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Electroweak corrections and unitarity in linear moose models

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We calculate the form of the corrections to the electroweak interactions in the class of Higgsless models which can be deconstructed to a chain of $SU(2)$ gauge groups adjacent to a chain of $U(1)$ gauge groups, and with the fermions coupled to any single $SU(2)$ group and to any single $U(1)$ group along the chain. The primary advantage of our technique is that the size of corrections to electroweak processes can be directly related to the spectrum of vector bosons (''KK modes''). In Higgsless models, this spectrum is constrained by unitarity. Our methods also allow for arbitrary background 5D geometry, spatially dependent gauge-couplings, and brane kinetic energy terms. We find that, due to the size of corrections to electroweak processes in any *unitary* theory, Higgsless models with localized fermions are disfavored by precision electroweak data. Although we stress our results as they apply to continuum Higgsless 5D models, they apply to any linear moose model including those with only a few extra vector bosons. Our calculations of electroweak corrections also apply directly to the electroweak gauge sector of 5D theories with a bulk scalar Higgs boson; the constraints arising from unitarity do not apply in this case.

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INTRODUCTION

The mechanism that spontaneously breaks the electroweak gauge symmetry remains unknown. One novel solution to the puzzle is embodied in the ''Higgsless'' models [1], which are based on five-dimensional gauge theories compactified on an interval. These models achieve unitarity of electroweak boson self-interactions through the exchange of a tower of massive vector bosons [2–4], rather than the exchange of a scalar Higgs boson [5].

In this paper, using deconstruction [6,7], we calculate the form of the corrections to the electroweak interactions in a large class of these models in which the fermions are localized in the extra dimension. Our analysis applies to any Higgsless model which can be deconstructed to a chain of $SU(2)$ gauge groups adjacent to a chain of $U(1)$ gauge groups, with the fermions coupled to any single $SU(2)$ group and to any single $U(1)$ group along the chain.[1] The analyses presented here extend and generalize those we have presented previously [10,11].

The primary advantage of our technique is that the size of corrections to electroweak processes can be directly related to the spectrum of vector bosons (''KK modes'') which, in Higgsless models, is constrained by unitarity. In addition, our results allow for arbitrary background 5D geometry, spatially dependent gauge-couplings, and brane kinetic energy terms. We find that, due to the size of corrections to electroweak processes in any unitary theory, Higgsless models with localized fermions are disfavored by precision electroweak data.

We find that the case (which we will designate ''case I'') in which the fermions' hypercharge interactions are with the $U(1)$ group at the interface between the $SU(2)$ and $U(1)$ groups are of particular phenomenological interest. Previous results [10] for the subset of case I models in which fermions couple only to the $SU(2)$ group at the leftmost end of the chain are recovered as a special limit of our general expressions, and the results of our analysis for case I was quoted in [11]. We also examine the subset of case I models where fermions couple only to the $SU(2)$ group at the interface; this is an extension of the generalized-BESS model [12,13]. We find that in this limit, S and T take on their minimum values and the quantity $S - 4\cos^2\theta_W T$ vanishes to leading order in the ratio $M_Z^2/\mathsf{m}_{Z_z}^2$ where m_{Z_z} is the mass of the extra Z' bosons in the model.

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[1] Recently, it has been proposed that the size of corrections to electroweak processes may be reduced by allowing for delocalized fermions [8,9]. Here we restrict our attention to the case of localized fermions.

Although we stress our results as they apply to continuum Higgsless 5D models, they apply also far from the continuum limit when only a few extra vector bosons are present. As such, these results form a generalization of phenomenological analyses [12] of models of extended electroweak gauge symmetries [13–15] motivated by models of hidden local symmetry [16–20]. Our calculations also apply directly to the electroweak gauge sector of 5D theories with a bulk scalar Higgs boson, although the constraints arising from unitarity no longer apply.

The results presented here are complementary to, and more general than, the analyses of the phenomenology of these models in the continuum [21–29]. Recently, using deconstruction, Perelstein [30] has argued that the higher-order corrections expected to be present in any QCD-like "high-energy" completion of a Higgsless theory are also likely to be large. In this work, we compute the tree-level corrections expected independent of the form of the high-energy completion.

In the next three sections, we introduce the models we will analyze, set notation for the correlation functions and vector-boson mass matrices, and carefully specify the electroweak parameters we will compute. In Sec. V, we show that one correlation function may be computed quite directly in the most general model. In Secs. VI and VII, we discuss the correlation functions and phenomenology of "case I" in detail, and then generalize this analysis in Secs. VIII and IX for a moose with fermions coupled to an arbitrary $U(1)$ group. The primary results of this paper, the form of electroweak corrections in the most general moose model with localized fermions, are summarized in Sec. IX. In Sec. X, we demonstrate that the size of the electroweak corrections is related to the unitarity of the theory, and show that these models are disfavored by precision electroweak data. Following a short discussion and summary, the appendixes present various technical and notational elaborations of the discussion given in the main body of the paper.

II. THE MODEL AND ITS RELATIVES

The model we analyze, shown diagrammatically (using "moose notation" [6,31]) in Fig. 1, incorporates an $SU(2)^{N+1} \times U(1)^{M+1}$ gauge group, and $N+1$ nonlinear $(SU(2) \times SU(2))/SU(2)$ sigma models adjacent to M $(U(1) \times U(1))/U(1)$ sigma models in which the global symmetry groups in adjacent sigma models are identified with the corresponding factors of the gauge group. The Lagrangian for this model at $O(p^2)$ is given by

$$\mathcal{L}_2 = \frac{1}{4}\sum_{j=1}^{N+M+1} f_j^2 \mathrm{tr}((D_\mu U_j)^\dagger (D^\mu U_j)) - \sum_{j=0}^{N+M+1} \frac{1}{2g_j^2}\mathrm{tr}(F^j_{\mu\nu}F^{j\mu\nu}), \tag{2.1}$$

with

$$D_\mu U_j = \partial_\mu U_j - i A_\mu^{j-1} U_j + i U_j A_\mu^j, \tag{2.2}$$

where all gauge fields A_μ^j ($j = 0, 1, 2, \ldots, N+M+1$) are dynamical. The first $N+1$ gauge fields ($j = 0, 1, \ldots, N$) correspond to $SU(2)$ gauge groups; the other $M+1$ gauge fields ($j = N+1, N+2, \ldots, N+M+1$) correspond to $U(1)$ gauge groups. The symmetry breaking between the A_μ^N and A_μ^{N+1} follows an $SU(2)_L \times SU(2)_R/SU(2)_V$ symmetry breaking pattern with the $U(1)$ embedded as the T_3-generator of $SU(2)_R$. In what follows, we will denote $N+M$ by K for brevity.

The fermions in this model take their weak interactions from the $SU(2)$ group at $j = p$ and their hypercharge interactions from the $U(1)$ group with $j = q$. The neutral-current couplings to the fermions are thus written as

$$J_3^\mu A_\mu^p + J_Y^\mu A_\mu^q, \tag{2.3}$$

while the charged-current couplings arise from

$$\frac{1}{\sqrt{2}} J_\pm^\mu A_\mu^{p\mp}. \tag{2.4}$$

We allow p to assume any value $0 \le p \le N$; we consider the special "case I" mooses in which $q = N+1$ and also the more general situation in which q can take on any value in the range $N+1 \le q \le K+1$.

Our analysis proceeds for arbitrary values of the gauge-couplings and f-constants, and therefore allow for arbitrary background 5D geometry, spatially dependent gauge-

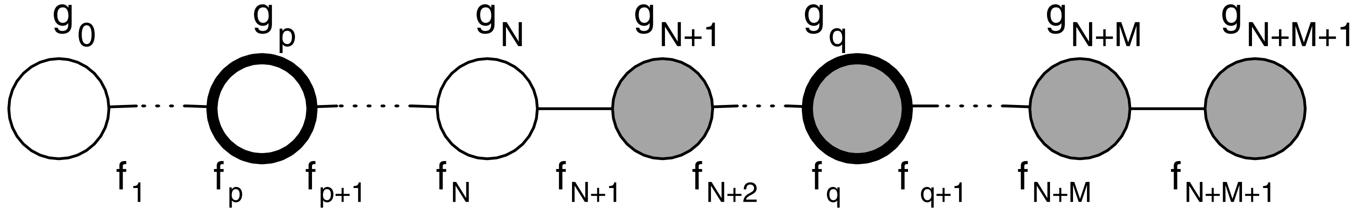

FIG. 1. Moose diagram for the class of models analyzed in this paper. $SU(2)$ gauge groups are shown as open circles; $U(1)$ gauge groups as shaded circles. The fermions couple to gauge groups p ($0 \le p \le N$) and q ($N+1 \le q \le N+M+1$). The values of the gauge-couplings g_i and f-constants f_i are arbitrary.

couplings, and brane kinetic energy terms for the gauge bosons.

III. NOTATION: PROPAGATORS, MASS-SQUARED MATRICES, AND CORRELATION FUNCTIONS

All four-fermion processes, including those relevant for the electroweak phenomenology of our model, depend on the neutral gauge field propagator matrix

$$D^Z(Q^2) \equiv [Q^2 \mathcal{I} + M_Z^2]^{-1}, \tag{3.1}$$

and the charged propagator matrix

$$D^W(Q^2) \equiv [Q^2 \mathcal{I} + M_W^2]^{-1}. \tag{3.2}$$

Here, M_Z^2 and M_W^2 are, respectively, the mass-squared matrices for the neutral and charged gauge bosons and $\mathcal{I}$ is the identity matrix. Consistent with [10], $Q^2 \equiv -q^2$ refers to the Euclidean momentum.

Recalling that fermions are charged under only a single SU(2) gauge group (at $j = p$ where $0 \leq p \leq N$) and a single U(1) group (at $j = q$ where $N+1 \leq q \leq K+1$), neutral-current four-fermion processes may be derived from the Lagrangian

$$\begin{aligned}\mathcal{L}_{nc} = &-\frac{1}{2} g_p^2 D^Z_{p,p}(Q^2) J_3^\mu J_{3\mu} - g_p g_q D^Z_{p,q}(Q^2) J_3^\mu J_{Y\mu} \\ &- \frac{1}{2} g_q^2 D^Z_{q,q}(Q^2) J_Y^\mu J_{Y\mu},\end{aligned} \tag{3.3}$$

and charged-current process from

$$\mathcal{L}_{cc} = -\frac{1}{2} g_p^2 D^W_{p,p}(Q^2) J_+^\mu J_{-\mu}, \tag{3.4}$$

where $D_{i,j}$ is the (i, j) element of the appropriate gauge field propagator matrix.

The neutral vector meson mass-squared matrix is of dimension $(K+2) \times (K+2)$

$$M_Z^2 = \frac{1}{4}\begin{pmatrix} g_0^2 f_1^2 & -g_0 g_1 f_1^2 & & & & \\ -g_0 g_1 f_1^2 & g_1^2(f_1^2 + f_2^2) & -g_1 g_2 f_2^2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -g_N g_{N+1} f_{N+1}^2 & g_{N+1}^2(f_{N+1}^2 + f_{N+2}^2) & -g_{N+1} g_{N+2} f_{N+2}^2 & \\ & & & -g_{N+1} g_{N+2} f_{N+2}^2 & g_{N+2}^2(f_{N+2}^2 + f_{N+3}^2) & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}. \tag{3.5}$$

and the charged-current vector bosons' mass-squared matrix is the upper $(N+1) \times (N+1)$ dimensional block of the neutral-current M_Z^2 matrix

$$M_W^2 = \frac{1}{4}\begin{pmatrix} g_0^2 f_1^2 & -g_0 g_1 f_1^2 & & & & \\ -g_0 g_1 f_1^2 & g_1^2(f_1^2 + f_2^2) & -g_1 g_2 f_2^2 & & & \\ & -g_1 g_2 f_2^2 & g_2^2(f_2^2 + f_3^2) & -g_2 g_3 f_3^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -g_{N-2} g_{N-1} f_{N-1}^2 & g_{N-1}^2(f_{N-1}^2 + f_N^2) & -g_{N-1} g_N f_N^2 \\ & & & & -g_{N-1} g_N f_N^2 & g_N^2(f_N^2 + f_{N+1}^2) \end{pmatrix}. \tag{3.6}$$

The neutral vector-boson mass matrix (3.5) is of a familiar form that has a vanishing determinant, due to a zero eigenvalue. Physically, this corresponds to a massless neutral gauge field—the photon. The nonzero eigenvalues of M_Z^2 are labeled by m_{Zz}^2 $(z = 0, 1, 2, \cdots, K)$, while those of M_W^2 are labeled by m_{Ww}^2 $(w = 0, 1, 2, \cdots, N)$. The lowest massive eigenstates corresponding to eigenvalues m_{Z0}^2 and m_{W0}^2 are, respectively, identified as the usual Z and W bosons. We will generally refer to these last eigenvalues by their conventional symbols M_Z^2, M_W^2; the distinction between these and the corresponding mass matrices should be clear from context.

It is useful to define an $M \times M$ matrix $\mathcal{M}_M$ which encompasses all of the $U(1)$ groups corresponding to $j > N+1$ [i.e., all except the leftmost $U(1)$ group in the linear moose].

$$\mathcal{M}_M^2 = \frac{1}{4} \times \begin{pmatrix} g_{N+2}^2(f_{N+2}^2 + f_{N+3}^2) & -g_{N+2}g_{N+3}f_{N+3}^2 & & & & \\ -g_{N+2}g_{N+3}f_{N+3}^2 & g_{N+3}^2(f_{N+3}^2 + f_{N+4}^2) & -g_{N+3}g_{N+4}f_{N+4}^2 & & & \\ & -g_{N+3}g_{N+4}f_{N+4}^2 & g_{N+4}^2(f_{N+4}^2 + f_{N+5}^2) & -g_{N+4}g_{N+5}f_{N+5}^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -g_{K-1}g_K f_K^2 & g_K^2(f_K^2 + f_{K+1}^2) & -g_K g_{K+1} f_{K+1}^2 \\ & & & & -g_K g_{K+1} f_{K+1}^2 & g_{K+1}^2 f_{K+1}^2 \end{pmatrix}, \tag{3.7}$$

(where $K \equiv N + M$). The neutral-current matrix (3.5) can be written more compactly as

$$M_Z^2 = \begin{pmatrix} M_W^2 & -g_N g_{N+1} f_{N+1}^2/4 & \\ -g_N g_{N+1} f_{N+1}^2/4 & g_{N+1}^2(f_{N+1}^2 + f_{N+2}^2)/4 & -g_{N+1}g_{N+2}f_{N+2}^2/4 \\ & -g_{N+1}g_{N+2}f_{N+2}^2/4 & \mathcal{M}_M^2 \end{pmatrix}. \tag{3.8}$$

The eigenvalues of $\mathcal{M}_M^2$, are labeled by m_m^2 ($m = 1, 2, \cdots, M$), and the associated propagator is denoted as $\mathcal{D}^M(Q^2) \equiv [Q^2 \mathcal{I} + \mathcal{M}_M^2]^{-1}$.

Generalizing the usual mathematical notation for "open" and "closed" intervals, we may denote the neutral-boson mass matrix M_Z^2 as $M^2_{[0,K+1]}$—i.e., it is the mass matrix for the entire moose running from site 0 to site $K + 1$ including the gauge-couplings of both end point groups. Analogously, the charged-boson mass matrix M_W^2 is $M^2_{[0,N+1)}$—it is the mass matrix for the moose running from site 0 to link $N + 1$, but not including the gauge coupling at site $N + 1$. This notation will be useful in thinking about the properties of the various submatrices and in keeping track of the end values of sums over eigenvalues. Using this notation, we can write $\mathcal{M}_M^2$ as

$$\mathcal{M}_M^2 = M^2_{(N+1,K+1]}. \tag{3.9}$$

We can also define other useful submatrices $\mathcal{M}_i^2$ and their eigenvalues m_i^2 as

$$\mathcal{M}_p^2 = M^2_{[0,p)} \qquad \mathsf{m}_{\hat{p}}^2 \ (\hat{p} = 1, 2, \cdots, p) \tag{3.10}$$

$$\mathcal{M}_r^2 = M^2_{(p,N+1)} \qquad \mathsf{m}_r^2 \ (r = p+1, p+2, \cdots, N) \tag{3.11}$$

$$\mathcal{M}_q^2 = M^2_{(q,K+1]} \qquad \mathsf{m}_{\hat{q}}^2 \ (\hat{q} = q+1, \cdots, K+1) \tag{3.12}$$

$$\mathcal{M}_L^2 = M^2_{(p,K+1]} \qquad \mathsf{m}_l^2 \ (l = 0, 1, \cdots, K-p). \tag{3.13}$$

The propagator matrix related to each $\mathcal{M}_i^2$ is written $\mathcal{D}^i(Q^2) \equiv [Q^2 \mathcal{I} + \mathcal{M}_i^2]^{-1}$. Note that $\mathcal{M}_M^2$ is the same as $\mathcal{M}^2_{q=N+1}$. We will show that the lowest ($l = 0$) eigenvalue of the matrix $\mathcal{M}_L^2$ is typically light (comparable to $M_{W,Z}^2$), and we will denote it M_L^2. For completeness, the matrices $\mathcal{M}^2_{p,q,r,L}$ are written out explicitly in Appendix A.

The propagator matrix elements related to M_Z^2 and M_W^2 may be written in a spectral decomposition in terms of the mass eigenstates as follows:

$$g_p^2 D^Z_{p,p}(Q^2) \equiv [G_{\rm NC}(Q^2)]_{WW} = \frac{[\xi_\gamma]_{WW}}{Q^2} + \frac{[\xi_Z]_{WW}}{Q^2 + M_Z^2} + \sum_{z=1}^{K} \frac{[\xi_{Zz}]_{WW}}{Q^2 + \mathsf{m}_{Zz}^2}, \tag{3.14}$$

$$g_p g_q D^Z_{p,q}(Q^2) \equiv [G_{\rm NC}(Q^2)]_{WY} = \frac{[\xi_\gamma]_{WY}}{Q^2} + \frac{[\xi_Z]_{WY}}{Q^2 + M_Z^2} + \sum_{z=1}^{K} \frac{[\xi_{Zz}]_{WY}}{Q^2 + \mathsf{m}_{Zz}^2}, \tag{3.15}$$

$$g_q^2 D^Z_{q,q}(Q^2) \equiv [G_{\rm NC}(Q^2)]_{YY} = \frac{[\xi_\gamma]_{YY}}{Q^2} + \frac{[\xi_Z]_{YY}}{Q^2 + M_Z^2} + \sum_{z=1}^{K} \frac{[\xi_{Zz}]_{YY}}{Q^2 + \mathsf{m}_{Zz}^2}, \tag{3.16}$$

$$g_p^2 D^W_{p,p}(Q^2) \equiv [G_{\rm CC}(Q^2)]_{WW} = \frac{[\xi_W]_{WW}}{Q^2 + M_W^2} + \sum_{w=1}^{N} \frac{[\xi_{Ww}]_{WW}}{Q^2 + \mathsf{m}_{Ww}^2}. \tag{3.17}$$

All poles should be simple (i.e., there should be no degenerate mass eigenvalues) because, in the continuum limit, we are analyzing a self-adjoint operator on a finite interval.

The pole residues ξ directly show the contributions of the various weak bosons to four-fermion processes. We can also write the couplings of the Z and W bosons in terms of the residues

$$\left(\sqrt{[\xi_Z]_{WW}}J_{3\mu} - \sqrt{[\xi_Z]_{YY}}J_{Y\mu}\right)Z^{\mu}, \tag{3.18}$$

$$\sqrt{\frac{[\xi_W]_{WW}}{2}}J^{\mp}_{\mu}W^{\pm\mu}, \tag{3.19}$$

and see how the existence of the heavy gauge bosons alters these couplings from their standard-model values.

