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Electromagnetic Field of Rotating Charged Bodies

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The fact that the electromagnetic field of the Kerr-Newman solution of the Einstein-Maxwell equations is independent of the gravitational constant calls attention to and is illuminated by a problem of flat-space physics: the field of a rotating charged oblate ellipsoid of revolution of infinite conductivity and either (a) magnetic susceptibility of vacuum or (b) infinite magnetic susceptibility. These problems are solved for the case of interest, when the angular velocity is $\omega_0 = a\tilde{R}^{-2}$, where $\tilde{R}^2 = R^2 + a^2$, R and $\tilde{R}$ being the semiminor and semimajor radii of the ellipsoid, respectively. It is shown that for small ω_0 the conductive surface current [case (a)] or the volume magnetization [case (b)] contributes to the total magnetic moment with twice the value of that generated by the convective current of the surface charge q. The multipole expansion of the fields $F_{\mu\nu}$ [same for (a) and (b)] is obtained and shown to be formally identical to one of the generalized Deutsch solutions obtained in the Appendix. These are the exact solutions for the electromagnetic field of a rotating charged spherical perfect conductor with the magnetic susceptibility of the vacuum. Implications of these results for the understanding of properties of the Kerr-Newman black hole, such as the value 2 for the gyromagnetic factor, are analyzed.

I. INTRODUCTION

The Kerr-Newman metric[1,2] of a rotating charged body, in the outside empty space, was given by Carter[3] in Boyer-Lindquist coordinates[4] (t, r, θ, ϕ). It may be expressed as $(c=1)$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -A\tilde{d}t^2 + \tilde{d}r_\phi{}^2 + (r^2 + a^2\cos^2\theta)(A^{-1}\tilde{r}^{-2}dr^2 + d\theta^2)\,, \tag{1}$$

$$\tilde{d}t = \gamma(dt - v\,dr_\phi),\quad \tilde{d}r_\phi = \gamma(dr_\phi - v\,dt)\,, \tag{2a}$$

$$dr_\phi = \tilde{r}\sin\theta\, d\phi,\quad \gamma = (1 - v^2)^{-1/2}\,, \tag{2b}$$

with

$$\tilde{r} = (r^2 + a^2)^{1/2},\quad v = (a\sin\theta)/\tilde{r} \tag{3}$$

and

$$A = 1 - G(2mr - q^2)\tilde{r}^{-2}\,. \tag{4}$$

The constants m, q, and $J = am$ are the mass, charge, and angular momentum of the body, respectively, and G is the gravitational constant. This solution is valid for $r > R$, the "radius" of the object. If $G^2m^2 > Gq^2 + a^2$ then R has a minimum possible value at r_H, the radius of the horizon where A vanishes, in which case the object is a black hole[3]

$$r_H = Gm + (G^2m^2 - Gq^2 - a^2)^{1/2}\,. \tag{5}$$

No internal solution has been obtained for this metric.

It is interesting to notice that ds^2 takes the diagonal form (1) with the help of the local Lorentz transformations (2). This system of Lorentz observers (not a coordinate system) which are instantly moving with velocity v given by (3) in the direction $\hat{\phi}$ define the angular velocity (or v^ϕ in curvilinear coordinates):

$$\omega = v^\phi = a\tilde{r}^{-2},\quad (v^\theta = v^r = 0)\,. \tag{6}$$

In what follows we shall assume that the body inside the surface $r = R$ is rigidly rotating with the constant angular velocity

$$\omega_0 = a\tilde{R}^{-2} = a(R^2 + a^2)^{-1}\,. \tag{7}$$

For a black hole $(R = r_H)$ this coincides with the expression

$$\omega_H = \frac{\partial m}{\partial J} = a(r_H{}^2 + a^2)^{-1} \tag{8}$$

obtained by Christodoulou and Ruffini.[5]

