 \Psi_3}). \quad (\text{B5})$$

As pointed out in Sec. III, we can thus do six different null rotations to transverse frames. For simplicity, let us first do the rotations for which $\Psi_1 a^{*2} - \Psi_3 = 0$. Specifically, we can do rotations with $a^* = \pm \sqrt{\Psi_3/\Psi_1}$ and $b = \mp \sqrt{\Psi_1/(4\Psi_3)}$. In the transverse frames we find that

$$\Psi_0'' = \frac{1}{8} \Psi_1 \left(3 \frac{\Psi_2}{\Psi_3} \mp 4 \sqrt{\frac{\Psi_1}{\Psi_3}} \right) \quad (\text{B6a})$$

$$\Psi_2'' = -\frac{1}{2} \Psi_2 \quad (\text{B6b})$$

$$\Psi_4'' = 6 \frac{\Psi_2 \Psi_3}{\Psi_1} \pm 8 \Psi_3 \sqrt{\frac{\Psi_3}{\Psi_1}}. \quad (\text{B6c})$$

For either choice of sign we find that the product $\Psi_0'' \Psi_4''$ is the same. Specifically, $\Psi_0'' \Psi_4'' = \frac{9}{4} \Psi_2^2 - 4 \Psi_1 \Psi_3$. Below, we show how to find the remaining two transverse frames.

The remaining two vectors of the null tetrad, namely, the complex null vectors m and $\bar{m}$ can be easily found up to a rotation in the $m\bar{m}$ plane by solving the following 7 equations for the 8 unknown components of the two vectors. These equations are the conditions that the frame is null, in addition to the normalization condition. Specifically, $m \cdot m = \bar{m} \cdot \bar{m} = 0$, $\ell \cdot m = \ell \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0$, $m \cdot \bar{m} = 1$. The indeterminate rotation parameter in the $m\bar{m}$ plane does not influence the two real null vectors ℓ, n , and affects the Weyl scalars only by a phase. In particular, Ψ_2 and the product $\Psi_0 \Psi_4$ (and also the product $\Psi_1 \Psi_3$) are invariant under spatial rotations in the $m\bar{m}$ plane (class III rotations).

2. The $\ell \leftrightarrow n$ degeneracy

In the preceding discussion we found that by setting $\Psi_1 a^{*2} - \Psi_3 = 0$ we find two transverse frames. Next, we show that the two choices of signs correspond to the degeneracy of $\ell \leftrightarrow n$ (up to a scale factor). Let us attach a subscript 1 to the choice of the sign $+$ in a^* , and a subscript 2 to the choice of $-$. Doing the two null rotations, the new real null vectors ℓ'' and n'' satisfy

$$\ell_{1,2}'' = \frac{1}{4} \ell \mp \frac{1}{4} \frac{\Psi_1^{*1/4}}{\Psi_3^{1/4}} m \mp \frac{1}{4} \frac{\Psi_1^{1/4}}{\Psi_3^{*1/4}} \bar{m} + \frac{1}{4} \frac{(\Psi_1 \Psi_1^*)^{1/4}}{(\Psi_3 \Psi_3^*)^{1/4}} n \quad (\text{B7})$$

and

$$n_{1,2}'' = n \pm \frac{\Psi_3^{1/4}}{\Psi_1^{1/4}} m \pm \frac{\Psi_3^{*1/4}}{\Psi_1^{*1/4}} \bar{m} + \frac{(\Psi_3 \Psi_3^*)^{1/4}}{(\Psi_1 \Psi_1^*)^{1/4}} \ell. \quad (\text{B8})$$

Then, we find that $n_1'' = K_1 \ell_2''$ and $\ell_1'' = K_1^{-1} n_2''$, where the scale factor $K_1 = \frac{1}{4} (\Psi_1 \Psi_1^*)^{1/4} / (\Psi_3 \Psi_3^*)^{1/4}$. That is, we find that by choosing different signs for a^* we arrive at the *same* transverse null frame: we only change the roles of ℓ and n . Also, the product $\Psi_0'' \Psi_4''$ (a radiation scalar) is invariant under this change of sign, although Ψ_0'' and Ψ_4'' are separately not.