In each case, the W and Z couplings will approach their tree-level standard-model values in the limit $\mathsf{m}^2_{Z,z>0}, \mathsf{m}^2_{W,w>0} \to \infty$. From a phenomenological point of view, therefore, it is particularly interesting to consider scenarios in which the only light gauge bosons are the photon, W, and Z, i.e., with all of the extra charged and neutral gauge bosons being much heavier. Ratios of physical masses such as M_Z^2/m^2_{Zz} will, then, be small. Phenomenological constraints will also require that the mass-squared eigenvalues m^2_m, $\mathsf{m}^2_{\hat{p}}$, m^2_r, $\mathsf{m}^2_{\hat{q}}$, and $\mathsf{m}^2_{l>0}$ (which do not directly correspond to the masses of any physical particles) be large. We will find it useful to define the following sums over the heavy masses[2] for these phenomenological discussions:

$$\Sigma_Z \equiv \sum_{z=1}^{K}\frac{1}{\mathsf{m}^2_{Zz}}, \qquad \Sigma_W \equiv \sum_{w=1}^{N}\frac{1}{\mathsf{m}^2_{Ww}}, \qquad \Sigma_M \equiv \sum_{m=1}^{M}\frac{1}{\mathsf{m}^2_{m}}, \tag{3.20}$$

$$\Sigma_p \equiv \sum_{\hat{p}=1}^{p}\frac{1}{\mathsf{m}^2_{\hat{p}}}, \qquad \Sigma_q \equiv \sum_{\hat{q}=q+1}^{K+1}\frac{1}{\mathsf{m}^2_{\hat{q}}}, \qquad \Sigma_r \equiv \sum_{r=p+1}^{N}\frac{1}{\mathsf{m}^2_{r}}, \qquad \Sigma_L \equiv \sum_{l=1}^{K-p}\frac{1}{\mathsf{m}^2_{l}}. \tag{3.21}$$

Note that for $p = 0$ we have $\Sigma_p = 0$ and likewise for $q = K + 1$ we have $\Sigma_q = 0$; this makes sense since the fermions then couple to the groups at the ends of the moose and the respective matrix $\mathcal{M}_p$ or $\mathcal{M}_q$ does not exist. When $q = N + 1$ we have $\Sigma_q = \Sigma_M$. Thus we expect that setting $\Sigma_p = 0$, $\Sigma_q = \Sigma_M$ the results of our general analysis will recover the results in Ref. [10].

Since the neutral bosons couple to only two currents, J_3^{μ} and J_Y^{μ}, the three sets of residues in Eqs. (3.14), (3.15), and (3.16) must be related. Specifically, they satisfy the $K + 1$ consistency conditions,

$$[\xi_Z]_{WW}[\xi_Z]_{YY} = ([\xi_Z]_{WY})^2, \qquad [\xi_{Zz}]_{WW}[\xi_{Zz}]_{YY} = ([\xi_{Zz}]_{WY})^2. \tag{3.22}$$

[2]Note that the sums in Σ_Z, Σ_W and Σ_L start, respectively, from $z = 1$, $w = 1$, and $l = 1$, excluding the lightest nonzero mass eigenvalue in each case.

In the case of the photon, charge universality further implies

$$e^2 = [\xi_\gamma]_{WW} = [\xi_\gamma]_{WY} = [\xi_\gamma]_{YY}. \tag{3.23}$$

IV. ELECTROWEAK PARAMETERS

Our goal is to analyze four-fermion electroweak processes in the general linear moose model. As we have shown in [11], the most general amplitude for low-energy four-fermion neutral weak current processes in any "universal" model [28] may be written as[3]

$$\begin{aligned}-\mathcal{A}_{NC} = {}& e^2\frac{\mathcal{Q}\mathcal{Q}'}{Q^2} + \frac{(I_3 - s^2\mathcal{Q})(I_3' - s^2\mathcal{Q}')}{(\frac{s^2c^2}{e^2} - \frac{S}{16\pi})Q^2 + \frac{1}{4\sqrt{2}G_F}(1 + \frac{\alpha\delta}{4s^2c^2} - \alpha T)} \\ &+ \sqrt{2}G_F\frac{\alpha\delta}{s^2c^2}I_3I_3' + 4\sqrt{2}G_F(\Delta\rho - \alpha T) \\ &\times(\mathcal{Q} - I_3)(\mathcal{Q}' - I_3'),\end{aligned} \tag{4.1}$$

and the matrix element for charged currents by

$$-\mathcal{A}_{\rm CC} = \frac{(I_+I_-' + I_-I_+')/2}{(\frac{s^2}{e^2} - \frac{S}{16\pi})Q^2 + \frac{1}{4\sqrt{2}G_F}(1 + \frac{\alpha\delta}{4s^2c^2})} + \sqrt{2}G_F\frac{\alpha\delta}{s^2c^2}\frac{(I_+I_-' + I_-I_+')}{2}. \tag{4.2}$$

Here $I_a^{(\prime)}$ and $\mathcal{Q}^{(\prime)}$ are weak isospin and charge of the corresponding fermion, $\alpha = e^2/4\pi$, G_F is the usual Fermi constant, and the weak mixing angle, as defined by the on-shell Z coupling, is denoted by s^2 ($c^2 \equiv 1 - s^2$).

We can read off the forms of the Z-pole and W-pole residues in the four-fermion amplitudes (4.1) and (4.2) in terms of the parameters S, T, and δ. For example,

$$[\xi_Z]_{WY} = -e^2\left[1 + \frac{\alpha S}{4s^2c^2}\right]. \tag{4.3}$$

By comparing the residues as written in this way with the expressions we will derive for them in terms of the gauge-boson spectrum in Secs. V, VI, and VIII, we will be able to solve for the values of S, T, $\Delta\rho$ and δ in terms of the Σ_i.

In order to make contact with experiment, we will need to use the more precisely measured G_F as an input instead of M_W. In this language, the standard-model weak mixing angle s_Z is defined as

$$c_Z^2(1 - c_Z^2) \equiv \frac{\pi\alpha}{\sqrt{2}G_FM_Z^2}, \qquad s_Z^2 \equiv 1 - c_Z^2. \tag{4.4}$$

This relationship is altered by the nonstandard elements of

[3]See [11] for a discussion of the correspondence between the "on-shell" parameters defined here, and the zero-momentum parameters defined in [28]. Note that U is shown in [11] to be zero to the order we consider in this paper.

our model; thus the weak mixing angle appearing in the amplitude for four-fermion processes is shifted from the standard-model value by an amount Δ_Z

$$c^2 = c_Z^2 + \Delta_Z. \tag{4.5}$$

The size of M_W relative to M_Z will likewise be altered from its standard-model value.

Deducing the value of M_Z^2 from the location of the $Q^2 = -M_Z^2$ pole of (4.1), and comparing this with Eqs. (4.4) and (4.5) yields the shift in the weak mixing angle:

$$\Delta_Z = \frac{\alpha}{c_Z^2 - s_Z^2}\left[-\frac{1}{4}(S+\delta) + s_Z^2 c_Z^2 T\right]. \tag{4.6}$$

Taking M_W^2 from (4.2) and incorporating the shift (4.5) in the weak mixing angle gives

$$M_W^2 = c_Z^2 M_Z^2\left[1 + \frac{\alpha}{c_Z^2 - s_Z^2}\left(-\frac{1}{2}S + c_Z^2 T - \frac{\delta}{4c_Z^2}\right)\right]. \tag{4.7}$$

We can now rewrite the residues in a consistent language in terms of α, G_F and M_Z. We use (4.5) and (4.6) to rewrite the residues as deduced from (4.1) and (4.2) in terms of c_Z:

$$\frac{1}{e^2}[\xi_Z]_{WW} = \frac{c_Z^2}{s_Z^2}\left[1 + \frac{1}{c_Z^2 s_Z^2}\left(\Delta_Z + \frac{\alpha S}{4}\right)\right] \tag{4.8}$$

$$\frac{1}{e^2}[\xi_Z]_{WY} = -1 - \frac{\alpha}{4 s_Z^2 c_Z^2} S. \tag{4.9}$$

$$\frac{1}{e^2}[\xi_Z]_{YY} = \frac{s_Z^2}{c_Z^2}\left[1 + \frac{1}{c_Z^2 s_Z^2}\left(-\Delta_Z + \frac{\alpha S}{4}\right)\right] \tag{4.10}$$

$$\frac{1}{e^2}[\xi_W]_{WW} = \frac{1}{s_Z^2}\left[1 + \frac{1}{s_Z^2}\left(\Delta_Z + \frac{\alpha S}{4}\right)\right] \tag{4.11}$$

and with Δ_Z as defined by Eq. (4.5).

Before calculating explicit expressions for all of the pole residues in terms of the sums over mass eigenvalues Σ_i, we can extract a few more pieces of information from the form of the weak amplitudes (4.1) and (4.2). Evaluating the matrix elements at $Q^2 = 0$, we find

$$[G_{CC}(0)]_{WW} = 4\sqrt{2}G_F = \frac{[\xi_W]_{WW}}{M_W^2} + \sum_{w=1}^{N}\frac{[\xi_{Ww}]_{WW}}{\mathsf{m}_{Ww}^2}, \tag{4.12}$$

and by removing the pole terms corresponding to the photon, W, and Z, we find

$$4\sqrt{2}G_F(\Delta\rho - \alpha T) = \sum_{z=1}^{K}\frac{[\xi_{Zz}]_{YY}}{\mathsf{m}_{Zz}^2}, \tag{4.13}$$

$$\sqrt{2}G_F\frac{\alpha\delta}{s^2c^2} = \sum_{w=1}^{N}\frac{[\xi_{Ww}]_{WW}}{\mathsf{m}_{Ww}^2} = \sum_{z=1}^{K}\frac{[\xi_{Zz}]_{WW}}{\mathsf{m}_{Zz}^2}. \tag{4.14}$$

Note also that for the amplitudes in Eqs. (4.1) and (4.2) to be consistent, we must have

$$\sum_{z=1}^{K}\frac{[\xi_{Zz}]_{WY}}{\mathsf{m}_{Zz}^2} = 0 \tag{4.15}$$

to this order. We will apply these findings in Secs. VI and VIII.

V. CORRELATION FUNCTION $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{WY}}$ IN A GENERAL LINEAR MOOSE

In this section, we will find $[G_{NC}(Q^2)]_{WY}$ for the general case with both p and q left arbitrary. We will discuss the other correlation functions and their phenomenology in subsequent sections.

We start with the weak-hypercharge interference term in Eq. (3.3). Direct calculation of the matrix inverse involved in $D^Z_{p,q}(Q^2)$ involves the computation of the cofactor related to the (p, q) element of the matrix $Q^2\mathcal{I} + M_Z^2$. Inspection of the neutral-boson mass-matrix equation (3.5) and subsequent definitions shows that the cofactor is the determinant of a matrix whose upper $p \times p$ block involves $\mathcal{M}_p^2$, and whose lower $(K - q + 1) \times (K - q + 1)$ block involves $\mathcal{M}_q^2$. The middle block is upper-diagonal and has diagonal entries independent of Q^2 and its determinant is therefore constant. Accordingly, we have

$$\begin{aligned}[G_{NC}(Q^2)]_{WY} &= \frac{C\det[\mathcal{D}^p(Q^2)]^{-1}\det[\mathcal{D}^q(Q^2)]^{-1}}{\det[D^Z(Q^2)]^{-1}} \\ &= \frac{C\prod_{\hat{p}=1}^{p}(Q^2 + \mathsf{m}_{\hat{p}}^2)\prod_{\hat{q}=q+1}^{K+1}(Q^2 + \mathsf{m}_{\hat{q}}^2)}{Q^2(Q^2 + M_Z^2)\prod_{z=1}^{K}(Q^2 + \mathsf{m}_{Zz}^2)},\end{aligned} \tag{5.1}$$

where C is a constant. Requiring the residue of the photon pole at $Q^2 = 0$ to equal e^2 determines the value of C and reveals

$$\begin{aligned}[G_{NC}(Q^2)]_{WY} = \frac{e^2 M_Z^2}{Q^2(Q^2 + M_Z^2)}\left[\prod_{z=1}^{K}\frac{\mathsf{m}_{Zz}^2}{Q^2 + \mathsf{m}_{Zz}^2}\right] \\ \times\left[\prod_{\hat{p}=1}^{p}\frac{Q^2 + \mathsf{m}_{\hat{p}}^2}{\mathsf{m}_{\hat{p}}^2}\right]\left[\prod_{\hat{q}=q+1}^{K+1}\frac{Q^2 + \mathsf{m}_{\hat{q}}^2}{\mathsf{m}_{\hat{q}}^2}\right].\end{aligned} \tag{5.2}$$

We can read off the Z-pole of the correlation function to find

$$\begin{aligned}[\xi_Z]_{WY} = -e^2\left[\prod_{z=1}^{K}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - M_Z^2}\right]\left[\prod_{\hat{p}=1}^{p}\frac{\mathsf{m}_{\hat{p}}^2 - M_Z^2}{\mathsf{m}_{\hat{p}}^2}\right] \\ \times\left[\prod_{\hat{q}=q+1}^{K+1}\frac{\mathsf{m}_{\hat{q}}^2 - M_Z^2}{\mathsf{m}_{\hat{q}}^2}\right].\end{aligned} \tag{5.3}$$

In the limit $\mathsf{m}_{Z,z>0}, \mathsf{m}_{\hat{p}}, \mathsf{m}_{\hat{q}} \to \infty$, $[\xi_Z]_{WY} \to -e^2$ as at tree-level in the standard model. Expanding the products in Eq. (5.3) to first nontrivial order in the ratio of M_Z^2/M^2 (where M is any of $[\mathsf{m}_{Z,z>0}, \mathsf{m}_p, \mathsf{m}_q]$), we can rewrite this in shorthand as

$$[\xi_Z]_{WY} = -e^2[1 + M_Z^2(\Sigma_Z - \Sigma_p - \Sigma_q)], \qquad (5.4)$$

where the Σ_i are defined in (3.20) and (3.21). If we set $p = 0$ and $q = N + 1$ we appropriately recover the leading-order value of $[\xi_Z]_{WY}$ in [10]. From Eq. (4.9), this immediately leads to a value for the S parameter

$$\alpha S = 4s_Z^2 c_Z^2 M_Z^2(\Sigma_Z - \Sigma_p - \Sigma_q). \qquad (5.5)$$

We can also read off the heavy Z pole residues

$$[\xi_{Zk}]_{WY} = e^2 \frac{M_Z^2}{\mathsf{m}_{Zk}^2 - M_Z^2}\left[\prod_{z\neq k}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - \mathsf{m}_{Zk}^2}\right] \times\left[\prod_{\hat{p}=1}^{p}\frac{\mathsf{m}_{\hat{p}}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{\hat{p}}^2}\right]\left[\prod_{\hat{q}=q+1}^{K+1}\frac{\mathsf{m}_{\hat{q}}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{\hat{q}}^2}\right]. \qquad (5.6)$$

Heavy boson exchange could make significant contributions to the four-fermion interaction processes between a weak current and a hypercharge current only if these residues were appreciable. Note that $[\xi_{Zk}]_{WY}$ is, in fact, of order $\mathcal{O}(M_{Z,W}^2/\mathsf{m}_{Zz,Ww}^2)$, and therefore Eq. (4.15) holds to this order. We will gather the heavy pole residues in Appendix C, since they will not be part of the main argument of the paper.

VI. CASE I: CORRELATION FUNCTIONS AND CONSISTENCY RELATIONS

In this section, we consider the phenomenologically important special case (dubbed case I) of the general linear moose in which p is left free but q is fixed as $q = N + 1$. We start by writing $[G_{\rm NC}(Q^2)]_{WY}$ for case I using Eq. (5.2) above. We then note that $[G_{\rm NC}(Q^2)]_{YY}$ is the same in case I as in the model of Ref. [10] because the latter model also has $q = N + 1$ (this correlation function depends on q but not on p). We next calculate $[G_{\rm NC}(Q^2)]_{WW}$ and $[G_{\rm CC}(Q^2)]_{WW}$ directly in terms of eigenvalues of submooses of the full linear moose. For comparison we then use $[\xi_Z]_{YY}$, $[\xi_Z]_{WY}$ and the consistency conditions (3.22) to deduce another way of writing $[G_{NC}(Q^2)]_{WW} - [G_{\rm CC}(Q^2)]_{WW}$ for case I. This allows us to derive a relationship among case I correlation functions

$$[G_{\rm NC}(Q^2) - G_{\rm CC}(Q^2)]_{WW}[G_{\rm NC}(Q^2)]_{YY} = [G_{\rm NC}(Q^2)]_{WY}^2 \qquad (6.1)$$

that echoes the consistency conditions (3.22) for the pole residues. This relationship was also noted in the specific case I model discussed in [10].

A. $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{WY}}$

This correlation function is as computed in Eq. (5.1), with q taking the value $N + 1$. We immediately see that Eq. (5.4) takes the form

$$[\xi_Z]_{WY} = -e^2[1 + M_Z^2(\Sigma_Z - \Sigma_p - \Sigma_M)]. \qquad (6.2)$$

B. $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{YY}}$

Since this correlation function depends on q but not p, it is the same as in [10]. Direct calculation of the matrix inverse involved in $D_{q,q}^Z(Q^2)$ involves the computation of the cofactor related to the $(N + 1, N + 1)$ element of the matrix $[Q^2\mathcal{I} + M_Z^2]$. Inspection of the neutral-boson mass-matrix equation (3.8) leads to

$$[D^Z(Q^2)]_{N+1,N+1} = \frac{\det[D^W(Q^2)]^{-1}\det[\mathcal{D}^M(Q^2)]^{-1}}{\det[D^Z(Q^2)]^{-1}} = \frac{(Q^2 + M_W^2)\prod_{w=1}^N(Q^2 + \mathsf{m}_{Ww}^2)\prod_{m=1}^M(Q^2 + \mathsf{m}_m^2)}{Q^2(Q^2 + M_Z^2)\prod_{z=1}^K(Q^2 + \mathsf{m}_{Zz}^2).}. \qquad (6.3)$$

As noted earlier, charge universality for the photon tells us that the residue of $[G_{\rm NC}(Q^2)]_{YY}$ at $Q^2 = 0$ is e^2, so that

$$[G_{\rm NC}(Q^2)]_{YY} = \frac{e^2}{Q^2}\frac{[Q^2 + M_W^2]M_Z^2}{M_W^2[Q^2 + M_Z^2]}\left[\prod_{w=1}^N\frac{Q^2 + \mathsf{m}_{Ww}^2}{\mathsf{m}_{Ww}^2}\right]\left[\prod_{z=1}^K\frac{\mathsf{m}_{Zz}^2}{Q^2 + \mathsf{m}_{Zz}^2}\right]\left[\prod_{m=1}^M\frac{Q^2 + \mathsf{m}_m^2}{\mathsf{m}_m^2}\right]. \qquad (6.4)$$

Reading off the residue of the Z-pole in Eq. (6.4), we obtain

$$[\xi_Z]_{YY} = e^2\frac{M_Z^2 - M_W^2}{M_W^2}\left[\prod_{w=1}^N\frac{\mathsf{m}_{Ww}^2 - M_Z^2}{\mathsf{m}_{Ww}^2}\right]\left[\prod_{z=1}^K\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - M_Z^2}\right]\left[\prod_{m=1}^M\frac{\mathsf{m}_m^2 - M_Z^2}{\mathsf{m}_m^2}\right]. \qquad (6.5)$$

Note that, in the limit $\mathsf{m}_{Z,z>0}, \mathsf{m}_{W,w>0}, \mathsf{m}_m \to \infty$, $[\xi_Z]_{YY} \to e^2(M_Z^2 - M_W^2)/M_W^2$, so that the correct standard-model tree-level value of the Z-boson coupling to hypercharge is recovered. If we expand the expression for $[\xi_Z]_{YY}$ to first nontrivial order, we can rewrite it as

$$[\xi_Z]_{YY} = \frac{e^2(M_Z^2 - M_W^2)}{M_W^2}[1 + M_Z^2(\Sigma_Z - \Sigma_W - \Sigma_M)]. \qquad (6.6)$$

C. $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{WW}}$ and $[\mathbf{G_{CC}(Q^2)}]_{\mathbf{WW}}$

In terms of the matrices defined previously, we may calculate the neutral-current correlation function directly (in terms of the relevant cofactors) as

$$[G_{\rm NC}(Q^2)]_{WW} = \frac{e^2 M_Z^2}{Q^2[Q^2 + M_Z^2]} \frac{[Q^2 + M_L^2]}{M_L^2} \left[\prod_{z=1}^{K} \frac{\mathsf{m}_{Zz}^2}{Q^2 + \mathsf{m}_{Zz}^2}\right] \left[\prod_{\hat{p}=1}^{p} \frac{Q^2 + \mathsf{m}_{\hat{p}}^2}{\mathsf{m}_{\hat{p}}^2}\right] \left[\prod_{l=1}^{K-p} \frac{Q^2 + \mathsf{m}_l^2}{\mathsf{m}_l^2}\right], \tag{6.7}$$

where we have displayed the contribution corresponding to the light eigenvalue M_L^2 of the matrix $\mathcal{M}_L^2$ explicitly, and charge universality determines the value at $Q^2 = 0$. Similarly, the charged-current correlation function may be written as

$$[G_{\rm CC}(Q^2)]_{WW} = \frac{4\sqrt{2} G_F M_W^2}{[Q^2 + M_W^2]} \left[\prod_{w=1}^{N} \frac{\mathsf{m}_{Ww}^2}{Q^2 + \mathsf{m}_{Ww}^2}\right] \left[\prod_{\hat{p}=1}^{p} \frac{Q^2 + \mathsf{m}_{\hat{p}}^2}{\mathsf{m}_{\hat{p}}^2}\right] \left[\prod_{r=p+1}^{N} \frac{Q^2 + \mathsf{m}_r^2}{\mathsf{m}_r^2}\right], \tag{6.8}$$

where we have imposed the relation $[G_{CC}(0)]_{WW} = 4\sqrt{2} G_F$.