It has been shown that the surfaces r = constant are not spherical.[6] This is the case even in flat space $(G = 0, A = 1)$. Indeed Eq. (1) becomes

$$ds^2 = -dt^2 + (r^2 + a^2\cos^2\theta)(\tilde{r}^{-2}dr^2 + d\theta^2) + \tilde{r}^2\sin^2\theta\, d\phi^2 \tag{9}$$

which takes Minkowski form if we define

$$x + iy = \tilde{r}\sin\theta\, e^{i\phi},\quad z = r\cos\theta\,. \tag{10}$$

Thus (r, θ, ϕ) are spheroidal coordinates and the surfaces r = constant are oblate ellipsoids of revolution around the z axis, along which is the semiminor axis r, $\tilde{r}$ being the semimajor axis. In particular our rotating charged body is also an oblate ellipsoid (in flat space), not a spherical body.

The components of the electromagnetic field $F_{\mu\nu}$ of the (external) Kerr-Newman solution are[3]

$$E_r = F_{10} = q(r^2 - a^2\cos^2\theta)\rho^{-4},$$
$$E_\theta = F_{20} = -aF_{23}\bar{r}^{-2}, \tag{11}$$
$$\bar{B}^r = F_{23} = qar\bar{r}^2\rho^{-4}\sin 2\theta,$$
$$\bar{B}^\theta = F_{31} = aE_r\sin\theta, \tag{12}$$
$$E_\phi = F_{30} = 0, \quad \bar{B}^\phi = F_{12} = 0, \tag{13}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad -g = -\det|g_{\mu\nu}| = \rho^4\sin^2\theta, \tag{14}$$

and, as in what follows, a bar indicates multiplication by $\sqrt{-g}$

$$\bar{f} = f\sqrt{-g}. \tag{15}$$

Similarly for $F^{\mu\nu}$ (and $\bar{F}^{\mu\nu}$)

$$D^r = F^{01} = E_r\bar{r}^2\rho^{-2}, \qquad D^\theta = F^{02} = E_\theta\rho^{-2}, \tag{16}$$

$$H_r = \bar{F}^{23} = B^r\rho^2\bar{r}^{-2}, \qquad H_\theta = \bar{F}^{31} = B^\theta\rho^2, \tag{17}$$

$$D^\phi = F^{03} = 0, \qquad H_\phi = \bar{F}^{12} = 0. \tag{18}$$

The interesting characteristic of this solution is that, although the $g_{\mu\nu}$'s depend on the gravitational constant G, the electromagnetic field components $(F_{\mu\nu}, F^{\mu\nu}, \bar{F}^{\mu\nu})$ and the "3-vectors" (E_i, B^i, H_i, D^i) defined by Eqs. (11)–(18) are *all independent of G*. Thus it is a solution even for a zero value of the gravitational constant (flat space). In Sec. II we shall find the internal electromagnetic field which matches this external field for a rotating charged body in flat space. We consider this as a promising step towards finding the internal, metric and electromagnetic, Kerr-Newman solution for such a body.

II. ELECTROMAGNETIC FIELD OF A ROTATING CONDUCTOR IN FLAT SPACE

We assume that the Kerr-Newman electromagnetic field (11)–(18) is the external solution ($r \geq R$) in flat space (spheroidal coordinates). We wish to extend this solution to the internal region of the rotating body whose electromagnetic properties we also wish to determine. The approach adopted here differs in one important respect from that employed in everyday problems of electricity and magnetism. There one knows the physical situation and seeks the electromagnetic field to which it gives rise. Here we take the external field to be known and ask what kind of a physical situation can give rise to such a field. We shall obtain indications of such properties using again the system of local Lorentz observers defined by Eq. (6). Thus if we define v_L^μ (not the four-velocity) as related to the four-velocity of the Lorentz local rotating observer as

$$\left(\frac{dx^\mu}{d\tau}\right)_L = v_L^\mu\left(\frac{dt}{d\tau}\right)_L, \quad v_L^\mu = (1, 0, 0, \omega). \tag{19}$$

We find immediately from (11)–(13) that

$$F_{2\mu}v_L^\mu = F_{20} + \omega F_{23} = E_\theta + \omega\bar{B}^r = 0, \tag{20a}$$
$$F_{3\mu}v_L^\mu \equiv 0. \tag{20b}$$