3. Finding the remaining two transverse frames

To find the remaining transverse null frames, for simplicity let us do null rotations on the frame we already found, instead of going back to the principal null frame.

(One could also do null rotations on the principal null frame, using $a^{*2} = -x_1/\Psi_1$ or $a^{*2} = -x_2/\Psi_1$ with the corresponding values for b . It is simpler, however, to find the remaining transverse null frame from the one we already found.) Specifically, let us assume that we are already in a transverse null frame, which will henceforth be denoted by unprimed quantities. Next, we do a class I null rotation with (a new) parameter a and a class II null rotation with (a new) parameter b . The composition of these two null rotations should preserve the transversality of the frame, i.e., we demand that both $\Psi_1'' = 0$ and $\Psi_3'' = 0$ simultaneously. Substituting $\Psi_1 = 0 = \Psi_3$ in Eq. (3.3), we find that the parameter

$$b = -a^* \frac{3\Psi_2 + a^{*2}\Psi_0}{\Psi_4 + 6a^{*2}\Psi_2 + a^{*4}\Psi_0} \quad (\text{B9})$$

makes $\Psi_3'' = 0$. We also find that Eq. (3.2) reduces under this situation to

$$\Psi_1'' = \frac{a^*(9\Psi_2^2 - \Psi_0\Psi_4)(\Psi_0 a^{*4} - \Psi_4)}{(\Psi_4 + 6a^{*2}\Psi_2 + a^{*4}\Psi_0)^2}. \quad (\text{B10})$$

The requirement that $\Psi_1'' = 0$ yields

$$a^{*4} = \frac{\Psi_4}{\Psi_0}. \quad (\text{B11})$$

(The case $9\Psi_2^2 = \Psi_0\Psi_4$ which also nullifies Ψ_1'' degenerates to Petrov type D space-time.) We thus find four solutions. Specifically,

$$a_{3,4}^* = \pm \left(\frac{\Psi_4}{\Psi_0} \right)^{1/4} \quad b_{3,4} = \mp \frac{1}{2} \left(\frac{\Psi_0}{\Psi_4} \right)^{1/4} \quad (\text{B12})$$

$$a_{5,6}^* = \pm i \left(\frac{\Psi_4}{\Psi_0} \right)^{1/4} \quad b_{5,6} = \pm \frac{i}{2} \left(\frac{\Psi_0}{\Psi_4} \right)^{1/4}. \quad (\text{B13})$$

The corresponding null vectors are

$$\ell_{3,4}'' = \frac{1}{4} \ell \mp \frac{1}{4} \frac{\Psi_0^{*1/4}}{\Psi_4^{1/4}} m \mp \frac{1}{4} \frac{\Psi_0^{1/4}}{\Psi_4^{*1/4}} \bar{m} + \frac{1}{4} \frac{(\Psi_0 \Psi_0^*)^{1/4}}{(\Psi_4 \Psi_4^*)^{1/4}} n \quad (\text{B14})$$

$$n_{3,4}'' = n \pm \frac{\Psi_4^{1/4}}{\Psi_0^{1/4}} m \pm \frac{\Psi_4^{*1/4}}{\Psi_0^{*1/4}} \bar{m} + \frac{(\Psi_4 \Psi_4^*)^{1/4}}{(\Psi_0 \Psi_0^*)^{1/4}} \ell \quad (\text{B15})$$

$$\ell_{5,6}'' = \frac{1}{4} \ell \mp \frac{i}{4} \frac{\Psi_0^{*1/4}}{\Psi_4^{1/4}} m \mp \frac{i}{4} \frac{\Psi_0^{1/4}}{\Psi_4^{*1/4}} \bar{m} + \frac{1}{4} \frac{(\Psi_0 \Psi_0^*)^{1/4}}{(\Psi_4 \Psi_4^*)^{1/4}} n \quad (\text{B16})$$

and

$$n_{5,6}'' = n \pm i \frac{\Psi_4^{1/2}}{\Psi_0^{1/4}} m \mp i \frac{\Psi_4^{*1/2}}{\Psi_0^{*1/4}} \bar{m} + \frac{(\Psi_4 \Psi_4^*)^{1/4}}{(\Psi_0 \Psi_0^*)^{1/4}} \ell. \quad (\text{B17})$$