Reading off the residues of the poles at M_Z^2 and M_W^2, we find

$$[\xi_W]_{WW} = 4\sqrt{2} G_F M_W^2 \left[\prod_{w=1}^{N} \frac{\mathsf{m}_{Ww}^2}{\mathsf{m}_{Ww}^2 - M_W^2}\right] \left[\prod_{\hat{p}=1}^{p} \frac{\mathsf{m}_{\hat{p}}^2 - M_W^2}{\mathsf{m}_{\hat{p}}^2}\right] \left[\prod_{r=p+1}^{N} \frac{\mathsf{m}_r^2 - M_W^2}{\mathsf{m}_r^2}\right], \tag{6.9}$$

$$[\xi_Z]_{WW} = -e^2 \frac{[M_L^2 - M_Z^2]}{M_L^2} \left[\prod_{z=1}^{K} \frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - M_Z^2}\right] \left[\prod_{\hat{p}=1}^{p} \frac{\mathsf{m}_{\hat{p}}^2 - M_Z^2}{\mathsf{m}_{\hat{p}}^2}\right] \left[\prod_{l=1}^{K-p} \frac{\mathsf{m}_l^2 - M_Z^2}{\mathsf{m}_l^2}\right]. \tag{6.10}$$

Expanding $[\xi_{W,Z}]_{WW}$ to lowest nontrivial order, we find

$$[\xi_W]_{WW} = 4\sqrt{2} G_F M_W^2 (1 + M_W^2(\Sigma_W - \Sigma_p - \Sigma_r)), \tag{6.11}$$

$$[\xi_Z]_{WW} = -e^2 \left[\frac{M_L^2 - M_Z^2}{M_L^2}\right] (1 + M_Z^2(\Sigma_Z - \Sigma_p - \Sigma_L)). \tag{6.12}$$

D. $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{WW}} - [\mathbf{G_{CC}(Q^2)}]_{\mathbf{WW}}$: Consistency relations

Next we turn to examining the difference

$$[G_{\rm NC}(Q^2)]_{WW} - [G_{\rm CC}(Q^2)]_{WW}. \tag{6.13}$$

In the $g_{N+1} \to 0$ limit, the matrix M_Z^2 in (3.8) becomes block-diagonal and the $U(1)$-sector fully decouples, leading to a vanishing difference in Eq. (6.13). As the custodial violation $g_{N+1} \neq 0$ originates from the $(N + 1)$th site of the moose chain, at high-Q^2 the asymptotic behavior of (6.13) arises from the diagram shown in Fig. 2 (cf. [10])

$$[G_{\rm NC}(Q^2)]_{WW} - [G_{\rm CC}(Q^2)]_{WW} \propto \frac{g_{N+1}^2}{(Q^2)^{2(N-p)+3}}. \tag{6.14}$$

Using Eqs. (3.14) and (3.17), we see that the only consistent way to achieve the high-energy behavior (6.14) is

$$[G_{\rm NC}(Q^2)]_{WW} - [G_{\rm CC}(Q^2)]_{WW} = \frac{e^2 M_W^2 M_Z^2 \mathcal{R}(Q^2)}{Q^2[Q^2 + M_W^2][Q^2 + M_Z^2]} \left[\prod_{w=1}^{N} \frac{\mathsf{m}_{Ww}^2}{Q^2 + \mathsf{m}_{Ww}^2}\right] \left[\prod_{z=1}^{K} \frac{\mathsf{m}_{Zz}^2}{Q^2 + \mathsf{m}_{Zz}^2}\right], \tag{6.15}$$

where the function $\mathcal{R}(Q^2)$ is a polynomial in Q^2 of order $M + 2p$ and $\mathcal{R}(0) = 1$ in order to have (6.15) satisfy charge universality.

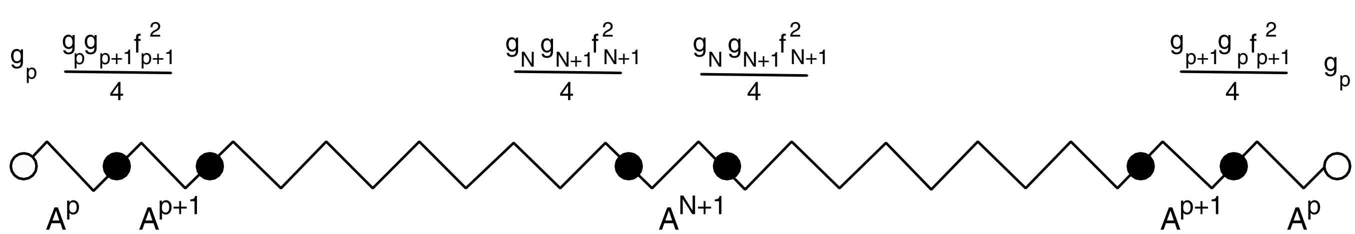

FIG. 2. Leading diagram at high-Q^2 ($Q^2 \gg \mathsf{m}_{Zz,Ww}^2$ for $w, z > 0$) which distinguishes $[G_{NC}(Q^2)]_{WW}$ from $[G_{CC}(Q^2)]_{WW}$.

From Eqs. (6.7) and (6.8), we see that the polynomial $\mathcal{R}(Q^2)$ must vanish at $Q^2 = -\mathsf{m}_{\hat{p}}^2$, yielding p conditions. Furthermore, the residues $[\xi_Z]_{WW}$ and $[\xi_{Zk}]_{WW}$ must satisfy the $K + 1$ consistency conditions of Eq. (3.22). Together, these yield $K + p + 1$ conditions, which is greater than the $M + 2p - 1$ free parameters in the polynomial $\mathcal{R}(Q^2)$. Therefore the polynomial $\mathcal{R}(Q^2)$ is (over)-constrained by these conditions. Motivated by the relation Eq. (6.1) found in [10], and recalling that $\mathcal{M}_q^2 \equiv \mathcal{M}_M^2$ in this case, we consider the ansatz

$$\mathcal{R}(Q^2) = \left[\prod_{m=1}^{M} \frac{Q^2 + \mathsf{m}_m^2}{\mathsf{m}_m^2}\right]\left[\prod_{\hat{p}=1}^{p} \frac{Q^2 + \mathsf{m}_{\hat{p}}^2}{\mathsf{m}_{\hat{p}}^2}\right]^2, \tag{6.16}$$

and find that the corresponding difference of correlation functions

$$[G_{\mathrm{NC}}(Q^2)]_{WW} - [G_{\mathrm{CC}}(Q^2)]_{WW} = \frac{e^2 M_W^2 M_Z^2}{Q^2[Q^2 + M_W^2][Q^2 + M_Z^2]}\left[\prod_{w=1}^{N} \frac{\mathsf{m}_{Ww}^2}{Q^2 + \mathsf{m}_{Ww}^2}\right]\left[\prod_{z=1}^{K} \frac{\mathsf{m}_{Zz}^2}{Q^2 + \mathsf{m}_{Zz}^2}\right] \times \left[\prod_{m=1}^{M} \frac{Q^2 + \mathsf{m}_m^2}{\mathsf{m}_m^2}\right]\left[\prod_{\hat{p}=1}^{p} \frac{(Q^2 + \mathsf{m}_{\hat{p}}^2)^2}{\mathsf{m}_{\hat{p}}^4}\right], \tag{6.17}$$

satisfies all of the consistency conditions. Therefore our ansatz for $\mathcal{R}(Q^2)$ is correct. Explicit examination of Eqs. (5.2), (6.4), and (6.17) reveals the promised relationship (6.1).

The pole residues are

$$[\xi_W]_{WW} = \frac{e^2 M_Z^2}{M_Z^2 - M_W^2}\left[\prod_{w=1}^{N} \frac{\mathsf{m}_{Ww}^2}{\mathsf{m}_{Ww}^2 - M_W^2}\right]\left[\prod_{z=1}^{K} \frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - M_W^2}\right]\left[\prod_{m=1}^{M} \frac{\mathsf{m}_m^2 - M_W^2}{\mathsf{m}_m^2}\right]\left[\prod_{\hat{p}=1}^{p} \frac{(\mathsf{m}_{\hat{p}}^2 - M_W^2)^2}{\mathsf{m}_{\hat{p}}^4}\right], \tag{6.18}$$

$$[\xi_Z]_{WW} = \frac{e^2 M_W^2}{M_Z^2 - M_W^2}\left[\prod_{w=1}^{N} \frac{\mathsf{m}_{Ww}^2}{\mathsf{m}_{Ww}^2 - M_Z^2}\right]\left[\prod_{z=1}^{K} \frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - M_Z^2}\right]\left[\prod_{m=1}^{M} \frac{\mathsf{m}_m^2 - M_Z^2}{\mathsf{m}_m^2}\right]\left[\prod_{\hat{p}=1}^{p} \frac{(\mathsf{m}_{\hat{p}}^2 - M_Z^2)^2}{\mathsf{m}_{\hat{p}}^4}\right], \tag{6.19}$$

If we expand the expressions for $[\xi_W]_{WW}$ and $[\xi_Z]_{WW}$ to first nontrivial order, we can rewrite them as

$$[\xi_W]_{WW} = \frac{e^2 M_Z^2}{M_Z^2 - M_W^2}[1 + M_W^2(\Sigma_Z + \Sigma_W - \Sigma_M - 2\Sigma_p)], \tag{6.20}$$

$$[\xi_Z]_{WW} = \frac{e^2 M_W^2}{M_Z^2 - M_W^2}[1 + M_Z^2(\Sigma_Z + \Sigma_W - \Sigma_M - 2\Sigma_p)]. \tag{6.21}$$

If we set $p = 0$ we recover our leading-order values from [10].

Comparing Eqs. (6.11) and (6.20), we find the relation

$$\frac{e^2}{4\sqrt{2}G_F} = M_W^2\left(1 - \frac{M_W^2}{M_Z^2}\right)[1 - M_W^2(\Sigma_Z + \Sigma_r - \Sigma_p - \Sigma_M)], \tag{6.22}$$

to this order. By comparing Eqs. (6.12) and (6.21), we find

$$M_L^2 = (M_Z^2 - M_W^2)[1 - M_W^2(\Sigma_W + \Sigma_L - \Sigma_M - \Sigma_p)], \tag{6.23}$$

which demonstrates our assertion that the matrix $\mathcal{M}_L^2$ has a light eigenvalue.

Using Eqs. (5.2), (6.4), (6.7), and (6.8), we see that the consistency relation of Eq. (6.1) can be written schematically as

$$\prod_{w,l}\left(1 + \frac{Q^2}{\mathsf{m}_{Ww}^2}\right)\left(1 + \frac{Q^2}{\mathsf{m}_l^2}\right) = \frac{\sqrt{2}G_F Q^2}{e^2}\prod_{r,z}\left(1 + \frac{Q^2}{\mathsf{m}_r^2}\right) \times \left(1 + \frac{Q^2}{\mathsf{m}_{Zz}^2}\right) + \prod_{\hat{p},\hat{q}}\left(1 + \frac{Q^2}{\mathsf{m}_{\hat{p}}^2}\right) \times \left(1 + \frac{Q^2}{\mathsf{m}_{\hat{q}}^2}\right), \tag{6.24}$$

where the products run over all the corresponding (nonzero) eigenvalues, including the light eigenvalues $\mathsf{m}_{Z0}^2 \equiv M_Z^2$, $\mathsf{m}_{W0}^2 \equiv M_W^2$, and $\mathsf{m}_{l=0}^2 \equiv M_L^2$. In principle this expression gives us many relations among the various eigenvalues. In practice, however, we are only interested in the consequences at momenta Q^2 much less than any of the heavy eigenvalues. Expanding Eq. (6.24) to lowest nontrivial order, we find

$$\left(1+\frac{Q^2}{M_L^2}\right)\left(1+\frac{Q^2}{M_W^2}\right)(1+Q^2\Sigma_L+Q^2\Sigma_W) = \frac{4\sqrt{2}G_F Q^2}{e^2}\left(1+\frac{Q^2}{M_Z^2}\right)(1+Q^2\Sigma_r+Q^2\Sigma_Z) + (1+Q^2\Sigma_p+Q^2\Sigma_q), \quad (6.25)$$

where we have explicitly kept all terms involving only light masses. Equating the terms proportional to Q^6, Q^4, and Q^2 (consistently to this order in the heavy mass expansion) we find

$$\Sigma_L+\Sigma_W=\Sigma_r+\Sigma_Z, \quad (6.26)$$

and reproduce Eqs. (6.22) and (6.23).

VII. CASE I: ELECTROWEAK PHENOMENOLOGY

A. $\Delta\rho$

Because case I models satisfy the consistency relation of Eq. (6.1), the low-energy ρ parameter will always equal 1 in this class of models—just as was found for the specific case I model of [10].

In discussing low-energy interactions it is conventional to rewrite the neutral-current Lagrangian of Eq. (3.3) in terms of weak and electromagnetic currents as

$$\mathcal{L}_{nc} = -\frac{1}{2}A(Q^2)J_3^\mu J_{3\mu} - B(Q^2)J_3^\mu J_{Q\mu} - \frac{1}{2}C(Q^2)J_Q^\mu J_{Q\mu}, \quad (7.1)$$

where

$$A(Q^2)=[G_{\rm NC}(Q^2)]_{WW}-2[G_{\rm NC}(Q^2)]_{WY}+[G_{\rm NC}(Q^2)]_{YY}, \quad (7.2)$$

$$B(Q^2)=[G_{\rm NC}(Q^2)]_{WY}-[G_{\rm NC}(Q^2)]_{YY}, \quad (7.3)$$

$$C(Q^2)=[G_{\rm NC}(Q^2)]_{YY}. \quad (7.4)$$

As required, the photon pole cancels in the expressions for $A(Q^2)$ and $B(Q^2)$. The low-energy ρ parameter is defined by

$$\rho = \lim_{Q^2\to 0}\frac{A(Q^2)}{[G_{\rm CC}(Q^2)]_{WW}}. \quad (7.5)$$

Consider

$$\frac{A(Q^2)}{[G_{\rm CC}(Q^2)]_{WW}}-1 = \frac{([G_{\rm NC}(Q^2)]_{WY}-[G_{\rm NC}(Q^2)]_{YY})^2}{[G_{\rm NC}(Q^2)]_{YY}[G_{\rm CC}(Q^2)]_{WW}}, \quad (7.6)$$

where the equality holds because of Eq. (6.1). As $Q^2\to 0$ in this expression, charge universality insures that the photon pole contribution cancels in the numerator, while the denominator diverges like e^2/Q^2. Therefore,

$$\Delta\rho = \lim_{Q^2\to 0}\frac{A(Q^2)}{[G_{\rm CC}(Q^2)]_{WW}} - 1 \equiv 0. \quad (7.7)$$

B. αS, αT, and $\alpha\delta$

To extract the on-shell electroweak parameters from our expressions for the residues of the poles in the correlations functions, we must first shift to a scheme where G_F is used as an input instead of M_W by applying Eq. (4.7) to Eqs. (6.2), (6.6), (6.20), and (6.21). Then comparing $[\xi_Z]_{YY}$, $[\xi_Z]_{WY}$, $[\xi_Z]_{WW}$, and $[\xi_W]_{WW}$ written in terms of the Σ_i to the forms derived earlier as (4.8), (4.9), (4.10), and (4.11), we can solve for S and T to leading order:

$$\alpha S = 4s_Z^2c_Z^2M_Z^2(\Sigma_Z-\Sigma_p-\Sigma_M), \quad (7.8)$$

$$\alpha T = s_Z^2M_Z^2(\Sigma_Z-\Sigma_W-\Sigma_M). \quad (7.9)$$

From Eqs. (4.12) and (4.14) we find

$$4\sqrt{2}G_F = \frac{[\xi_W]_{WW}}{M_W^2}+\sqrt{2}G_F\frac{\alpha\delta}{s^2c^2}. \quad (7.10)$$

Applying (6.11) to the right-hand side of (7.10) we find

$$\alpha\delta = -4s_Z^2c_Z^4M_Z^2(\Sigma_W-\Sigma_r-\Sigma_p), \quad (7.11)$$

where we have substituted $c_Z^2M_Z^2$ for M_W^2 and s_Z^2 for s^2 to this order since the Σ_i and δ are small quantities.

C. Zero-momentum parameters

Barbieri *et al.* have introduced [28] a set of electroweak parameters defined at $Q^2=0$ that describe leading-order and higher-order effects of physics beyond the standard model. In Ref. [11], we not only derived the relationships between the zero-momentum parameters and the on-shell parameters S, T, δ and $\Delta\rho$, but also quoted expressions for the zero-momentum parameters in terms of the Σ_i. We now show how the expressions we have obtained for the correlation functions in Sec. VI support the results described in [11].

Barbieri *et al.* [28] choose parameters to describe four-fermion electroweak processes using the transverse gauge-boson polarization amplitudes. Formally, all such processes can be summarized in momentum space (at tree-level in the electroweak interactions, having integrated out all heavy states, and ignoring external fermion masses) by the neutral-current Lagrangian

$$\mathcal{L}_{nc} = \frac{1}{2}(J_{3\mu}\quad J_{B\mu})\begin{bmatrix}\Pi_{W^3W^3}(Q^2) & \Pi_{W^3B}(Q^2)\\ \Pi_{W^3B}(Q^2) & \Pi_{BB}(Q^2)\end{bmatrix}^{-1}\begin{pmatrix}J_3^\mu\\ J_B^\mu\end{pmatrix}, \quad (7.12)$$

and the charged-current Lagrangian

$$\mathcal{L}_{cc} = \frac{1}{2}[\Pi_{W^+W^-}(Q^2)]^{-1} J_+^{\mu} J_{-\mu}, \tag{7.13}$$

where the $\vec{J}^{\mu}$ and J_B^{μ} are the weak isospin and hypercharge fermion currents, respectively. All two-point correlation functions of fermionic currents—and therefore all four-fermion scattering amplitudes at tree-level—can be read off from the appropriate element(s) of the inverse gauge-boson polarization matrix.