Equations (20) state that the force on a charged test particle at rest in the instantaneously rotating local Lorentz frame is purely "radial" (for any r). More precisely it is orthogonal to the ellipsoidal surface r = constant. In particular this is true for the surface ($r = R$) of the rotating body. Therefore we *may* assume the body to be a perfect conductor, for which the fields inside the body must satisfy in flat space

$$\vec{E} + \vec{v}\times\vec{B} = 0, \tag{21}$$

where $\vec{v}$ is the velocity of rotation of the (rigid) conductor at the point considered:

$$\vec{v} = \vec{\omega}_0 \times \vec{r}' \quad \text{(Cartesian coordinates)}. \tag{22}$$

Condition (21) indeed implies that only the component of $F_{\mu\nu}v_L^\nu$ normal to the surface may not vanish for an external point near the surface.

A quite unexpected result is obtained if we examine the expression $^*F_{\mu\nu}v_L^\nu$, where the dual electromagnetic field $^*F_{\mu\nu}$ is given by

$$^*F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{F}^{\rho\sigma}, \quad ^*F_{0i} = H_i, \quad ^*F_{ij} = \bar{D}^k. \tag{23}$$

That expression is proportional to the force acting on a fictitious magnetic monopole moving with velocity v_L^i (or instantaneously at rest in the local Lorentz frame):

$$^*F_{i\nu}v^\nu = -H_i + (\vec{v}\times\vec{D})_i. \tag{24}$$

Thus we find

$$^*F_{2\mu}v_L^\mu = -H_\theta + \omega\bar{D}^r = 0, \tag{25a}$$
$$^*F_{3\mu}v_L^\mu \equiv 0. \tag{25b}$$

Therefore also this force is purely "radial." In other terms, we have shown that both the electric and the magnetic fields as measured by the local

Lorentz observers, whose velocities are given by (6), are normal to the surface $r=$ constant and therefore are *parallel to each other*.

The magnetic monopole being only a helpful fiction, we cannot assume that the body is also a "magnetic conductor." Instead we find that *if* the body has infinite magnetic susceptibility (isotropic perfect ferromagnet), thus satisfying for interior points (in flat space)

$$\vec{H}-\vec{v}\times\vec{D}=0\,, \tag{26}$$

then relations (25) at $r=R$ (outside) must result if the surface current is purely convective,

$$\vec{J}_S=\sigma\vec{v}_L(R) \tag{27}$$

(and thus vanishes in the local Lorentz frame).

It should be emphasized that neither condition (21) nor (26) is compulsory. Indeed we shall consider several possibilities for the internal solution. Before examining the internal solution we shall examine the electromagnetic potentials of the external field. It can be easily verified that the components $F_{\mu\nu}$ given by Eqs. (11)–(13) derive from the four-potential:

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}\,, \tag{28a}$$

where

$$A_r=A_\theta=0,\qquad A_0=-qr\rho^{-2},\qquad A_\phi=-A_0 a\sin^2\theta. \tag{28b}$$

We may also introduce, instead of the vector potential A_i, the scalar magnetic potential

$$E_j+iH_j=(A_0+i\Phi)_{,j}\,, \tag{29a}$$

$$A_0+i\Phi=-q(r-ia\cos\theta)^{-1}. \tag{29b}$$

It will be also helpful to remember that the "potential"

$$\chi=A_\mu v_L^\mu=-qr\tilde{r}^{-2}=\chi(r) \tag{30}$$

is constant at the surfaces $r=$ constant which are normal to the lines of force $F_{i\nu}v_L^\nu$ and $^*F_{i\nu}v_L^\nu$.