Again, we find that $n_3'' = K_2 \ell_4''$, $\ell_3'' = K_2^{-1} n_4''$, $n_5'' = K_3 \ell_6''$ and $\ell_5'' = K_3^{-1} n_6''$, where $K_2 = K_3 = 4(\Psi_4 \Psi_4^*)^{1/4}/(\Psi_0 \Psi_0^*)^{1/4}$. That is, the four frames are just *two* additional distinct frames, where we interchange the roles of ℓ, n (up to a scale factor). The Weyl scalars in the new frame are

$$\begin{aligned} \Psi_0'' &= [\Psi_0 \Psi_4^3 + 6a^{*2} \Psi_4 \Psi_2 (9\Psi_2^2 + \Psi_4 \Psi_0) + a^{*4} (81\Psi_2^4 \\ &\quad - \Psi_4^2 \Psi_0^2 + 54\Psi_0 \Psi_4 \Psi_2^2) + 6a^{*6} \Psi_0 \Psi_2 (9\Psi_2^2 \\ &\quad + \Psi_4 \Psi_0) + a^{*8} \Psi_4 \Psi_0^3] / (\Psi_4 + 6a^{*2} \Psi_2 + a^{*4} \Psi_0)^3, \end{aligned} \quad (\text{B18})$$

$$\Psi_2'' = \frac{\Psi_4 \Psi_2 - 3a^{*2} \Psi_2^2 + a^{*4} \Psi_2 \Psi_0 + a^{*2} \Psi_0 \Psi_4}{\Psi_4 + 6a^{*2} \Psi_2 + a^{*4} \Psi_0}, \quad (\text{B19})$$

and

$$\Psi_4'' = \Psi_4 + 6a^{*2} \Psi_2 + a^{*4} \Psi_0. \quad (\text{B20})$$

Note, that Ψ_0'' , Ψ_2'' and Ψ_4'' are unchanged if we choose a_3^* , a_5^* or a_4^* , a_6^* , respectively, to get the two new frames, because Ψ_0'' , Ψ_2'' and Ψ_4'' are even functions of a^* . In particular, the product $\Psi_0'' \Psi_4''$ is invariant under the change of sign in a^* . On the other hand, if we change a^* by a multiplication by i , i.e. change a_3^* to a_5^* (or a_4^* to a_6^*) the Weyl scalars will in general change, because they include terms which are not quartic in a^* .

We showed that we can find all the three distinct transverse null frames for type I space-times, and in general the product $\Psi_0'' \Psi_4''$ will be different in these three transverse frames. The above analysis allows us to find all the three unique radiation scalars $\Psi_0 \Psi_4$ in all the transverse frames of type I space-times.

APPENDIX C: TRANSVERSE FRAMES FOR ALGEBRAICALLY SPECIAL SPACE-TIMES

Algebraically special space-times are not likely to arise in numerical simulation, unless sought explicitly. For completeness, we discuss in this appendix transverse frames in algebraically special space-times. A summary of the properties of transverse frames in algebraically special space-times appears in Table I.