Comparing Eqs. (3.3) and (3.4) to Eqs. (7.12) and (7.13), we find

$$\Pi^{-1}(Q^2) = \begin{bmatrix} \Pi_{W^3W^3}(Q^2) & \Pi_{W^3B}(Q^2) \\ \Pi_{W^3B}(Q^2) & \Pi_{BB}(Q^2) \end{bmatrix}^{-1} \equiv -\begin{bmatrix} [G_{\rm NC}(Q^2)]_{WW} & [G_{\rm NC}(Q^2)]_{WY} \\ [G_{\rm NC}(Q^2)]_{WY} & [G_{\rm NC}(Q^2)]_{YY} \end{bmatrix}, \tag{7.14}$$

and

$$\Pi_{W^+W^-} \equiv \frac{-1}{[G_{\rm CC}(Q^2)]_{WW}}. \tag{7.15}$$

Using the consistency relation, Eq. (6.1), we find

$$\Pi(Q^2) = \begin{bmatrix} \frac{-1}{[G_{\rm CC}(Q^2)]_{WW}} & \frac{[G_{\rm NC}(Q^2)]_{WY}}{[G_{\rm CC}(Q^2)]_{WW}[G_{\rm NC}(Q^2)]_{YY}} \\ \frac{[G_{\rm NC}(Q^2)]_{WY}}{[G_{\rm CC}(Q^2)]_{WW}[G_{\rm NC}(Q^2)]_{YY}} & \frac{-[G_{\rm NC}(Q^2)]_{WW}}{[G_{\rm CC}(Q^2)]_{WW}[G_{\rm NC}(Q^2)]_{YY}} \end{bmatrix}. \tag{7.16}$$

Barbieri *et al.* proceed by defining the (approximate) electroweak couplings

$$\frac{1}{g^2} \equiv \left[\frac{d\Pi_{W^+W^-}(Q^2)}{d(-Q^2)}\right]_{Q^2=0}, \qquad \frac{1}{g'^2} \equiv \left[\frac{d\Pi_{BB}(Q^2)}{d(-Q^2)}\right]_{Q^2=0}, \tag{7.17}$$

and the electroweak scale

$$v^2 \equiv -4\Pi_{W^+W^-}(0) = (\sqrt{2}G_F)^{-1} \approx (246~\text{GeV})^2. \tag{7.18}$$

(Our definition of v differs from that used in Ref. [28] by $\sqrt{2}$.) In terms of the polarization functions and these constants, the authors of [28] define the parameters

$$\hat{S} \equiv g^2 \left[\frac{d\Pi_{W^3B}(Q^2)}{d(-Q^2)}\right]_{Q^2=0}, \tag{7.19}$$

$$\hat{T} \equiv \frac{g^2}{M_W^2}(\Pi_{W^3W^3}(0) - \Pi_{W^+W^-}(0)), \tag{7.20}$$

$$W \equiv \frac{g^2 M_W^2}{2}\left[\frac{d^2\Pi_{W^3W^3}(Q^2)}{d(-Q^2)^2}\right]_{Q^2=0}, \tag{7.21}$$

$$Y \equiv \frac{g'^2 M_W^2}{2}\left[\frac{d^2\Pi_{BB}(Q^2)}{d(-Q^2)^2}\right]_{Q^2=0}. \tag{7.22}$$

In any nonstandard electroweak model in which all of the relevant effects occur *only* in the correlation functions of fermionic electroweak gauge currents,[4] the values of these four parameters [28] summarize the leading deviations in all four-fermion processes from the standard-model predictions. In addition, the quantities [28]

$$\hat{U} \equiv -g^2\left[\frac{d\Pi_{W^3W^3}}{d(-Q^2)} - \frac{d\Pi_{W^+W^-}}{d(-Q^2)}\right]_{Q^2=0}, \tag{7.23}$$

$$\hat{V} \equiv \frac{g^2 M_W^2}{2}\left[\frac{d^2\Pi_{W^3W^3}}{d(-Q^2)^2} - \frac{d^2\Pi_{W^+W^-}}{d(-Q^2)^2}\right]_{Q^2=0}, \tag{7.24}$$

$$X = \frac{g g' M_W^2}{2}\left[\frac{d^2\Pi_{W^3B}(Q^2)}{d(-Q^2)^2}\right]_{Q^2=0} \tag{7.25}$$

describe higher-order effects.

We now evaluate these expressions for the on-shell parameters in case I linear moose models. Initially, we notice that

$$\Pi_{W^+W^-}(Q^2) = \Pi_{W^3W^3}(Q^2), \tag{7.26}$$

and therefore the zero-momentum parameters

$$\hat{T} = \hat{U} = \hat{V} = 0, \tag{7.27}$$

vanish identically.

Using the forms of the correlation functions derived previously, and the identities (7.16) which we have just derived from the consistency relations, we may compute $\hat{S}$, W, and Y directly in terms of the Σ_i. Alternatively, as shown in [11], we may compute the zero-momentum parameters from the on-shell parameters αS, αT, and $\alpha\delta$, which we have already derived in terms of the Σ_i in Sec. VI. In either case, we find

$$\hat{S} = M_W^2 \Sigma_r, \tag{7.28}$$

$$W = -M_W^2(\Sigma_W - \Sigma_p - \Sigma_r), \tag{7.29}$$

$$Y = -M_W^2(\Sigma_Z - \Sigma_W - \Sigma_M). \tag{7.30}$$

We see that $\hat{S}$ is strictly positive—we will return to the phenomenological consequences of this in Sec. XI.

Finally, computing the parameter X in the same way, we see that it is suppressed by four powers of the heavy masses and is, as expected, zero to this order [28].

[4] And not, for example, through extra gauge bosons or compositeness operators involving the $B - L$ or weak isosinglet currents.

VIII. GENERAL LINEAR MOOSE MODELS: CORRELATION FUNCTIONS AND CONSISTENCY CONDITIONS

We now study the correlation functions of the general linear moose. We have already derived $[G_{NC}(Q^2)]_{WY}$ for the general case in Sec. V. Since in both case I and the general model the value of p is arbitrary (and the charged currents do not depend on q), the expressions for $[G_{NC}(Q^2)]_{WW}$ and $[G_{CC}(Q^2)]_{WW}$ and their residues are the same in the general model as in case I. We therefore reapply the consistency conditions (3.22) and deduce the residues of $[G_{NC}(Q^2)]_{YY}$ in the general model. We then discuss how the consistency relations among correlation functions found for case I are modified in the general linear moose.

A. $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{WY}}$ in the general case

The expressions for $[G_{NC}(Q^2)]_{WY}$ and the associated residues are as derived in Sec. V.

B. $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{WW}}$ and $[\mathbf{G_{CC}(Q^2)}]_{\mathbf{WW}}$ in the general case

As previously noted, the form of $[G_{NC}(Q^2)]_{WW}$ and $[G_{CC}(Q^2)]_{WW}$ depend only on the identity of the $SU(2)$ group to which fermions couple and not on the $U(1)$ to which they couple. The expressions for $[G_{NC}(Q^2)]_{WW}$, $[G_{CC}(Q^2)]_{WW}$ and their residues derived in Secs. VI C and VI D for case I therefore apply equally well in the general case. Note that there is dependence on the $\mathcal{M}_m$ [which cover all of the $U(1)$ groups with $j > N+1$] rather than on $\mathcal{M}_q$.

C. $[\mathbf{G_{NC}(Q^2)}]_{\mathbf{YY}}$ in the general case: The related case I moose

We can calculate the pole residues of $[G_{NC}(Q^2)]_{YY}$ in the general case by applying the consistency conditions (3.22) to our previous results for $[\xi_Z]_{WY}$ (5.3) and $[\xi_Z]_{WW}$ (6.19). We find

$$[\xi_Z]_{YY} = \frac{e^2(M_Z^2 - M_W^2)}{M_W^2}\left[\prod_{w=1}^{N}\frac{\mathsf{m}_{Ww}^2 - M_Z^2}{\mathsf{m}_{Ww}^2}\right] \times\left[\prod_{z=1}^{K}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - M_Z^2}\right]\left[\prod_{m=1}^{M}\frac{\mathsf{m}_m^2}{\mathsf{m}_m^2 - M_Z^2}\right] \times\left[\prod_{\hat{q}=q+1}^{K+1}\frac{(\mathsf{m}_{\hat{q}}^2 - M_Z^2)^2}{\mathsf{m}_{\hat{q}}^4}\right]. \tag{8.1}$$

This reduces to the case I (and [10]) expression (6.5) if we set $q = N+1$, since the $\mathsf{m}_{\hat{q}}$ then take on the values $\mathsf{m}_{\hat{m}}$. To leading order in mass-squared ratios, one has

$$[\xi_Z]_{YY} = \frac{e^2(M_Z^2 - M_W^2)}{M_W^2}[1 + M_Z^2(\Sigma_Z - \Sigma_W + \Sigma_M - 2\Sigma_q)]. \tag{8.2}$$

Note that for $q = N+1$ we have $\Sigma_q = \Sigma_M$ and we recover the case I result (6.6); if we set $p = 0$ and $q = N+1$ we recover the result in [10].

Alternatively, we may use a different kind of extrapolation from a case I moose to calculate $[G_{NC}(Q^2)]_{YY}$. Consider the case I moose shown in Fig. 3: in this model, the values of the couplings and f constants are the same as in our original general moose, but the sites corresponding to $SU(2)$ groups now include those all the way to site $q-1$. All of the properties of the neutral bosons are the same in this "related case I moose" as in our original moose. Therefore, we can use our previous knowledge of the form of $[G_{NC}(Q^2)]_{YY}$ for case I to find this correlation function for the moose in Fig. 3, and this will also be the answer for our original moose.

Furthermore, as we shall see in the next subsection, the properties of the charged bosons and the corresponding correlation functions in the two models are related by a new consistency condition which will allow us to extract to the electroweak parameters more easily. Defining the matrix and eigenvalues for the charged-boson masses in the related case I moose

$$M_{W'}^2 = M_{[0,q)}^2 \qquad \mathsf{m}_{W'w}^2 \quad (w = 0, 1, \cdots, q-1), \tag{8.3}$$

we will show that the lowest eigenvalue of this matrix, $\mathsf{m}_{W'0}^2$, is light. As before, we will denote this *light* eigenvalue by $M_{W'}^2$, and the distinction between this and the mass matrix will be clear from context.

In terms of the related case I moose, we may evaluate $[G_{NC}(Q^2)]_{YY}$ directly, and find

$$[G_{\mathrm{NC}}(Q^2)]_{YY} = \frac{e^2}{Q^2}\frac{[Q^2 + M_{W'}^2]M_Z^2}{M_{W'}^2[Q^2 + M_Z^2]}\left[\prod_{w=1}^{q-1}\frac{Q^2 + \mathsf{m}_{W'w}^2}{\mathsf{m}_{W'w}^2}\right] \times\left[\prod_{z=1}^{K}\frac{\mathsf{m}_{Zz}^2}{Q^2 + \mathsf{m}_{Zz}^2}\right]\left[\prod_{\hat{q}=q+1}^{K+1}\frac{Q^2 + \mathsf{m}_{\hat{q}}^2}{\mathsf{m}_{\hat{q}}^2}\right]. \tag{8.4}$$

Reading off the residues of the poles in Eq. (8.4), we obtain

$$[\xi_Z]_{YY} = e^2\frac{M_Z^2 - M_{W'}^2}{M_{W'}^2}\left[\prod_{w=1}^{q-1}\frac{\mathsf{m}_{W'w}^2 - M_Z^2}{\mathsf{m}_{W'w}^2}\right] \times\left[\prod_{z=1}^{K}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - M_Z^2}\right]\left[\prod_{\hat{q}=q+1}^{K+1}\frac{\mathsf{m}_{\hat{q}}^2 - M_Z^2}{\mathsf{m}_{\hat{q}}^2}\right], \tag{8.5}$$

If we expand the expression for $[\xi_Z]_{YY}$ to first nontrivial order, we can rewrite it as

$$[\xi_Z]_{YY} = \frac{e^2(M_Z^2 - M_{W'}^2)}{M_{W'}^2}[1 + M_Z^2(\Sigma_Z - \Sigma_{W'} - \Sigma_q)] \tag{8.6}$$

where we have defined

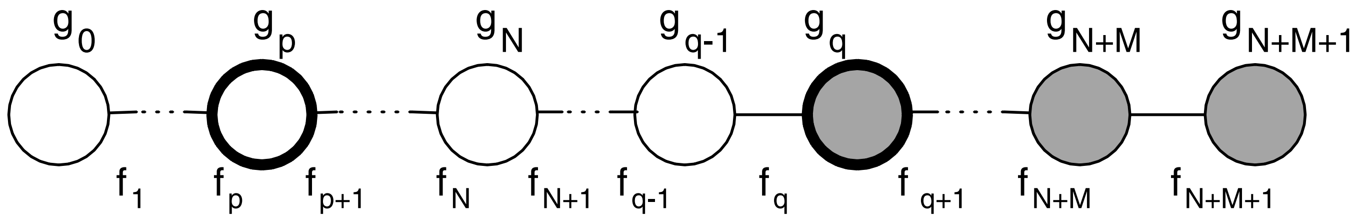

FIG. 3. "Related" case I moose corresponding to the moose in Fig. 1. In this moose, the $SU(2)$ groups continue all the way to site $q-1$.

$$\Sigma_{W'} \equiv \sum_{w=1}^{q-1} \frac{1}{\mathsf{m}^2_{W'w}}. \tag{8.7}$$

Equating the two expressions in Eqs. (8.2) and (8.6), we find the relation

$$M^2_{W'} = M^2_W[1 - (M^2_Z - M^2_W)(\Sigma_{W'} - \Sigma_W + \Sigma_M - \Sigma_q)], \tag{8.8}$$

justifying our assumption that the matrix $M^2_{W'}$ had a light eigenvalue.

D. The custodial consistency conditions and $\Delta\rho$

We now relate the properties of the charged-boson correlation functions in the original and related case I moose. To facilitate the discussion we define the following general propagators

$$G^{x,y}_{[I,J]}(Q^2), \quad G^{x,y}_{[I,J)}(Q^2), \quad G^{x,y}_{(I,J]}(Q^2), \quad G^{x,y}_{(I,J)}(Q^2), \tag{8.9}$$

where, for example,

$$G^{x,y}_{[I,J]}(Q^2) = g_x g_y \langle x| \frac{1}{Q^2 + M^2_{[I,J]}} |y\rangle, \tag{8.10}$$

is the Euclidean space propagator for the correlation function of the current at site x with that at site y in the linear moose with vector-boson mass-squared matrix $M^2_{[I,J]}$, and similarly for the other correlation functions. Clearly, we must have that $I \le x, y \le J$.

Using this notation, $[G_{CC}(Q^2)]_{WW} = G^{p,p}_{[0,N+1)}(Q^2)$ is the charged-current correlation function in the original moose and $[\tilde{G}_{CC}(Q^2)]_{WW} = G^{p,p}_{[0,q)}(Q^2)$ is the charged-boson correlation function in the related case I moose. By definition,

$$[G_{CC}(Q^2 = 0)]_{WW} = g^2_p(M^{-2}_W)_{p,p} \equiv 4\sqrt{2}G_F, \tag{8.11}$$

while

$$[\tilde{G}_{CC}(Q^2 = 0)]_{WW} = g^2_p(M^{-2}_{W'})_{p,p} \equiv 4\sqrt{2}\rho G_F, \tag{8.12}$$

where the second equality follows from the fact that the zero-momentum charged-current correlation function of the related case I moose has the same strength as the T^2_3 part of the neutral-current correlation function in the original moose (i.e., $\rho \equiv 1$ in case I). In these equations $(M^{-2}_W)_{p,p}$ and $(M^{-2}_{W'})_{p,p}$ represent, respectively, the (p, p) element of the inverse of the mass-squared matrices M^2_W and $M^2_{W'}$. As quoted in Appendix B, these individual elements are calculated to be

$$\begin{aligned}(M^{-2}_W)_{i,j} &= \frac{4}{g_i g_j} \sum_{k=\max(i,j)+1}^{N+1} \frac{1}{f^2_k}, \\ (M^{-2}_{W'})_{i,j} &= \frac{4}{g_i g_j} \sum_{k=\max(i,j)+1}^{q} \frac{1}{f^2_k}.\end{aligned} \tag{8.13}$$

Then using (8.13) in Eq. (8.11) shows that the weak scale is related to the f-constants of the links between the $SU(2)$ group to which the fermions couple (p) and the $U(1)$ group at the interface ($N + 1$):

$$\sum_{j=p+1}^{N+1} \frac{4}{f^2_j} \equiv \frac{4}{v^2} = 4\sqrt{2}G_F, \tag{8.14}$$

while applying (8.13) to the difference of Eqs. (8.11) and (8.12) yields a simple expression for $\Delta\rho$:

$$\Delta\rho = \rho - 1 = \sum_{k=N+2}^{q} \frac{v^2}{f^2_k} > 0. \tag{8.15}$$

We see that the low-energy ρ parameter will be identically 1 when $q = N + 1$ and will be greater than 1 otherwise. Furthermore, using these formulas we find that

$$G^{p,N+1}_{[0,q)}(Q^2 = 0) = G^{N+1,N+1}_{[0,q)}(Q^2 = 0) = 4\sqrt{2}\Delta\rho G_F. \tag{8.16}$$

By direct inversion, we compute the following correlation functions[5] (where the products are over *all* nonzero eigenvalues)

$$[G_{CC}(Q^2)]_{WW} = 4\sqrt{2}G_F \frac{\prod_{p,r}(1 + \frac{Q^2}{\mathsf{m}^2_p})(1 + \frac{Q^2}{\mathsf{m}^2_r})}{\prod_w(1 + \frac{Q^2}{\mathsf{m}^2_{Ww}})}, \tag{8.17}$$

[5]The form of these correlation functions is determined entirely by the zeros and poles which arise from examining the relevant inverse matrix, and by fixing the value at $Q^2 = 0$.

$$[\tilde{G}_{CC}(Q^2)]_{WW} = 4\sqrt{2}\rho G_F \frac{\prod_{p,r'}(1+\frac{Q^2}{\mathsf{m}_p^2})(1+\frac{Q^2}{\mathsf{m}_{r'}^2})}{\prod_{w'}(1+\frac{Q^2}{\mathsf{m}_{w'}^2})}, \quad (8.18)$$

and

$$G^{p,N+1}_{[0,q)}(Q^2) = 4\sqrt{2}\Delta\rho G_F \frac{\prod_{p,s}(1+\frac{Q^2}{\mathsf{m}_p^2})(1+\frac{Q^2}{\mathsf{m}_s^2})}{\prod_{w'}(1+\frac{Q^2}{\mathsf{m}_{w'}^2})},$$

$$G^{N+1,N+1}_{[0,q)}(Q^2) = 4\sqrt{2}\Delta\rho G_F \frac{\prod_{w,s}(1+\frac{Q^2}{\mathsf{m}_{Ww}^2})(1+\frac{Q^2}{\mathsf{m}_s^2})}{\prod_{w'}(1+\frac{Q^2}{\mathsf{m}_{w'}^2})}. \quad (8.19)$$

In these expressions, we have defined the matrices and corresponding eigenvalues

$$M^2_{r'} = M^2_{(p,q)} \qquad \mathsf{m}^2_{r'}\ (r' = p+1, \cdots, q-1), \quad (8.20)$$

$$M^2_{s} = M^2_{(N+1,q)} \qquad \mathsf{m}^2_{s}\ (s = N+2, \cdots, q-1). \quad (8.21)$$

Phenomenological considerations will require that these eigenvalues be large, $\mathsf{m}^2_{r',s} \gg M^2_{W,Z}$.