III. INTERNAL ELECTROMAGNETIC FIELD

The internal four-potential must be adjusted with continuity to the external one given by Eq. (28b). We cannot use spheroidal coordinates in the interior region where Eq. (9) is not valid. Thus we rewrite (28b) in Minkowski coordinates with $\vec{r}'=(x,y,z)$ and $\vec{\nabla}=\partial/\partial\vec{r}'$. For $r>R$ we have

$$A_0=\chi(r)(1-a^2\tilde{r}^{-2}\sin^2\theta)^{-1}=\chi(r)(1-\omega^2\vec{\rho}^{\,2})^{-1}\,, \tag{31}$$

with

$$\vec{\rho}^{\,2}=x^2+y^2=\tilde{r}^2\sin^2\theta=r'^2\sin^2\theta'\,, \tag{32a}$$

$$r'^2=|\vec{r}'|^2=r^2+a^2\sin^2\theta\,, \tag{32b}$$

and $\omega=\omega(r)$ given by (6) and $\chi(r)$ by (30). The hybrid form (31), depending both on the spheroidal coordinate r and on the Cartesian coordinates $\vec{r}'$, will be convenient for adjustment to the internal solution at $r=R$. We also find for $r>R$

$$\vec{A}=-A_0\vec{v}_L=-A_0\vec{\omega}\times\vec{\rho}\,, \tag{33a}$$

where

$$\vec{\rho}=(x,y,0)\,. \tag{33b}$$

We can guess that the internal solution is still given by (31) and (33) with R instead of r:

$$A_0=\chi(R)(1-\omega_0^{2}\vec{\rho}^{\,2})^{-1}=\chi(R)(1-v^2)^{-1}\,, \tag{34a}$$

$$\vec{A}=-A_0\vec{v}=-A_0\vec{\omega}_0\times\vec{\rho}\,. \tag{34b}$$

These solutions match the external ones at the surface ($r=R$). Thus we find for the fields $\vec{E},\vec{B}$ in the internal region

$$\vec{E}=\vec{\nabla}A_0=2A_0\omega_0^{2}\vec{\rho}(1-v^2)^{-1}=-\vec{v}\times\vec{B}, \tag{35a}$$

$$\vec{B}=\mathrm{curl}\vec{A}=-2A_0\vec{\omega}_0(1-v^2)^{-1}=v^{-2}\vec{v}\times\vec{E}\,. \tag{35b}$$

Therefore condition (21) is indeed satisfied. Actually (34)–(35) are possibly the simplest of an infinite set of internal solutions which can be matched to the external one.

Thus we shall assume that the body is a perfect conductor and we are left with the alternative of imposing (26) or not. We consider now the following two cases.

a. Perfect conductor with the susceptibility of the vacuum. Maxwell's equations

$$\mathrm{div}\vec{E}=4\pi\rho',\quad \mathrm{curl}\vec{B}=4\pi\vec{j} \tag{36}$$

lead to the charge and current volume densities

$$\rho'=-\frac{q}{\pi}R\tilde{R}^{-2}\omega_0^{2}(1-v^2)^{-2}(1+2v^2)\,, \tag{37a}$$

$$\vec{j} = 2\rho' \vec{v}(1+2v^2)^{-1} . \tag{37b}$$

Similarly the surface charge and current density are obtained from the discontinuities of the normal component of $\vec{E}/(4\pi)$ and of the tangential component of $\vec{B}/(4\pi)$. This solution corresponds to the Deutsch solution[6] for a rotating spherical conductor with a magnetic moment produced by conduction currents. Also here, since $\vec{j}_S \neq \sigma\vec{v}$ and $\vec{j} \neq \rho\vec{v}$, we have conduction currents besides the convective ones. This solution is very similar to the case derived in the Appendix of a generalized Deutsch solution for a conductor with a magnetic moment μ and a charge $q = \mu(\omega_0 R^2)^{-1}$.

b. Perfect conductor with infinite magnetic susceptibility. Here we still have A_μ and $F_{\mu\nu}$ given by (34) and (35) (inside the conductor). Thus we find the distribution of charges, currents, magnetization, and polarization densities using Eq. (26) and Maxwell's equations (in flat space with the coordinates appropriate to an inertial frame):

$$\operatorname{div}\vec{D} = 4\pi\rho', \quad \vec{D} = \vec{E} + 4\pi\vec{P}, \tag{38a}$$

$$\operatorname{curl}\vec{H} = 4\pi\vec{j}, \quad \vec{H} = \vec{B} - 4\pi\vec{M} . \tag{38b}$$