1. Type II

We can always find a standard form frame in which only Ψ_2 and Ψ_3 are nonzero. In that frame do a class I null rotation with parameter a and a subsequent class II rotation with parameter b . Demanding that in the new frame $\Psi_3'' = 0$ implies that

$$b = -\frac{1}{2a^*} \frac{\Psi_3 + 3a^* \Psi_2}{2\Psi_3 + 3a^* \Psi_2}. \quad (\text{C1})$$

TABLE I. The number of distinct equivalency classes of transverse frames in space-times of each Petrov type and the Weyl scalars in the transverse frames in terms of the Weyl scalars of the standard forms (see text). For type I space-times we list the Weyl scalars in Appendix B. In the case of Petrov type D we emphasize that we have the singled out Kinnersley frame in addition to infinitely many non-Kinnersley frames. (The table lists the Weyl scalar only for the non-Kinnersley cases.) In this table the unprimed Weyl scalars are in the standard form frames, and the double-primed scalars are in the transverse frames (TFs).

Petrov type	No. of TFs	Ψ_0''	Ψ_1''	Ψ_2''	Ψ_3''	Ψ_4''
I	3		0		0	
D	$\infty \oplus 1$	$\frac{3}{8} \Psi_2 / a^{*2}$	0	$-\frac{1}{2} \Psi_2$	0	$6a^{*2} \Psi_2$
II	1	0	0	Ψ_2	0	$-\frac{2}{3} \Psi_3^2 / \Psi_2$
III	0	...	...	...	...	...
N	∞	0	0	0	0	Ψ_4
0	∞	0	0	0	0	0

Then, $\Psi_1'' = 0$ if either $\Psi_3 = 0$ (type D), or if

$$a^* = -\frac{\Psi_3}{2\Psi_2}. \quad (\text{C2})$$

Thus we find that there is a *unique* transverse frame (up to rotations in the $m, \bar{m}$ plane). In that frame, $\Psi_0'' = 0$, $\Psi_1'' = 0$, $\Psi_2'' = \Psi_2$, $\Psi_3'' = 0$, and $\Psi_4'' = -\frac{2}{3} \Psi_3^2 / \Psi_2$, such that $\Psi_0'' \Psi_4'' = 0$.

2. Type D

We can always find a standard form frame in which only Ψ_2 is nonzero. Notice, that this is already a transverse frame. In fact, this is the Kinnersley frame, in which the real null vectors coincide with the directions of the (repeated) principal null directions of the Weyl tensor. For any nonzero a , if we choose $b = -1/(2a^*)$, both the new Ψ_1'' and Ψ_3'' will be zero. That is, there is an *infinite number* of transverse frames. We can parametrize all these frames with a^* . In all these frames $\Psi_0'' = \frac{3}{8} \Psi_2 / a^{*2}$, $\Psi_1'' = 0$, $\Psi_2'' = -\frac{1}{2} \Psi_2$, $\Psi_3'' = 0$, and $\Psi_4'' = 6a^{*2} \Psi_2$, such that in all these frames the product $\Psi_0'' \Psi_4'' = \frac{3}{2} \Psi_2^2$ is independent of a^* . Notice that among the infinitely many transverse frames for type D space-times, there is a unique frame that is singled out, specifically, the Kinnersley frame. In the Kinnersley frame the radiation scalar vanishes, whereas in the continuum of non-Kinnersley transverse frames the radiation scalar is nonzero.

3. Type III

We can always find a standard form frame in which only Ψ_3 is nonzero. If we choose $b = -1/(4a^*)$ we can make $\Psi_3'' = 0$, but then $\Psi_1'' \neq 0$ (unless $\Psi_3 = 0$, which is type 0). Alternatively, we can choose $b = -3/(4a^*)$ which makes $\Psi_1'' = 0$, but $\Psi_3'' \neq 0$ (unless it is type 0). That is, we

cannot nullify both $\Psi_1'' = 0$ and $\Psi_3'' = 0$ simultaneously. There are *no* transverse frames for type III space-times.

4. Type N

We can always find a standard form frame in which only Ψ_4 is nonzero. Note, that this is already a transverse frame. No matter which a^* we choose, we remain in a transverse

frame. That is, there is an *infinite number* of transverse frames, in all of which $\Psi_0''\Psi_4'' = 0$.

5. Type 0

In type 0 space-times all the Weyl scalars are zero, and all null rotations will preserve this. There are *infinitely many* transverse null frames.

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