The arguments which lead to the consistency condition of Eq. (6.1) may be used to show that

$$\tilde{G}^{WW}_{CC}(Q^2) - G^{WW}_{CC}(Q^2) = G^{p,p}_{[0,q)}(Q^2) - G^{p,p}_{[0,N+1)}(Q^2) = \frac{[G^{p,N+1}_{[0,q)}(Q^2)]^2}{G^{N+1,N+1}_{[0,q)}(Q^2)}. \quad (8.22)$$

and also to prove the more general consistency conditions listed in Appendix D. Performing a low-Q^2 expansion on Eq. (8.22), we find the relation

$$4\sqrt{2}\rho G_F\left(1+\frac{Q^2}{M_W^2}\right)(1+Q^2\Sigma_{r'}+Q^2\Sigma_W) = 4\sqrt{2}G_F\left(1+\frac{Q^2}{M_{W'}^2}\right)(1+Q^2\Sigma_r+Q^2\Sigma_{W'}) + 4\sqrt{2}\Delta\rho G_F(1+Q^2\Sigma_p+Q^2\Sigma_s), \quad (8.23)$$

where we have defined

$$\Sigma_{r'} = \sum_{r'=p+1}^{q-1}\frac{1}{\mathsf{m}^2_{r'}}, \qquad \Sigma_s = \sum_{s=N+2}^{q-1}\frac{1}{\mathsf{m}^2_s}. \quad (8.24)$$

We know from phenomenologically considerations that $\Delta\rho < \mathcal{O}(10^{-2})$—therefore we may neglect the last terms in this expression, which are proportional to $\Delta\rho \cdot \Sigma_{p,s} \cdot Q^2$. Equating powers of Q^2 and Q^4 we find (to lowest nontrivial order in deviation from the standard model):

$$\Sigma_{r'} + \Sigma_W = \Sigma_r + \Sigma_{W'}, \quad (8.25)$$

$$M_W^2 = \rho M_{W'}^2. \quad (8.26)$$

Equations (8.25) and (8.26) are the primary results of this subsection, and will allow us to compute the electroweak parameters of the original general model in terms of the properties of the related case I moose.

IX. GENERAL LINEAR MOOSE MODELS: ELECTROWEAK PHENOMENOLOGY

A. αS and αT

Comparing our pairs of expressions from Secs. IV and VIII for $[\xi_Z]_{YY}$, $[\xi_Z]_{WY}$, $[\xi_Z]_{WW}$, and $[\xi_W]_{WW}$ (after moving consistently to the scheme with G_F as an experimental input) we can solve for S and T to leading order:

$$\alpha S = 4s_Z^2c_Z^2M_Z^2(\Sigma_Z - \Sigma_p - \Sigma_q), \quad (9.1)$$

$$\alpha T = s_Z^2M_Z^2(\Sigma_Z - \Sigma_W + \Sigma_M - 2\Sigma_q). \quad (9.2)$$

The linear combination of parameters computed in [10] now looks like

$$\alpha(S - 4c_Z^2T) = 4s_Z^2c_Z^2M_Z^2[(\Sigma_W - \Sigma_p) - (\Sigma_M - \Sigma_q)] \quad (9.3)$$

and depends on the differences of the inverse-square-sums of eigenvalues of the full W and M matrices and the reduced-rank p and q matrices.

If we take $q = N+1$ we recover the equivalent expressions for case I; if we take $p = 0$ along with $q = N+1$ we recover the expressions from [10]. Note that models in which $p = N$ and $q = N+1$ are an extension of the generalized-BESS-type-I models studied in [12,13] and our results are consistent with these earlier papers. In this case we have $\Sigma_q = \Sigma_M$, and Σ_p takes on a value which we denote $\Sigma_{p=N}$. Therefore Eqs. (9.1) and (9.2) become

$$\alpha S = 4s_Z^2c_Z^2M_Z^2(\Sigma_Z - \Sigma_{p=N} - \Sigma_M),$$

$$\alpha T = s_Z^2M_Z^2(\Sigma_Z - \Sigma_W - \Sigma_M),$$

$$\alpha S - 4c_Z^2\alpha T = 4s_Z^2c_Z^2M_Z^2(\Sigma_W - \Sigma_{p=N}). \quad (9.4)$$

References [12,13] focused on the case in which g_p and g_q were small. In this limit, all of the parameters above are of order $\mathcal{O}(M_W^4/M_H^4)$, and the first nonzero contributions to electroweak corrections were found to arise as the 4th power of mass ratios rather than the 2nd power.

B. $\alpha\delta$

We find a general expression for δ by exploiting our ability to compute δ for any case I moose, including the related case I moose for our general linear moose (shown in Fig. 3). In Eq. (4.14) we related the nonresonant contribution to four-fermion processes (arising from heavy neutral-boson exchange) to the pole residues and masses by

$$\sqrt{2}G_F \frac{\alpha\delta}{s^2c^2} = \sum_{z=1}^{K} \frac{[\xi_{Zz}]_{WW}}{\mathsf{m}_{Zz}^2}. \tag{9.5}$$

As the matrix M_Z^2 is the same in the original moose and its related case I moose, we see that the value of δ is the same[6] in the original moose and its related case I moose. From our calculation in Sec. VII [see (7.11)], then, we can write δ in terms of the Σ_i for the related case I moose

$$\frac{\alpha\delta}{c^2} = -4s_Z^2c_Z^2M_Z^2(\Sigma_{W'} - \Sigma_{r'} - \Sigma_p) \tag{9.6}$$

and by applying Eq. (8.25) we obtain δ in terms of the Σ_i for the original general linear moose

$$\frac{\alpha\delta}{c^2} = -4s_Z^2c_Z^2M_Z^2(\Sigma_W - \Sigma_r - \Sigma_p). \tag{9.7}$$

C. $\Delta\rho$

From Eqs. (8.8) and (8.26), we find

$$\Delta\rho = s_Z^2M_Z^2(\Sigma_{W'} - \Sigma_W + \Sigma_M - \Sigma_q). \tag{9.8}$$

Recall that we also found [Eq. (8.15)] that $\Delta\rho \equiv \rho - 1 = \sum_{k=N+2}^{q}\frac{v^2}{f_k^2} > 0$. The results of Appendix B may be used to show the equivalence of Eqs. (8.15) and (9.8). Note that for any case I moose, $\Sigma_W = \Sigma_{W'}$, $\Sigma_M = \Sigma_q$, and $q = N + 1$—therefore using either expression we see that $\Delta\rho$ vanishes.

Furthermore, the experimental bounds on $\rho - 1$ place limits on the size of some of the f_k. As shown[7] in [28], the current limit is $\rho - 1 < 0.004$. In a model with $q = N + 2$ [i.e., the fermions couple to the $U(1)$ to the right of the one at the $SU(2)$–$U(1)$ interface], we find $f_{N+2} > v/\sqrt{0.004} \sim 3.9$ TeV. In a model with $q > N + 2$, each f_k contributing to $\rho - 1$ must be greater than 3.9 TeV. This suggests that the case I models, with $\Delta\rho = 0$ will be of greatest phenomenological interest.

D. Zero-momentum parameters

Using the results of [11] we can write the zero-momentum parameters as

$$\hat{S} = \frac{1}{4s^2}\left[\alpha S + 4c^2(\Delta\rho - \alpha T) + \frac{\alpha\delta}{c^2}\right], \tag{9.9}$$

$$\hat{T} = \Delta\rho, \tag{9.10}$$

$$W = \frac{\alpha\delta}{4s^2c^2}, \tag{9.11}$$

$$Y = \frac{c^2}{s^2}(\Delta\rho - \alpha T). \tag{9.12}$$

Inserting our expressions for the on-shell parameters in terms of the Σ_i from Secs. IX A and IX B and applying the custodial consistency relation of Eq. (8.25), we find

$$\hat{S} = M_W^2\Sigma_{r'} > 0, \tag{9.13}$$

is strictly greater than zero, and

$$\hat{T} = \frac{s_Z^2}{c_Z^2}M_W^2(\Sigma_{W'} - \Sigma_W + \Sigma_M - \Sigma_q), \tag{9.14}$$

$$W = -M_W^2(\Sigma_{W'} - \Sigma_{r'} - \Sigma_p), \tag{9.15}$$

$$Y = -M_W^2(\Sigma_Z - \Sigma_{W'} - \Sigma_q). \tag{9.16}$$

The values of $\hat{S}$, W and Y are precisely the same as the values one would calculate for the related case I moose.

To better understand the nonzero value of $\hat{T}$ in the general linear moose (recall $\hat{T} = 0$ in the related case I moose) we can return to the polarization amplitudes of Eqs. (7.12) and (7.13). If we consider the related case I moose shown in Fig. 3, the consistency relation of Eq. (6.1) implies

$$([G_{NC}(Q^2)]_{WW} - [\tilde{G}_{CC}(Q^2)]_{WW}) \cdot [G_{NC}(Q^2)]_{YY} = ([G_{NC}(Q^2)]_{WY})^2. \tag{9.17}$$

From Eq. (9.17), we compute

$$\Pi_{W^3W^3}(Q^2) = \frac{-1}{[\tilde{G}_{CC}(Q^2)]_{WW}}, \tag{9.18}$$

while

$$\Pi_{W^+W^-}(Q^2) = \frac{-1}{[G_{CC}(Q^2)]_{WW}}, \tag{9.19}$$

and therefore $\Pi_{W^+W^-}(Q^2) - \Pi_{W^3W^3}(Q^2) \neq 0$ so that $\hat{T}$ will not vanish. Using Eq. (8.22), and the forms of the correlation functions in Eqs. (8.17), (8.18), and (8.19), we compute

$$\Pi_{W^3W^3}(Q^2) - \Pi_{W^+W^-}(Q^2) = \frac{\Delta\rho}{4\sqrt{2}\rho G_F} \times \frac{\prod_s(1 + \frac{Q^2}{\mathsf{m}_s^2})}{\prod_{r,r'}(1 + \frac{Q^2}{\mathsf{m}_r^2})(1 + \frac{Q^2}{\mathsf{m}_{r'}^2})}. \tag{9.20}$$

This expression reproduces the value of $\hat{T}$ in Eq. (9.14) when evaluated at $Q^2 = 0$.

[6]In principle, the value of G_F in Eq. (9.5) changes to ρG_F in the related case I moose—however, since $\Delta\rho = \mathcal{O}(10^{-2})$, this change is higher-order in deviation from the standard model.

[7]See [11] for a discussion of the correspondence between the notation of Barbieri *et al.* and the notation used here.

Given that the variables $\hat{U}$ and V depend on higher-order derivatives of the difference in Eq. (9.20), when the $\mathsf{m}^2_{r,r',s}$ are large we see that $\hat{U}$ and V will be much smaller than $\hat{T}$.

X. WHICH MODELS ARE VIABLE?

The preceding sections of this paper have analyzed the corrections to precisely measured electroweak quantities in the moose shown in Fig. 1, subject to the following phenomenologically motivated assumptions:

(1) The matrix $M^2_Z = M^2_{[0,N+M+1]}$ has only one light nonzero mass eigenvalue, which is associated with the ordinary Z boson.

(2) The matrices $M^2_W = M^2_{[0,N+1)}$ and $M^2_{W'} = M^2_{[0,q)}$ each have only one light eigenvalue. The light eigenvalue of M^2_W is associated with the ordinary W boson.

(3) None of the submatrices $\mathcal{M}^2_p = M^2_{[0,p)}$, $\mathcal{M}^2_r = M^2_{(p,N+1)}$, $M^2_{r'} = M^2_{(p,q)}$, $\mathcal{M}^2_q = M^2_{(q,K+1]}$, and $\mathcal{M}^2_M = M^2_{(N+1,K+1]}$ has a light eigenvalue with mass of order M^2_W or M^2_Z.

(4) The f-constants and couplings of the moose are constrained to obey

$$\sqrt{2}G_F = \frac{1}{v^2} = \sum_{k=p+1}^{N+1} \frac{1}{f_k^2}, \qquad \frac{1}{e^2} = \sum_{i=0}^{N+M+1} \frac{1}{g_i^2}. \tag{10.1}$$

We find that the deviations in electroweak parameters from their standard-model values may be summarized by the following four quantities, each of which is constrained by experiment [28] to be less than of order 10^{-3} in magnitude:

$$\hat{S} = M^2_W \Sigma_{r'} > 0, \tag{10.2}$$

$$\hat{T} = \frac{s_Z^2}{c_Z^2} M^2_W (\Sigma_{W'} - \Sigma_W + \Sigma_M - \Sigma_q), \tag{10.3}$$

$$W = -M^2_W(\Sigma_{W'} - \Sigma_{r'} - \Sigma_p) = -M^2_W(\Sigma_W - \Sigma_r - \Sigma_p), \tag{10.4}$$

$$Y = -M^2_W(\Sigma_Z - \Sigma_{W'} - \Sigma_q). \tag{10.5}$$

Because the Σ_i are sums over the inverse-squares of mass eigenvalues, barring some enhancement of a subleading contribution that cancels against these terms, the above constraints on electroweak parameters imply that the ratio M^2_W/m^2 for any heavy eigenvalue of any of the mass matrices must likewise be less than or of order 10^{-3}.

In this section, we explore two further questions. First, we determine which configurations of the f_i and g_i result in a linear moose satisfying all of the constraints above. Second, we ask whether such a moose can also be consistent with unitarity, which places an upper bound on the masses of the extra W bosons that unitarize high-energy longitudinal W-boson scattering.

A. Couplings and f-constants

Let us determine which values of the f_k and g_i will result in a linear moose that meets the phenomenological constraints above (except unitarity which we will address shortly). As discussed in Appendix B, the mass eigenvalues of the charged-boson portion of the moose and its submatrices are related as

$$\left[\prod_{\hat{p}=1}^{p} \mathsf{m}^2_{\hat{p}}\right] g_p^2 \left[\frac{1}{4} F^2_{W,p} \prod_{r=p+1}^{N} \mathsf{m}^2_r\right] = M^2_W \prod_{\hat{w}=1}^{N} \mathsf{m}^2_{W\hat{w}}, \tag{10.6}$$

where

$$\frac{1}{F^2_{W,p}} = \sum_{i=p+1}^{N+1} \frac{1}{f_i^2}. \tag{10.7}$$

Having g_p be small is sufficient to enable the W to be light without having any of the $\mathsf{m}^2_{\hat{p}}$ or m^2_r be comparably light. If any other g_i were small, some related $\mathsf{m}^2_{\hat{i}}$ would also be light. Likewise, having g_p and g_q small is sufficient to enable the full moose to have a light Z eigenstate without having any of the $\mathsf{m}^2_{\hat{p}}$, $\mathsf{m}^2_{\hat{q}}$, or m^2_r be similarly light. This demonstrates the existence of acceptable configurations of the f_k and g_i. We would like to go further and determine the nature of all acceptable configurations. We will focus on the charged-boson portion of the linear moose (Fig. 4) because it will directly relate to our eventual concerns about satisfying unitarity bounds.

Our discussion employs the intuition derived from Georgi's "spring analogy" [32], which is reviewed in Appendix E. Namely, we know that the eigenvalues of the mass-squared matrix for any linear moose correspond to those of a system of coupled springs with masses $1/g_i^2$ and spring constants f_k^2. The amplitudes of the mass-squared eigenvectors correspond to the normal-mode displacements of the spring system. Note that we must satisfy the constraints of Eq. (10.1), and therefore the couplings obey $g_i > e$ (for all i) and the f-constants obey $f_k > v$ ($k = p+1, \cdots, N+1$). The global symmetry at site $N+1$ is a fixed boundary, equivalent to an infinite mass or $g^2_{N+1} \equiv 0$. Satisfying the condition that $M^2_W/\mathsf{m}^2 < \mathcal{O}(10^{-3})$ corresponds to our system's having one and only one low-frequency mode. We proceed by examining several possibilities in turn.

If all of the couplings and f-constants are comparable, with no hierarchies among them, there is no way to establish a large hierarchy between the lightest eigenvalue and the next. This is analogous to the KK spectrum for the gauge fields in a flat background with spatially independent coupling: the tower of KK states is linear in mass-squared. This is not phenomenologically acceptable.

Suppose, instead, that one of the g_i is smaller than all the rest, but all the f_k's are comparable in size. Such a situation clearly has one light eigenvalue—in the spring analogy, there is a low-frequency mode in which site i moves slowly

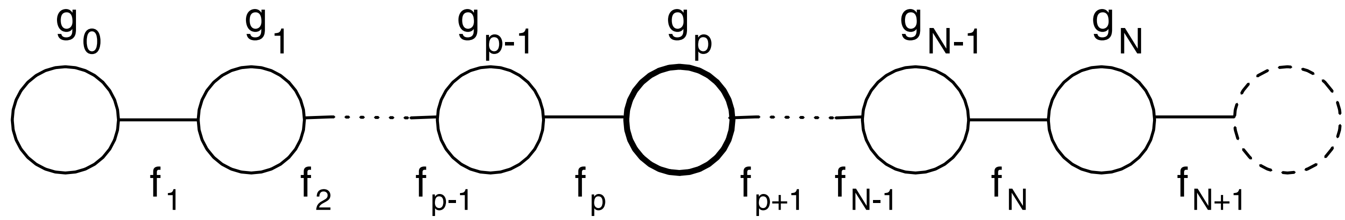

FIG. 4. Moose diagram corresponding to the charged-boson mass matrix $M_W^2 = M_{[0,N+1)}^2$.

and there are approximate heavy modes with site i remaining still. Then there will be a single light eigenstate of M_W^2. However, recall that all of the parameters in Eqs. (10.2), (10.3), (10.4), and (10.5) are constrained to be small. Examining Eq. (10.4), we see that neither of the matrices $\mathcal{M}_p^2 = M_{[0,p)}^2$ and $\mathcal{M}_r^2 = M_{(p,N+1)}^2$ may have a mass eigenvalue of order M_W^2, or else the parameter W would be larger than experiment allows (since Σ_W does not include the light W mass and could not cancel a large Σ_r or Σ_p). If the small coupling g_i corresponds to a site with $i < p$, the matrix $\mathcal{M}_p^2$ will have a small eigenvalue; if $i > p$, the matrix $\mathcal{M}_r^2$ will have a small eigenvalue. We therefore conclude that $g_i = g_p$, is the only viable option when all of the f_k are of similar magnitude.

A small extension of the above analysis shows that we cannot have hierarchical g_i's with more than one small coupling when all f_k are comparable. Suppose there are two couplings much smaller than the rest: $g_{i,i'}$. The high-frequency, large mass-squared, modes are those where the displacements at sites i, i' are approximately zero. Conversely, there are now two low-frequency modes involving the displacements at sites i, i', implying there is more than one light mass-squared eigenvalue for matrix M_W^2. This contradicts observation.

Now let us look at the opposite extreme in which all of the g_i are comparable and one of the f_k is small. Since the f-constants link pairs of gauge groups, it is not possible for an f_k to form part of matrix M_W^2 without being included in either $\mathcal{M}_p^2$ or $\mathcal{M}_r^2$. Hence, an f_k small enough to produce a light eigenvalue in M_W^2 will also produce a small eigenvalue in one of the submatrices.