We must find $\vec{H}$ and $\vec{D}$. A further simplification comes from the fact that we are assuming for the conductor the dielectric constant of vacuum. Thus the polarization $\vec{P}$ is induced by rotation from the magnetization $\vec{M}_0$ (measured in the comoving frame) and we must have

$$\vec{P} = -\vec{v}\times\vec{M}, \quad \vec{M} = M\hat{\omega}_0 . \tag{39}$$

The expression for M can be obtained from (26), (38), and (39). Thus

$$\vec{B} = \vec{v}\times\vec{E} + 4\pi\vec{M}(1+v^2)$$

and using (35)

$$4\pi\vec{M} = \vec{B}(1-v^2)(1+v^2)^{-1} . \tag{40}$$

Thus we find

$$M = M_0(1-v^4)^{-1} , \tag{41a}$$

$$4\pi M_0 = qaR\tilde{R}^{-4} \tag{41b}$$

and

$$\begin{aligned}\vec{H} &= 8\pi\vec{M}_0 v^2(1-v^2)^{-1}(1-v^4)^{-1} \\ &= O(\omega_0{}^3R^3) ,\end{aligned} \tag{42a}$$

$$\begin{aligned}\vec{D} &= -|\vec{H}|\,\omega_0\vec{\rho}v^{-2} \\ &= O(\omega_0{}^2R^2) .\end{aligned} \tag{42b}$$

Maxwell's equations (38) give now for the volume charge and current densities

$$\begin{aligned}\rho' &= -2|\vec{H}|\,\omega_0(1+v^4-2v^6)v^{-2} \\ &= O(\omega_0{}^2R^2) ,\end{aligned} \tag{43a}$$

$$\begin{aligned}\vec{j} &= \rho'\vec{v} \\ &= O(\omega_0{}^3R^3) .\end{aligned} \tag{43b}$$

Similarly we may obtain the surface densities from the discontinuities at the surface $r = R$ of the normal component of $\vec{D}$ and the tangential component of $\vec{H}$. Actually as in the comoving frame the magnetic field at the surface ($\vec{H}'$) has no tangential components either inside or outside the body, the surface current must vanish in that frame, in the same way as the volume current density (42b) vanishes. Thus, although conductive currents might exist in the comoving frames (as the conductivity was assumed to be infinite) they actually vanish. We have purely convective currents both at the surface and inside the body.

IV. MULTIPOLE EXPANSION

We shall consider here the multipole expansion of the external potentials up to terms of the order $(a/r')^4$ with r' given by (32b). Thus we invert that relation as

$$\begin{aligned}r^2 = r'^2 - a^2 + \frac{(\vec{a}\cdot\vec{r}')^2}{r'^2} + \frac{a^2(\vec{a}\cdot\vec{r}')^2}{r'^4} \\ - \frac{(\vec{a}\cdot\vec{r}')^4}{r'^6} + O((a/r')^6) ,\end{aligned} \tag{44a}$$

where we have also used

$$\begin{aligned}\vec{a}\cdot\vec{r} &= ar\cos\theta = \vec{a}\cdot\vec{r}' \\ &= ar'\cos\theta' .\end{aligned} \tag{44b}$$

We find (for $r > R$) the multipole expansions

$$\begin{aligned}-A_0 = \frac{q}{r'} - \frac{qa^2(3\cos^2\theta - 1)}{2r'^3} \\ + \frac{qa^4(\frac{35}{2}\cos^4\theta' - 15\cos^2\theta' + \frac{3}{2})}{4r'^5} ,\end{aligned} \tag{45a}$$

$$-\Phi = \frac{qa\cos\theta'}{r'^2} - \frac{(5\cos\theta' - 3)\cos\theta'\,qa^3}{2r'^4} . \tag{45b}$$