The related case in which all of the g_i are comparable and two or more f_k are small is also forbidden. To see this, consider the dual moose [33] of our charged-gauge-boson moose in which the roles of sites (g_i) and links (f_i) are exchanged. As reviewed in Appendix E, this moose has the same mass-squared eigenvalues as the original moose. In the dual moose, we would have all f_i comparable and two or more g_k small. But we saw earlier that this necessarily leads to more than one low-frequency eigenmode.

The only viable case identified so far is one in which all of the f_k are of comparable size and the coupling $g_{i=p}$ is small. We now ask whether introducing hierarchies in the values of the f_k would allow one to have the small coupling at a site $g_{i\neq p}$. Again, the goal is to have a mass-spring system with one heavy mass whose motion provides a single low-frequency eigenmode in M_W^2 and no such eigenmodes in $\mathcal{M}_r^2$ or $\mathcal{M}_p^2$. Suppose that the small coupling is at site $i \neq p$ and that all of the f-constants linking sites between i and p are large; this is equivalent to having a heavy mass at i connected to the mass at site p by very stiff springs. The masses between i and p will oscillate as a single heavy unit; if one integrated out the large f-constants, one would obtain an effective site with a smaller coupling than the original g_i. Because the fermions originally coupled to site p, which is subsumed in the new effective low-coupling site $p^{\rm eff}$, the new effective site will play the role of the original site p. In particular, coupling $g_{p,\rm eff}$ appears in the W mass-squared matrix, but not in $\mathcal{M}_{p,\rm eff}^2$ or $\mathcal{M}_{r,\rm eff}^2$. Hence, the large f-constants have made a situation with small $g_{i\neq p}$ viable.

Note that several extensions are possible. The chain of large f_k must include both site i and site p, but neither of those sites is required to be an end point. Also, it is possible to have more than one small-coupling site among those linked to i and p by large f-constants. The constraint (10.1) on the values of the f_k lying between sites p and $N+1$ can always be satisfied by making link f_{N+1} of order v since site $N+1$ is associated with a $U(1)$ group and does not contribute to M_W^2 or its submatrices. Finally, it is possible to have one or more small-coupling sites embedded in a chain of large f-constants which includes the rightmost link f_{N+1}—this corresponds in the oscillator language to a "thick" wall for the fixed boundary.

B. Unitarity

Having identified which configurations of f_k and g_i can in principle satisfy the precision electroweak constraints, we now investigate whether they can simultaneously be compatible with unitarity bounds.

Recall that $\sqrt{8\pi}v$ is the scale at which $W_L W_L$ spin-0 isospin-0 scattering would violate unitarity in the absence of a Higgs boson. From the equivalence theorem, we know that the scattering of high-energy longitudinal W bosons is identical to the scattering of the R_ξ-gauge pions eaten by the W-bosons. The eaten pions are necessarily a linear combination of the pions implicit in the $N+1$ $SU(2) \times SU(2)/SU(2)$ nonlinear sigma-model "links" in the moose shown in Fig. 1. Unitarity, therefore, guarantees that the mass-squared M_{W1}^2, of the next lightest state after the W-boson, is bounded from above by $8\pi v^2$.

In the phenomenologically relevant linear mooses defined in the previous subsection, the pion eaten by the W-boson is largely a combination of the pions coming from links $p+1$ to $N+1$. This is easiest to see in terms of the dual moose. As shown in Appendix E, the mass-squared matrix of the dual moose corresponds to the R_ξ-gauge mass matrix of the eaten Goldstone bosons—precisely the objects related to high-energy longitudinal vector-boson scattering via the equivalence theorem.

Consider first the case of all f-constants comparable and one small coupling, $g_p \ll g_{i\neq p}$. The dual moose, as shown in Fig. 5, corresponds to a spring-mass system with a fixed boundary at the left-hand side and a "weak" spring corresponding to the small g_p. The smallest eigenvalue, which corresponds to the Goldstone boson eaten by the W, arises from a normal mode in which the masses corresponding to the sites from f_{p+1} to f_{N+1} move in unison while those at sites f_1 to f_p do not—and the only spring being stretched corresponds to the coupling g_p. The amplitudes of the displacements of this normal mode correspond to the Goldstone boson eaten by the W, which is therefore made up entirely (to leading order) of the pions coming from links $p+1$ to $N+1$.

In the limit $g_p \ll g_{i\neq p}$, the moose in Fig. 4 approximately cleaves at site p into separate mooses with mass-squared matrices $\mathcal{M}_p^2$ and $\mathcal{M}_r^2$. Hence, in this limit, the heavy mass eigenstates of M_W^2 are approximately given by the eigenstates of $\mathcal{M}_p^2$ and $\mathcal{M}_r^2$. As the pions corresponding to the longitudinal polarization states of the W come almost entirely from sites $p+1$ to $N+1$, their scattering can only be unitarized by the vector bosons coming from $\mathcal{M}_r^2$. Hence, the unitarity bound also applies to the lightest eigenstate of $\mathcal{M}_r^2$:

$$\Sigma_r > \frac{1}{8\pi v^2}. \tag{10.8}$$

The case in which g_0 is small is discussed in detail in Appendix F; as anticipated by the argument of the previous subsection, it is shown explicitly that satisfying the precision electroweak constraints (specifically, keeping W or δ small) requires that the f_k between sites 0 and p be large. Therefore, the pion eaten by the light W is concentrated on links f_{p+1} through f_{N+1}. Again, the scattering of such a pion can only be unitarized by a vector meson concentrated at sites $p+1$ through N. And, again, because the masses from sites 0 to p remain approximately fixed in the high-frequency modes, the second lightest eigenstate of M_W^2 is approximately an eigenstate of $\mathcal{M}_r^2$ and the bound of Eq. (10.8) applies.

This analysis applies directly to any of the realistic cases considered in the previous subsection, as any weak-coupling sites must be embedded in a chain of adjacent links with large f-constants which includes either site p or link $N+1$. Unitarity therefore tells us that $M_W^2\Sigma_r \geq 0.004$. The sum $\Sigma_{r'}$ appearing in parameter $\hat{S}$ [Eq. (10.2)] is strictly larger than Σ_r, as can be shown by direct evaluation of these sums using the explicit form of the inverse matrices (in Appendix B). We therefore conclude that unitarity requires $\hat{S} \geq 0.004$, a value too large to be phenomenologically acceptable. Moose models which do not have extra light vector bosons (with masses of the order of the W and Z masses) cannot simultaneously satisfy the constraints of precision electroweak data and unitarity bounds.

XI. DISCUSSION AND SUMMARY

In this paper, we have calculated the form of the corrections to the electroweak interactions in the class of Higgsless models which can be "deconstructed" to a chain of $SU(2)$ gauge groups adjacent to a chain of $U(1)$ gauge groups, and with the fermions coupled to any single $SU(2)$ group and to any single $U(1)$ group along the chain. We have related the size of corrections to electroweak processes in these models to the spectrum of vector bosons which, in turn, is constrained by unitarity in Higgsless models. In particular, we find that the electroweak parameter $\hat{S}$ is bounded by

$$\hat{S} = \frac{1}{4s^2}\left[\alpha S + 4c^2(\Delta\rho - \alpha T) + \frac{\alpha\delta}{c^2}\right] \geq M_W^2\Sigma_r \geq \frac{M_W^2}{8\pi v^2} \simeq 4\times 10^{-3}, \tag{11.1}$$

which is disfavored by precision electroweak data. We have shown that this bound is true for arbitrary background 5D geometry, spatially dependent gauge-couplings, and brane kinetic energy terms.

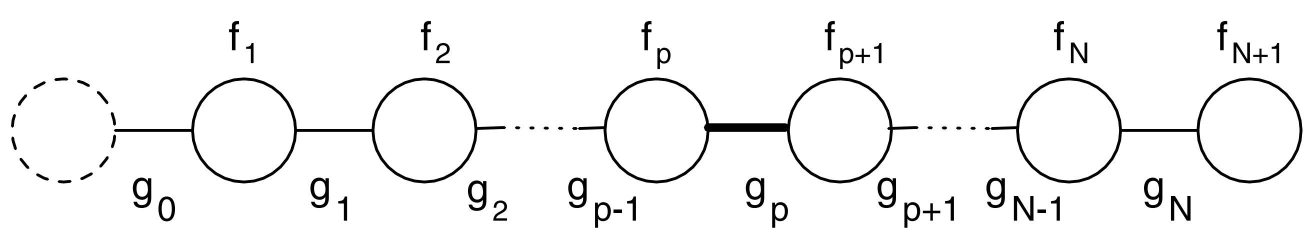

FIG. 5. Dual moose diagram corresponding to Goldstone bosons eaten by the charged vector bosons. The bold link labeled g_p corresponds to the site to which fermions couple in the original moose.

Although we have stressed our results as they apply to continuum Higgsless 5D models, they apply to any linear moose model including models motivated by hidden local symmetry. Our calculations also apply directly to the electroweak gauge sector of 5D theories with a bulk scalar Higgs boson, although the constraints arising from unitarity [the rightmost inequality in Eq. (11.1) above] no longer applies. The extension of these calculations to consider delocalized fermions is under investigation.

ACKNOWLEDGMENTS

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APPENDIX A: THE SUBMATRICES $\mathcal{M}^2_{p,q,r,L}$

For the reader's convenience, we write out explicitly the matrices $\mathcal{M}_{p,q,r,L}$ introduced in Sec III:

$$4\mathcal{M}_p^2 = 4M^2_{[0,p)} = \begin{pmatrix} g_0^2 f_1^2 & -g_0 g_1 f_1^2 & & & & & \\ -g_0 g_1 f_1^2 & g_1^2(f_1^2+f_2^2) & -g_1 g_2 f_2^2 & & & & \\ & -g_1 g_2 f_2^2 & g_2^2(f_2^2+f_3^2) & -g_2 g_3 f_3^2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -g_{p-3}g_{p-2}f_{p-2}^2 & g_{p-2}^2(f_{p-2}^2+f_{p-1}^2) & -g_{p-2}g_{p-1}f_{p-1}^2 \\ & & & & -g_{p-2}g_{p-1}f_{p-1}^2 & g_{p-1}^2(f_{p-1}^2+f_p^2) \end{pmatrix}, \quad \text{(A1)}$$

$$4\mathcal{M}_r^2 = 4M^2_{(p,N+1)} = \begin{pmatrix} g_{p+1}^2(f_{p+1}^2+f_{p+2}^2) & -g_{p+1}g_{p+2}f_{p+2}^2 & & & & \\ -g_{p+1}g_{p+2}f_{p+2}^2 & g_{p+2}^2(f_{p+2}^2+f_{p+3}^2) & -g_{p+2}g_{p+3}f_{p+3}^2 & & & \\ & -g_{p+2}g_{p+3}f_{p+3}^2 & g_{p+3}^2(f_{p+3}^2+f_{p+4}^2) & -g_{p+3}g_{p+4}f_{p+4}^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -g_{N-2}g_{N-1}f_{N-1}^2 & g_{N-1}^2(f_{N-1}^2+f_N^2) & -g_{N-1}g_N f_N^2 \\ & & & & -g_{N-1}g_N f_N^2 & g_N^2(f_N^2+f_{N+1}^2) \end{pmatrix}, \quad \text{(A2)}$$

$$4\mathcal{M}_q^2 = 4M^2_{(q,K+1]} = \begin{pmatrix} g_{q+1}^2(f_{q+1}^2+f_{q+2}^2) & -g_{q+1}g_{q+2}f_{q+2}^2 & & & & \\ -g_{q+1}g_{q+2}f_{q+2}^2 & g_{q+2}^2(f_{q+2}^2+f_{q+3}^2) & -g_{q+2}g_{q+3}f_{q+3}^2 & & & \\ & -g_{q+2}g_{q+3}f_{q+3}^2 & g_{q+3}^2(f_{q+3}^2+f_{q+4}^2) & -g_{q+3}g_{q+4}f_{q+4}^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -g_{K-1}g_K f_K^2 & g_K^2(f_K^2+f_{K+1}^2) & -g_K g_{K+1}f_{K+1}^2 \\ & & & & -g_K g_{K+1}f_{K+1}^2 & g_{K+1}^2 f_{K+1}^2 \end{pmatrix}, \quad \text{(A3)}$$

$$4\mathcal{M}_L^2 = 4M^2_{(p,K+1]} = \begin{pmatrix} g_{p+1}^2(f_{p+1}^2+f_{p+2}^2) & -g_{p+1}g_{p+2}f_{p+2}^2 & & & & & \\ -g_{p+1}g_{p+2}f_{p+2}^2 & g_{p+2}^2(f_{p+2}^2+f_{p+3}^2) & -g_{p+2}g_{p+3}f_{p+3}^2 & & & & \\ & -g_{p+2}g_{p+3}f_{p+3}^2 & g_{p+3}^2(f_{p+3}^2+f_{p+4}^2) & -g_{p+3}g_{p+4}f_{p+4}^2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -g_{K-1}g_Kf_K^2 & g_K^2(f_K^2+f_{K+1}^2) & -g_Kg_{K+1}f_{K+1}^2 \\ & & & & -g_Kg_{K+1}f_{K+1}^2 & g_{K+1}^2f_{K+1}^2 \end{pmatrix}. \tag{A4}$$

APPENDIX B: MASS-SQUARED MATRIX INVERSE AND OTHER IDENTITIES

This appendix contains relationships between the couplings and f-constants of linear moose models with either gauged or global groups at their end points and the mass eigenvalues of these same mooses. The relationships are written as for mooses with $N+2$ groups, but can clearly be adapted for application to the submooses of varying length into which our analysis divides our general linear moose model.

Consider the vector-boson mass-squared matrices for the moose models (each with $N+2$ groups) shown in Figs. 6–9 Defining

$$\frac{1}{F^2} = \sum_{l=1}^{N+1} \frac{1}{f_l^2}, \tag{B1}$$

$$\frac{1}{F_i^2} = \sum_{l=i+1}^{N+1} \frac{1}{f_l^2}, \tag{B2}$$

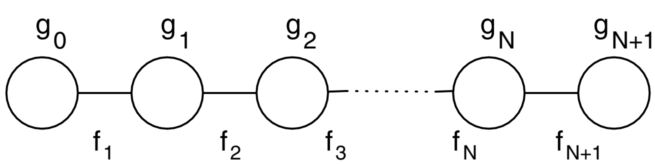

FIG. 6. Linear moose with vector-boson mass-squared matrix $M^2_{[0,N+1]}$.

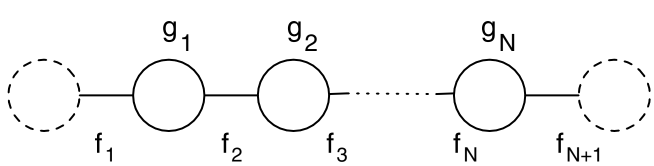

FIG. 7. Linear moose with vector-boson mass-squared matrix $M^2_{(0,N+1)}$.

$$\frac{1}{\tilde{F}_i^2} = \sum_{l=1}^{i} \frac{1}{f_l^2}, \tag{B3}$$

the elements of the corresponding nonsingular inverse mass-squared matrices may be written

$$\{M^{-2}_{[0,N+1)}\}_{i,j} = \frac{4}{g_i g_j F^2_{\max(i,j)}}, \tag{B4}$$

$$\{M^{-2}_{(0,N+1]}\}_{i,j} = \frac{4}{g_i g_j \tilde{F}^2_{\min(i,j)}}, \tag{B5}$$

$$\{M^{-2}_{(0,N+1)}\}_{i,j} = \frac{4F^2}{g_i g_j F^2_{\max(i,j)}\tilde{F}^2_{\min(i,j)}}. \tag{B6}$$

Denoting the eigenvalues of the matrix $M^2_{[0,N+1)}$ by $[\mathsf{m}^2_{[0,N+1)}]^\ell$, and similarly for the other matrices, we have the relations

$$\prod_{\ell=0}^{N}[\mathsf{m}^2_{[0,N+1)}]^\ell = \prod_{m=0}^{N} g_m^2 \prod_{p=1}^{N+1} \frac{f_p^2}{4}, \tag{B7}$$

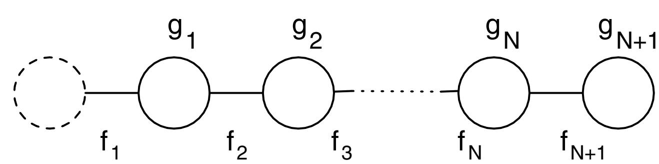

FIG. 8. Linear moose with vector-boson mass-squared matrix $M^2_{(0,N+1]}$.

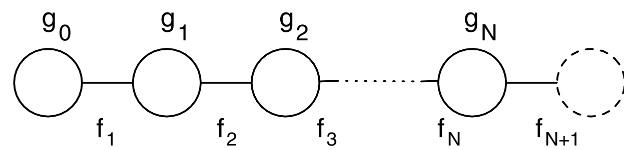

FIG. 9. Linear moose with vector-boson mass-squared matrix $M^2_{[0,N+1)}$.

$$\prod_{\ell=1}^{N+1}[\mathsf{m}^2_{(0,N+1]}]^\ell = \prod_{m=1}^{N+1} g_m^2 \prod_{p=1}^{N+1} \frac{f_p^2}{4}, \tag{B8}$$

$$\prod_{\ell=1}^{N}[\mathsf{m}^2_{(0,N+1)}]^\ell = \frac{4}{F^2}\prod_{m=1}^{N} g_m^2 \prod_{p=1}^{N+1} \frac{f_p^2}{4}, \tag{B9}$$

$$\prod_{\ell\neq 0}[\mathsf{m}^2_{[0,N+1]}]^\ell = \frac{1}{g^2}\prod_{m=0}^{N+1} g_m^2 \prod_{p=1}^{N+1} \frac{f_p^2}{4}, \tag{B10}$$

where the last product is understood to run only over the nonzero eigenvalues of $M^2_{[0,N+1]}$, and where we have defined

$$\frac{1}{g^2} = \sum_{m=0}^{N+1}\frac{1}{g_m^2}. \tag{B11}$$

It is interesting to note that the forms of Eqs. (B9) and (B10) are dual to one another (in the sense of exchanging couplings for f-constants [33], as discussed in the next Appendix E), while the equations in (B7) and (B8) are each self-dual in this sense.

One can apply these relations to illustrate that it is possible to ensure that the eigenstates of the full linear moose contain a light Z and W (and a massless photon) while the $\mathsf{m}_{Z\hat{k}}$, $\mathsf{m}_{W\hat{w}}$, $\mathsf{m}_{\hat{p}}$, m_r, and $\mathsf{m}_{\hat{q}}$ are all significantly more massive. Consider what happens to our general linear moose when $g_p, g_q \equiv 0$. As shown in Fig. 10, groups p and q become global, dividing the moose into three separate mooses of kinds represented in Figs. 7–9. Then we can rewrite the relationship between the couplings, f-constants and mass eigenvalues of the full moose in terms of mass eigenvalues of the submooses:

$$\begin{aligned} e^2 M_Z^2 \prod_{\hat{z}=1}^{K+1}\mathsf{m}^2_{Z\hat{z}} &= \prod_{\hat{i}=0}^{K+1} g_{\hat{i}}^2 \prod_{\hat{j}=1}^{K+1}\frac{1}{4}f_{\hat{j}}^2 \\ &= \left[\prod_{\hat{p}=1}^{p}\mathsf{m}^2_{\hat{p}}\right] g_p^2 \left[\prod_{\hat{q}=q+1}^{K+1}\mathsf{m}^2_{\hat{q}}\right] \\ &\quad\times g_q^2\left[\frac{1}{4}\bar{F}^2\prod_{r'=p+1}^{q-1}\mathsf{m}^2_{r'}\right], \end{aligned} \tag{B12}$$

where

$$\frac{1}{\bar{F}^2} = \sum_{i=p+1}^{q}\frac{1}{f_i^2}. \tag{B13}$$

Clearly, having g_p and g_q small is sufficient to enable the full moose to have a light Z eigenstate without having any of the $\mathsf{m}^2_{\hat{p}}$, $\mathsf{m}^2_{\hat{q}}$, or m^2_r be light. One can similarly show that small g_p is sufficient to enable the W to be light without having any of the $\mathsf{m}^2_{\hat{p}}$ or $\mathsf{m}^2_{r'}$ be light.

APPENDIX C: HEAVY Z AND W BOSON POLE RESIDUES

We collect here, for completeness, the residues of the poles corresponding to the Z' and W' bosons in the correlation functions derived in Secs. V, VI, and VIII.

1. Case I moose

Correlation function $[G_{NC}(Q^2)]_{WY}$ as written in Eq. (5.2) has the following heavy pole residue in case I

$$\begin{aligned} [\xi_{Zk}]_{WY} = e^2 \frac{M_Z^2}{\mathsf{m}^2_{Zk} - M_Z^2}&\left[\prod_{z\neq k}\frac{\mathsf{m}^2_{Zz}}{\mathsf{m}^2_{Zz}-\mathsf{m}^2_{Zk}}\right] \\ \times&\left[\prod_{\hat{p}=1}^{p}\frac{\mathsf{m}^2_{\hat{p}}-\mathsf{m}^2_{Zk}}{\mathsf{m}^2_{\hat{p}}}\right]\left[\prod_{m=1}^{M}\frac{\mathsf{m}^2_m-\mathsf{m}^2_{Zk}}{\mathsf{m}^2_m}\right]. \end{aligned} \tag{C1}$$

Correlation function $[G_{NC}(Q^2)]_{YY}$ as written in Eq. (6.4) has the following heavy pole residue in case I

$$\begin{aligned} [\xi_{Zk}]_{YY} = -e^2\frac{[\mathsf{m}^2_{Zk}-M_W^2]M_Z^2}{M_W^2[\mathsf{m}^2_{Zk}-M_Z^2]}&\left[\prod_{w=1}^{N}\frac{\mathsf{m}^2_{Ww}-\mathsf{m}^2_{Zk}}{\mathsf{m}^2_{Ww}}\right] \\ \times&\left[\prod_{z\neq k}\frac{\mathsf{m}^2_{Zz}}{\mathsf{m}^2_{Zz}-\mathsf{m}^2_{Zk}}\right]\left[\prod_{m=1}^{M}\frac{\mathsf{m}^2_m-\mathsf{m}^2_{Zk}}{\mathsf{m}^2_m}\right]. \end{aligned} \tag{C2}$$

Correlation functions $[G_{CC}(Q^2)]_{WW}$ and $[G_{NC}(Q^2)]_{WW}$ as written, respectively, in Eqs. (6.8) and (6.7) have the following heavy pole residues in case I

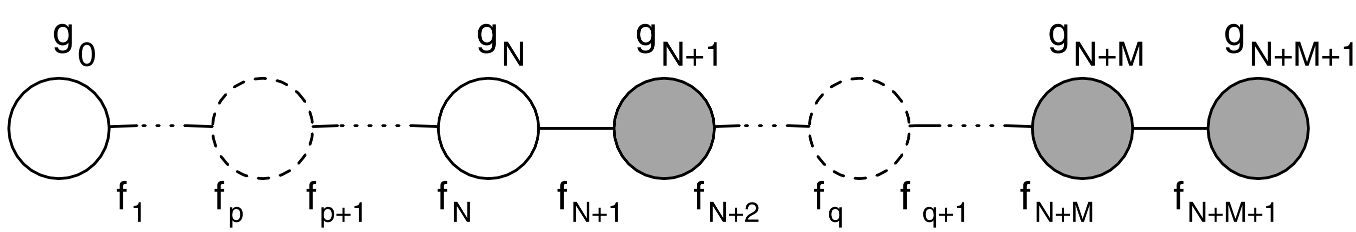

FIG. 10. General moose in limit $g_{p,q} \equiv 0$. The dashed circles represent the global symmetry groups remaining when $g_{p,q} \to 0$, and are to be understood to (each) belong to both the smaller moose to the immediate left and to the immediate right.

$$[\xi_{Wk}]_{WW} = \frac{4\sqrt{2}G_F M_W^2 \mathsf{m}_{Wk}^2}{[M_W^2 - \mathsf{m}_{Wk}^2]}\left[\prod_{w\neq k}\frac{\mathsf{m}_{Ww}^2}{\mathsf{m}_{Ww}^2 - \mathsf{m}_{Wk}^2}\right] \times\left[\prod_{\hat{p}=1}^{p}\frac{\mathsf{m}_{\hat{p}}^2 - \mathsf{m}_{Wk}^2}{\mathsf{m}_{\hat{p}}^2}\right]\left[\prod_{r=p+1}^{N}\frac{\mathsf{m}_r^2 - \mathsf{m}_{Wk}^2}{\mathsf{m}_r^2}\right], \tag{C3}$$

$$[\xi_{Zk}]_{WW} = -e^2\frac{M_Z^2[M_L^2 - \mathsf{m}_{Zk}^2]}{M_L^2[M_Z^2 - \mathsf{m}_{Zk}^2]}\left[\prod_{z\neq k}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - \mathsf{m}_{Zk}^2}\right] \times\left[\prod_{\hat{p}=1}^{p}\frac{\mathsf{m}_{\hat{p}}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{\hat{p}}^2}\right]\left[\prod_{l=1}^{K-p}\frac{\mathsf{m}_l^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_l^2}\right]. \tag{C4}$$

The difference of correlation functions $[G_{NC}(Q^2)]_{WW} - [G_{CC}(Q^2)]_{WW}$ as written in Eq. (6.17) has the following heavy pole residues in case I

$$[\xi_{Wk}]_{WW} = \frac{e^2 M_W^2 M_Z^2}{(M_W^2 - \mathsf{m}_{Wk}^2)(M_Z^2 - \mathsf{m}_{Wk}^2)} \times\left[\prod_{w\neq k}^{N}\frac{\mathsf{m}_{Ww}^2}{\mathsf{m}_{Ww}^2 - \mathsf{m}_{Wk}^2}\right]\left[\prod_{z=1}^{K}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - \mathsf{m}_{Wk}^2}\right] \times\left[\prod_{m=1}^{M}\frac{\mathsf{m}_m^2 - \mathsf{m}_{Wk}^2}{\mathsf{m}_m^2}\right]\left[\prod_{\hat{p}=1}^{p}\frac{(\mathsf{m}_{\hat{p}}^2 - \mathsf{m}_{Wk}^2)^2}{\mathsf{m}_{\hat{p}}^4}\right], \tag{C5}$$

$$[\xi_{Zk}]_{WW} = -\frac{e^2 M_W^2 M_Z^2}{(M_W^2 - \mathsf{m}_{Zk}^2)(M_Z^2 - \mathsf{m}_{Zk}^2)} \times\left[\prod_{w=1}^{N}\frac{\mathsf{m}_{Ww}^2}{\mathsf{m}_{Ww}^2 - \mathsf{m}_{Zk}^2}\right]\left[\prod_{z\neq k}^{K}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - \mathsf{m}_{Zk}^2}\right] \times\left[\prod_{m=1}^{M}\frac{\mathsf{m}_m^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_m^2}\right]\left[\prod_{\hat{p}=1}^{p}\frac{(\mathsf{m}_{\hat{p}}^2 - \mathsf{m}_{Zk}^2)^2}{\mathsf{m}_{\hat{p}}^4}\right]. \tag{C6}$$

2. General linear moose

Correlation function $[G_{NC}(Q^2)]_{WY}$ as written in Eq. (5.2) has the following heavy pole residue for the general linear moose

$$[\xi_{Zk}]_{WY} = e^2\frac{M_Z^2}{\mathsf{m}_{Zk}^2 - M_Z^2}\left[\prod_{z\neq k}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - \mathsf{m}_{Zk}^2}\right] \times\left[\prod_{\hat{p}=1}^{p}\frac{\mathsf{m}_{\hat{p}}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{\hat{p}}^2}\right]\left[\prod_{\hat{q}=q+1}^{K+1}\frac{\mathsf{m}_{\hat{q}}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{\hat{q}}^2}\right]. \tag{C7}$$

The correlation functions $[G_{NC}(Q^2)]_{WW}$ and $[G_{CC}(Q^2)]_{WW}$ are the same in the general moose as they were in case I. Therefore, their heavy pole residues are as given in the previous subsection.

Applying the consistency relations (3.22) to the WY and WW neutral-current heavy pole residues allows us to deduce the YY heavy pole residues:

$$[\xi_{Zk}]_{YY} = -\frac{e^2(M_W^2 - \mathsf{m}_{Zk}^2)M_Z^2}{M_W^2(M_Z^2 - \mathsf{m}_{Zk}^2)}\left[\prod_{w=1}^{N}\frac{\mathsf{m}_{Ww}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{Ww}^2}\right] \times\left[\prod_{z\neq k}^{K}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - \mathsf{m}_{Zk}^2}\right]\left[\prod_{m=1}^{M}\frac{\mathsf{m}_m^2}{\mathsf{m}_m^2 - \mathsf{m}_{Zk}^2}\right] \times\left[\prod_{\hat{q}=q+1}^{K+1}\frac{(\mathsf{m}_{\hat{q}}^2 - \mathsf{m}_{Zk}^2)^2}{\mathsf{m}_{\hat{q}}^4}\right]. \tag{C8}$$

Alternatively, we can read off the YY heavy pole residues from the correlation function $[G_{NC}(Q^2)]_{YY}$ as written in Eq. (8.4)

$$[\xi_{Zk}]_{YY} = -e^2\frac{[\mathsf{m}_{Zk}^2 - M_{W'}^2]M_Z^2}{M_{W'}^2[\mathsf{m}_{Zk}^2 - M_Z^2]}\left[\prod_{w=1}^{q-1}\frac{\mathsf{m}_{W'w}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{W'w}^2}\right] \times\left[\prod_{z\neq k}\frac{\mathsf{m}_{Zz}^2}{\mathsf{m}_{Zz}^2 - \mathsf{m}_{Zk}^2}\right]\left[\prod_{\hat{q}=q+1}^{K+1}\frac{\mathsf{m}_{\hat{q}}^2 - \mathsf{m}_{Zk}^2}{\mathsf{m}_{\hat{q}}^2}\right]. \tag{C9}$$

APPENDIX D: GENERALIZED CONSISTENCY CONDITIONS

Using the notation of Eqs. (8.9) and (8.10), the methods used to derive the consistency relations of Secs. VI D and VIII D can be used to prove the following general relations:

$$G^{p,p}_{[0,r]}(Q^2) - G^{p,p}_{[0,q)}(Q^2) = \frac{[G^{p,q}_{[0,r]}(Q^2)]^2}{G^{q,q}_{[0,r]}(Q^2)}, \qquad \text{for } 0 \le p < q \le r, \tag{D1}$$

$$G^{p,p}_{[0,r)}(Q^2) - G^{p,p}_{[0,q)}(Q^2) = \frac{[G^{p,q}_{[0,r)}(Q^2)]^2}{G^{q,q}_{[0,r)}(Q^2)}, \qquad \text{for } 0 \le p < q < r, \tag{D2}$$

$$G^{p,p}_{(0,r]}(Q^2) - G^{p,p}_{(0,q)}(Q^2) = \frac{[G^{p,q}_{(0,r]}(Q^2)]^2}{G^{q,q}_{(0,r]}(Q^2)}, \qquad \text{for } 0 < p < q \le r, \tag{D3}$$

$$G^{p,p}_{(0,r)}(Q^2) - G^{p,p}_{(0,q)}(Q^2) = \frac{[G^{p,q}_{(0,r)}(Q^2)]^2}{G^{q,q}_{(0,r)}(Q^2)}, \qquad \text{for } 0 < p < q < r. \tag{D4}$$

APPENDIX E: GOLDSTONE BOSONS, DUALITY, AND OSCILLATOR MODELS

1. Goldstone bosons

Consider an arbitrary $(K+2)$-site linear moose model at $O(p^2)$, given by

$$\mathcal{L}_2 = \frac{1}{4}\sum_{k=1}^{K+1} f_k^2 \mathrm{tr}((D_\mu U_k)^\dagger (D^\mu U_k)) - \sum_{k=0}^{K+1}\frac{1}{2}\mathrm{tr}(F^k_{\mu\nu}F^{k\mu\nu}), \tag{E1}$$

with

$$D_\mu U_k = \partial_\mu U_k - i g_{k-1} A^{k-1}_\mu U_k + i g_k U_k A^k_\mu, \tag{E2}$$

where all gauge fields A^k_μ $(k = 0, 1, 2, \cdots, K+1)$ are dynamical and we have used canonical normalization for the gauge fields.

The chiral fields U_k $(k = 1, 2, \cdots, K+1)$ may be written

$$U_k = \exp\left(\frac{2i\tilde{\pi}_k}{f_k}\right), \qquad \tilde{\pi}_k \equiv \frac{\pi^a_k \tau^a}{2}, \tag{E3}$$

where the π^a_k are the Goldstone boson fields and the τ^a are the usual Pauli matrices. Expanding the sigma-model kinetic terms Eq. (E1) to quadratic order in the fields, we find

$$\mathcal{L}_{KE} = \frac{1}{2}\sum_{k=1}^{K+1}\left[\partial_\mu \pi^a_k - \frac{f_k}{2}(g_{k-1}A^{a,k-1}_\mu - g_k A^{a,k}_\mu)\right]^2. \tag{E4}$$

Defining the vectors

$$\vec{\pi}^{\,a} = \begin{pmatrix} \pi^a_1 \\ \pi^a_2 \\ \vdots \\ \pi^a_{K+1} \end{pmatrix}, \qquad \vec{A}^a_\mu = \begin{pmatrix} A^{a,0}_\mu \\ A^{a,1}_\mu \\ \vdots \\ A^{a,K+1}_\mu \end{pmatrix}, \tag{E5}$$

in "link" and "site" space we see that the terms in Eq. (E4) may be written

$$\mathcal{L}_{KE} = \frac{1}{2}[\partial_\mu \vec{\pi}^a - (Q\cdot\vec{A}_\mu)^a]^T \cdot [\partial^\mu \vec{\pi}^a - (Q\cdot\vec{A}^\mu)^a]. \tag{E6}$$

Here the matrix Q is $(K+1)\times(K+2)$ dimensional, and may be written

$$Q = F\cdot D\cdot G, \tag{E7}$$

where F is the $(K+1)\times(K+1)$ matrix with diagonal elements $(f_1/2, f_2/2, \ldots, f_{K+1}/2)$, G is the $(K+2)\times(K+2)$ dimensional coupling-constant matrix with diagonal elements $(g_0, g_1, \ldots, g_{K+1})$, and D is the $(K+1)\times(K+2)$ dimensional difference matrix

$$D = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & & 1 & -1 \end{pmatrix}. \tag{E8}$$

Examining Eq. (E7), we see that the $(K+2)\times(K+2)$ vector meson mass matrix

$$M^2_K = \frac{1}{4}\begin{pmatrix} g_0^2 f_1^2 & -g_0 g_1 f_1^2 & & & & & \\ -g_0 g_1 f_1^2 & g_1^2(f_1^2+f_2^2) & -g_1 g_2 f_2^2 & & & & \\ & -g_1 g_2 f_2^2 & g_2^2(f_2^2+f_3^2) & -g_2 g_3 f_3^2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -g_{K-1}g_K f_K^2 & g_K^2(f_K^2+f_{K+1}^2) & -g_K g_{K+1} f_{K+1}^2 \\ & & & & -g_K g_{K+1} f_{K+1}^2 & g_{K+1}^2 f_{K+1}^2 \end{pmatrix}, \tag{E9}$$

may be written

$$M^2_K = Q^T Q. \tag{E10}$$

The pion kinetic energy terms in Eq. (E6) contain those which mix the Goldstone boson modes with the gauge bosons

$$\mathcal{L}_{\text{mixing}} = -\partial^\mu \vec{\pi}^{aT}\cdot(Q\cdot\vec{A}^a_\mu). \tag{E11}$$

Each massive vector-boson eigenstate "eats" a linear combination of Goldstone bosons, and the mixing terms are eliminated by the introduction of gauge-fixing

$$\mathcal{L}_\xi = -\frac{1}{2\xi}[\partial^\mu \vec{A}^a_\mu + \xi(Q^T\cdot\vec{\pi})^a]^T\cdot[\partial^\mu\vec{A}^a_\mu + \xi(Q^T\cdot\vec{\pi})^a], \tag{E12}$$

shown here for an arbitrary R_ξ gauge. From $\mathcal{L}_\xi$, we read off the mass matrix for the unphysical "eaten" Goldstone bosons

$$M^2_\xi = \xi Q Q^T \equiv \xi N^2_K. \tag{E13}$$

2. Duality

Consider the $(K+1)\times(K+1)$ dimensional matrix $N_K^2 = QQ^T$. Inspection of Eq. (E7) shows that N_K^2 may be interpreted as the boson mass matrix of a "dual" moose, in which the couplings and F-constants are interchanged (of course, some overall dimensionful constant will have to be factored out to interpret the F's as "coupling constants"—however, this factor will just be reabsorbed when we form N). Sfetsos [33] has shown that every nonzero eigenvalue of N_K^2 is also an eigenvalue of M_K^2.

Consider an eigenvector of N_K^2, $|\hat{l}_N\rangle$, with eigenvalue $M_{\hat{l}}^2 \neq 0$. Note that $|\hat{l}_N\rangle$ is a $K+1$-dimensional vector. Form the $K+2$ dimensional vector

$$|\hat{l}\rangle = \frac{1}{M_{\hat{l}}} Q^T |\hat{l}_N\rangle. \tag{E14}$$

Acting on $|\hat{l}\rangle$ with M_K^2 and regrouping terms we see that $|\hat{l}\rangle$ is an eigenvector of M_K^2 with eigenvalue $M_{\hat{l}}^2$. Correspondingly, given an eigenvector $|\hat{l}\rangle$ of M_K^2 with eigenvalue $M_{\hat{l}}^2 \neq 0$, we find that

$$|\hat{l}_N\rangle = \frac{1}{M_{\hat{l}}} Q|\hat{l}\rangle \tag{E15}$$

is an eigenvector of N_K^2 with the same eigenvalue. Therefore, all nonzero eigenvalues of N_K^2 and M_K^2 are equal.

From Eq. (E13) we see that the equality of the nonzero eigenvalues of N_K^2 and M_K^2 is equivalent to the statement that in 't-Hooft-Feynman gauge, where $\xi = 1$, the masses of the unphysical Goldstone bosons are equal to those of the massive vector bosons. Furthermore, comparing Eqs. (E11) and (E15), we see that the linear combination of Goldstone bosons eaten by a given vector-boson eigenstate is given by the eigenvector of N_K^2 with the same eigenvalue—and this relationship is true independent of the gauge-fixing parameter ξ.

The results above apply directly to the neutral-boson mass matrix in the general linear moose model. We note, however, that the construction of the unphysical Goldstone boson mass matrix proceeds analogously for an arbitrary linear moose, whether or not there are gauge fields at both end points. All that changes is the form of the matrix D in Eq. (E8). One finds that if the end point of the model is gauged the corresponding end point in the dual model corresponds to an global (ungauged) symmetry group, and vice versa. Again, one finds that nonzero eigenvalues for the linear moose and its dual are equal. Duality, therefore, applies to the charged-boson mass matrix as well.

3. Oscillator models

Consider a system of $K+2$ masses m_i, with adjacent masses coupled by $K+1$ massless springs with spring constant k_j (the jth spring couples masses m_{j-1} to m_j). We now demonstrate that, as shown by Georgi [32], the eigenfrequencies of this oscillator system are proportional to the eigenvalues of the vector-boson mass-squared matrix M_K^2 if we choose the masses $m_i \propto 1/g_i^2$ and the spring constants $k_i \propto f_i^2$.

Consider the equation of motion for an arbitrary mass m_i:

$$m_i \frac{d^2 x_i}{dt^2} = -k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i). \tag{E16}$$

If we introduce the vector

$$\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{K+1} \end{pmatrix}, \tag{E17}$$

the equations for the eigenmodes may be written

$$(D^T \cdot K \cdot D) \cdot \vec{x} = \omega^2 M \cdot \vec{x}. \tag{E18}$$

Here M is a $(K+2)\times(K+2)$ dimensional matrix with diagonal entries $(m_0, m_1, \cdots, m_{K+1})$, K is a $(K+1)\times(K+1)$ dimensional matrix with diagonal entries $(k_1, k_2, \cdots, k_{K+1})$, and D is the matrix of Eq. (E8).

Defining $\vec{v} = M^{1/2}\cdot\vec{x}$, we see that the eigenvalue equation can be written

$$[\tilde{Q}^T\tilde{Q}]\vec{v} = \omega^2\vec{v}, \tag{E19}$$

where

$$\tilde{Q} = K^{\frac{1}{2}} \cdot D \cdot M^{-1/2}. \tag{E20}$$

Comparing Eqs. (E7) and (E20), we see that the eigenvalues ω^2 of the spring system correspond to the vector-boson mass-squared values if we make the associations [32]

$$K^{1/2} \leftrightarrow F, \qquad M^{-1/2} \leftrightarrow G. \tag{E21}$$

A fixed boundary condition at either end of the spring system corresponds to taking the mass at that site to infinity, and hence the corresponding coupling in the linear moose to zero—corresponding to a global (ungauged) symmetry group at that site. The duality observed above for linear moose models implies a relation between oscillator models in which one exchanges the spring constants with the inverse of the masses. By reducing the analysis of the mass-squared matrices of the vector bosons to a coupled spring system, we may use physical intuition to understand which sets of couplings and f-constants that are phenomenologically acceptable.

Finally, we note that the eigenvalue equation [Eq. (E18)] may be rewritten

$$(D \cdot M^{-1} \cdot D^T) \cdot (K \cdot D \cdot \vec{x}) = \omega^2 K^{-1} \cdot (K \cdot D \cdot \vec{x}), \tag{E22}$$

and we see that all nonzero eigenvalues of the original oscillator equation are the same in an oscillator where one exchanges

$$M \leftrightarrow K^{-1}, \quad D \leftrightarrow D^T, \quad \vec{x} \leftrightarrow K \cdot D \cdot \vec{x}. \tag{E23}$$

These expressions are the oscillator analog of the duality noted in the previous subsection, and Eq. (E23) implies that the displacements of the dual oscillator are related to the Hooke's law forces experienced by each spring of the original oscillator.

APPENDIX F: CASE I WITH LARGE "BULK" COUPLING

The analyses presented in this paper are fairly abstract, and it is interesting to consider a special case in which the results can be checked analytically. In this appendix, we consider the case of large bulk coupling, i.e., $g_{0,K+1} \ll g_m$ for $0 < m < K+1$. In the case $M = 0$, and for fermions coupling to the weakly coupled groups at the ends of the moose ($p = 0$, $q = N+1$), this case has been analyzed previously in Ref. [32]. In this appendix we will find explicit formulas for the masses and "wave functions" of the W and Z bosons, and directly compute the parameters αS, αT, and $\alpha\delta$ to leading nontrivial order and obtain a connection between the formulas we have derived previously.

We should note at the outset that although this analysis is interesting for illustrative purposes, the requirement that the overall coupling be of electroweak strength [and hence $g_{0,K+1} = \mathcal{O}(1)$] and that the bulk couplings cannot be too large [$g_m < \mathcal{O}(4\pi)$] imply that corrections of subleading order are expected to be of order 10^{-2} and are therefore phenomenologically relevant.

1. Approximate mass eigenstates

We begin by considering what happens when $g_{0,K+1} \equiv 0$, as shown in Fig. 11. The remaining moose has global symmetry groups on both sides (formerly sites 0 and $K+1$), and the gauge groups from sites 1 to K. This moose has a massless particle—an exact Goldstone boson with decay constant

$$\frac{1}{F_Z^2} = \sum_{l=1}^{K+1} \frac{1}{f_l^2}. \tag{F1}$$

Therefore if we consider sites 0 and $K+1$ to be weakly gauged, the usual Higgs mechanism causes the mass of the Z boson to be

$$M_Z^2 \approx \frac{g_0^2 + g_{K+1}^2}{4} F_Z^2, \tag{F2}$$

to leading order in $g_{0,K+1}$.

If we similarly consider the linear moose for the charged currents (this includes all of the SU(2) gauged groups, and has the $N+1$ group as global), the charged-current moose has a massless particle—an exact Goldstone boson with decay constant

$$\frac{1}{F_W^2} = \sum_{l=1}^{N+1} \frac{1}{f_l^2}, \tag{F3}$$

and therefore

$$M_W^2 \approx \frac{g_0^2}{4} F_W^2. \tag{F4}$$

As in Appendix B, let us define

$$\frac{1}{F_i^2} = \sum_{l=i+1}^{K+1} \frac{1}{f_l^2}, \qquad \frac{1}{\tilde{F}_i^2} = \sum_{l=1}^{i} \frac{1}{f_l^2}. \tag{F5}$$

Note that these parameters satisfy

$$\frac{1}{F_i^2} + \frac{1}{\tilde{F}_i^2} \equiv \frac{1}{F_Z^2}, \qquad \frac{1}{F_0^2} = \frac{1}{\tilde{F}_{K+1}^2} \equiv \frac{1}{F_Z^2}, \tag{F6}$$

and

$$\frac{1}{\tilde{F}_{N+1}^2} \equiv \frac{1}{F_W^2}. \tag{F7}$$

Following the previous calculation, we begin by defining two vectors, $|a\rangle$ and $|b\rangle$, in the space of neutral gauge-boson states $(A_\mu^{3,0}, \ldots, A_\mu^{3,K+1})$. We define $|a\rangle$ by

$$\langle i|a\rangle = \frac{g_0}{g_i}\frac{F_Z^2}{F_i^2}, \qquad i = 0, \ldots, K, \qquad \langle i|a\rangle = 0, \qquad i = K+1, \tag{F8}$$

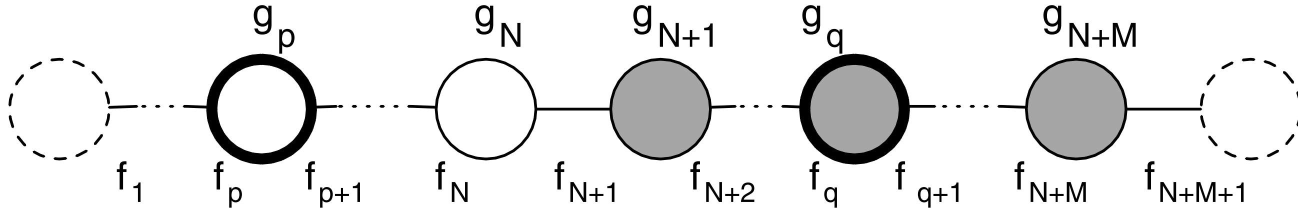

FIG. 11. The moose of Fig. 1 in limit $g_{0,K+1} \equiv 0$. The dashed circles represent the global symmetry groups remaining when $g_{0,K+1} \to 0$.

and $|b\rangle$ by

$$\begin{aligned} \langle i|b\rangle &= 0, \qquad i = 0, \\ \langle i|b\rangle &= \frac{g_{K+1}}{g_i}\frac{F_Z^2}{\tilde{F}_i^2}, \qquad i = 1, \ldots, K+1. \end{aligned} \tag{F9}$$

The vector $|a\rangle$ has the following properties: $\langle 0|a\rangle \equiv 1$, the amplitudes $\langle i|a\rangle$ for $i = 1, \ldots, K$ are suppressed by $g_0/g_i \ll 1$, and the vector satisfies "Dirichlet" boundary conditions (i.e., the amplitude vanishes) at site $K+1$. The vector $|b\rangle$ is precisely the same as $|a\rangle$, exchanging the moose left for right and g_0 with g_{K+1}. Also, to leading order in $g_{0,K+1}/g_i$, these vectors are normalized and orthogonal to one and other.

Denoting the neutral gauge-boson mass matrix by M_Z^2 (unfortunately, the same notation we have used for the Z-boson mass-squared, though the distinction should be clear from context), we find that

$$\langle a|M_Z^2|a\rangle = \frac{g_0^2 F_Z^2}{4}, \qquad \langle b|M_Z^2|b\rangle = \frac{g_{K+1}^2 F_Z^2}{4}, \tag{F10}$$

and

$$\langle a|M_Z^2|b\rangle = -\frac{g_0 g_{K+1} F_Z^2}{4}. \tag{F11}$$

To the order to which we have worked,

$$\frac{1}{e^2} \approx \frac{1}{g_0^2} + \frac{1}{g_{K+1}^2}, \tag{F12}$$

so we may define[8]

$$g_0 = \frac{e}{\tilde{s}}, \qquad g_{K+1} = \frac{e}{\tilde{c}}, \tag{F13}$$

where $\tilde{s}^2 + \tilde{c}^2 \equiv 1$. Using Eqs. (F10) and (F11), we immediately see that the state

$$|\gamma\rangle = \tilde{s}|a\rangle + \tilde{c}|b\rangle, \tag{F14}$$

is massless. From Eqs. (2.3), (F8), (F9), (F13), and (F14), we find that the photon couples to charge, $Q^\mu = J_3^\mu + J_Y^\mu$, with strength e.

The approximate Z eigenstate is the orthogonal combination

$$|z\rangle = \tilde{c}|a\rangle - \tilde{s}|b\rangle. \tag{F15}$$

Note that, using Eqs. (F10) and (F11), we may explicitly verify that

$$\langle z|M_Z^2|z\rangle = \frac{g_0^2 + g_{K+1}^2}{4} F_Z^2 \equiv \frac{e^2}{4\tilde{s}^2\tilde{c}^2} F_Z^2. \tag{F16}$$

Similarly, we find that the approximate W eigenstate is given by

$$\langle i|w\rangle = \frac{g_0}{g_i}\frac{F_W^2}{F_{W,i}^2}, \qquad i = 0, \ldots, N, \tag{F17}$$

where

$$\frac{1}{F_{W,i}^2} = \sum_{l=i+1}^{N+1} \frac{1}{f_l^2}. \tag{F18}$$

Note that $\langle 0|w\rangle \equiv 1$, and the amplitude at all other sites is suppressed by $g_0/g_i \ll 1$. It is easy to verify that

$$\langle w|M_W^2|w\rangle = \frac{g_0^2}{4} F_W^2 = \frac{e^2}{4\tilde{s}^2} F_W^2, \tag{F19}$$

where M_W^2 is the charged-boson mass matrix.

The Z coupling to fermions takes the form

$$\frac{e}{\tilde{s}\,\tilde{c}}\left(\frac{F_Z^2}{F_{W,p}^2} J_{3\mu} - J_{Q\mu}\left[\tilde{s}^2 - \left(1 - \frac{F_Z^2}{F_W^2}\right)\right]\right) Z^\mu, \tag{F20}$$

while the W coupling to fermions is

$$\frac{e}{\sqrt{2}\tilde{s}}\frac{F_W^2}{F_{W,p}^2} J_\pm^\mu W_\mu^\mp. \tag{F21}$$

In the above, we have used the relationships

$$\frac{1}{\tilde{F}_p^2} = \frac{1}{F_W^2} - \frac{1}{F_{W,p}^2}, \tag{F22}$$

$$\frac{1}{F_p^2} = \frac{1}{F_Z^2} - \frac{1}{F_W^2} + \frac{1}{F_{W,p}^2}, \tag{F23}$$

which derive from our previous definitions of the F-constants in Eqs. (F5) and (F18).

2. Low-energy charged- and neutral-currents

Low-energy W-exchange is given by

$$\mathcal{L}_W = -\frac{2F_W^2}{F_{W,p}^4} J_+^\mu J_{-\mu} \tag{F24}$$

and exchange of the heavy W bosons contributes approximately

$$\mathcal{L}_{Wh} = -\frac{1}{2} g_p^2 \{M_{(0,N+1)}^{-2}\}_{p,p} J_+^\mu J_{-\mu}, \tag{F25}$$

where we can calculate directly (see Appendix B)

$$\{M_{(0,N+1)}^{-2}\}_{p,p} = \frac{4}{g_p^2}\frac{F_W^2}{F_{W,p}^2 \tilde{F}_p^2}. \tag{F26}$$

Combining these results, we find

$$\mathcal{L}_{cc} = \mathcal{L}_W + \mathcal{L}_{Wh} = -\frac{2}{F_{W,p}^2} J_+^\mu J_{-\mu} \tag{F27}$$

[8]The distinction between $\tilde{s}$ defined here, and s as defined in our phenomenological analyses will become clear in what follows.

so that the definition of G_F is as we expect:

$$\sqrt{2}G_F = \frac{1}{F_{W,p}^2}. \tag{F28}$$

The J_3^2 portion of low-energy Z exchange is given by

$$\mathcal{L}_Z = -\frac{2F_Z^2}{F_{W,p}^4} J_3^\mu J_{3\mu} \tag{F29}$$

and the exchange of heavy Z bosons contributes approximately

$$\begin{aligned}\mathcal{L}_{Zh} = &-\frac{1}{2} g_p^2 \{M_{(0,K+1)}^{-2}\}_{p,p} J_3^\mu J_{3\mu} \\ &- g_p g_{N+1} \{M_{(0,K+1)}^{-2}\}_{p,N+1} J_3^\mu J_{Y\mu} \\ &- \frac{1}{2} g_{N+1}^2 \{M_{(0,K+1)}^{-2}\}_{N+1,N+1} J_Y^\mu J_{Y\mu}.\end{aligned} \tag{F30}$$

Computing the matrix elements as in Appendix B, and exchanging J_Y for $J_Q - J_3$, we can rewrite the low-energy effects of heavy Z exchange as

$$\begin{aligned}\mathcal{L}_{Zh} = &-\frac{2}{F_{W,p}^2}\left(1 - \frac{F_Z^2}{F_{W,p}^2}\right) J_3^\mu J_{3\mu} \\ &+ \frac{4}{F_{W,p}^2}\left(1 - \frac{F_Z^2}{F_W^2}\right) J_3^\mu J_{Q\mu} - \frac{2}{F_W^2}\left(1 - \frac{F_Z^2}{F_W^2}\right) J_Q^\mu J_{Q\mu}.\end{aligned} \tag{F31}$$

Note that if we combine the J_3^2 pieces of Eqs. (F30) and (F31) we find the total low-energy exchange in this channel is

$$-\frac{2}{F_{W,p}^2} J^{3\mu} J_{3\mu} \tag{F32}$$

and comparing this with Eq. (F27) shows that $\rho = 1$. In the limit $N + 1 = K + 1$ isospin is a good symmetry, $F_W = F_Z$ and the contributions from heavy boson exchange ($\mathcal{L}_{Wh,Zh}$) are proportional to $\vec{J}^{\mu 2}$.

For a phenomenologically viable model, we must require that the contributions from heavy boson exchange are small

$$\begin{aligned} 0 < \left(\frac{F_{W,p}^2}{F_W^2} - 1\right) \equiv A \ll 1, \\ 0 < \left(\frac{F_W^2}{F_Z^2} - 1\right) \equiv B \ll 1. \end{aligned} \tag{F33}$$

In this case we may expand the low-energy four-fermion operator in Eq. (F31) to lowest order in A and B. Doing so, we find

$$\mathcal{L}_{Zh} \approx -\frac{2}{F_Z^2}\left(\frac{F_{W,p}^2}{F_W^2} - 1\right) J_3^\mu J_{3\mu} - \frac{2}{F_Z^2}\left(\frac{F_W^2}{F_Z^2} - 1\right) J_Y^\mu J_{\mu Y}. \tag{F34}$$

Comparing this Lagrangian to the forms of the amplitudes, we find

$$\alpha T = -\left(\frac{F_W^2}{F_Z^2} - 1\right), \tag{F35}$$

$$\alpha\delta = 4s^2c^2\left(\frac{F_{W,p}^2}{F_W^2} - 1\right) > 0. \tag{F36}$$

In the limit $p = 0$, $F_{W,p} = F_W$ and the contributions from heavy boson exchange are proportional to $J_Y^\mu J_{Y\mu}$ as expected in the case of large isospin-violation.

3. Z-pole observables

The general analyses the formulas for αS, αT, and $\alpha\delta$ are given by Eqs. (7.8), (7.9), and (7.11):

$$\alpha S = 4s_z^2c_z^2M_Z^2[\Sigma_Z - \Sigma_p - \Sigma_q], \tag{F37}$$

$$\alpha T = s_z^2M_Z^2[\Sigma_Z - \Sigma_W + \Sigma_M - 2\Sigma_q], \tag{F38}$$

$$\alpha\delta = -4s_Z^2c_Z^2M_W^2[\Sigma_W - \Sigma_r - \Sigma_p]. \tag{F39}$$

To leading order in the strong bulk coupling expansion all of the mass-squareds appearing in Σ_W, Σ_Z, and Σ_r are large—and are proportional to squares of the bulk couplings. On the other hand, the $\Sigma_{p,q}$ have contributions proportional to $1/g_{0,K+1}^2$:

$$\Sigma_p = \frac{4}{g_0^2}\left(\frac{1}{F_W^2} - \frac{1}{F_{W,p}^2}\right), \tag{F40}$$

$$\Sigma_M = \frac{4}{g_{K+1}^2}\left(\frac{1}{F_Z^2} - \frac{1}{F_W^2}\right), \tag{F41}$$

while, since this is a model with $q = N + 1$, we also have $\Sigma_M = \Sigma_q$. As a result, we find[9]

$$\alpha S = -4F_Z^2\left[\tilde{s}^2\left(\frac{1}{F_W^2} - \frac{1}{F_{W,p}^2}\right) + \tilde{c}^2\left(\frac{1}{F_Z^2} - \frac{1}{F_W^2}\right)\right], \tag{F43}$$

$$\alpha T = \left(1 - \frac{F_W^2}{F_Z^2}\right) < 0, \tag{F44}$$

[9]To this order, we may substitute s_Z and c_Z for $\tilde{s}$ and $\tilde{c}$, and F_W for F_Z, in the coefficients of the expressions for αT and $\alpha\delta$. Also, for $A, B \ll 1$,

$$\left(\frac{F_W^2}{F_Z^2} - 1\right) \approx \left(1 - \frac{F_Z^2}{F_W^2}\right), \qquad \left(\frac{F_{W,p}^2}{F_W^2} - 1\right) \approx \left(1 - \frac{F_W^2}{F_{W,p}^2}\right), \tag{F42}$$

and the values of αT and $\alpha\delta$ are in agreement with the low-energy analysis of the previous section.

$$\alpha\delta = 4\tilde{s}^2\tilde{c}^2\left(1 - \frac{F_W^2}{F_{W,p}^2}\right), \tag{F45}$$

with the consequences

$$\alpha S - 4c^2\alpha T \approx -4F_W^2\tilde{s}^2\left(\frac{1}{F_W^2} - \frac{1}{F_{W,p}^2}\right) < 0 \tag{F46}$$

and

$$\hat{S} \propto \alpha S - 4c^2\alpha T + \frac{\alpha\delta}{c^2} \simeq 0. \tag{F47}$$

As shown by Eq. (9.12), the next order corrections to $\hat{S}$ are known to be positive.

Note that the requirement that $\alpha\delta$ be small implies that the constants f_i, $0 \leq i \leq p$, must be large.

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