The angle-dependent terms here are proportional to the spherical harmonics Y_{lm} (for $m = 0$) as should be expected. The coefficients indicate the strength of the multipoles up to electric octupole. In particular the magnetic moment is equal to $q\vec{a}$. The internal four-potential becomes, to the present order of approximation,

$$A_0 = -q/R + qa^2R^3(1-\vec{\rho}^2/R^2) , \tag{46a}$$

$$\vec{A} = (q/R^3)\vec{a}\times\vec{\rho} . \tag{46b}$$

We are indebted to Christodoulou and Ruffini for contact with work unpublished[5] as of this writing, concerned with the multipole expansion of the external Kerr-Newman electromagnetic field. Dif-

ficulties arose in this work, associated with the presentation of tensors in nonorthogonal coordinate systems in curved space. The problem recalled to the author earlier work of Tamm which offers a simple way out of these difficulties and has led to the present work. Tamm[7] provides a "flat-space formulation" of Maxwell's equations in curved space. In this formulation we consider the electromagnetic field components $\bar{D}^r$, $\bar{B}^r$ given by (11)–(18). They can be expanded in multipoles up to electric octupole (for $r>R$) as

$$\frac{\bar{D}^r}{r^2\sin\theta}=\frac{q}{r^2}-\frac{qa^2(3\cos^2\theta-1)}{r^4}, \tag{47a}$$

$$\frac{\bar{D}^\theta}{r^2\sin\theta}=-\frac{2qa^2\cos\theta}{r^5}, \tag{47b}$$

$$\bar{D}^\phi=0, \tag{47c}$$

$$\frac{\bar{B}^r}{r^2\sin\theta}=\frac{2qa\cos\theta}{r^3}, \tag{48a}$$

$$\frac{\bar{B}^\theta}{r^2\sin\theta}=\frac{qa\sin\theta}{r^4}, \tag{48b}$$

$$\bar{B}^\phi=0. \tag{48c}$$

These expressions, valid in *curved* space, are *formally* identical with those derived from the potentials (45) in *flat space* except that we have here r and θ instead of r' and θ'. They are also *formally* identical with a particular case ($\mu=q\omega_0R^2$) of the generalized Deutsch solution[8] derived in the Appendix as quoted in Ref. 5. However, (r,θ,ϕ) are spherical coordinates in the last case, this not being the case in (47)–(48).

V. DISCUSSION

We have found two solutions of Maxwell's equations in flat space for a rotating charged conductor, which outside the body are identical to the electromagnetic field of the Kerr-Newman solution of the Einstein-Maxwell equations. The internal solution $F_{\mu\nu}$ is found and is the same for the two cases considered for the magnetic susceptibility of the body: (a) infinite, (b) equal to that of vacuum. $\vec{D}$ and $\vec{H}$ are now different in the two cases, as well as the distributions of charges and currents. In case (a) we found the charge-current distribution. In case (b) the magnetization, charges, and convective currents were obtained.

We turn now again to the Kerr-Newman solution in curved space ($G\neq0$) in order to understand the origin of the value two for the gyromagnetic factor.[3] We consider two possibilities: (i) a solution for which the Kerr-Newman form is valid in the external region even when the gravitational field is weak ($Gm\ll R$); (ii) solutions for which the external Kerr-Newman form results only in the limiting case of a black-hole ($R\to r_H$).

(i) Let us consider case (b) of a conductor with infinite magnetic susceptibility. If ω_0 (or a) is very small, the departure from sphericity is negligible and the charge is nearly uniformly distributed on the surface. The purely convective current leads to the value $\vec{\mu}_q$ for the corresponding magnetic moment:

$$\vec{\mu}_q=\tfrac{1}{3}qR^2\vec{\omega}_0\approx\tfrac{1}{3}q\vec{a}. \tag{49a}$$

But the uniform magnetization (41) produces an additional magnetic moment $\vec{\mu}_M$, exactly twice $\vec{\mu}_q$. Thus

$$\vec{\mu}=q\vec{a} \tag{49b}$$

gives the value of the total magnetic moment.

If we now add a distribution of masses to this body we find in the weak-field case for the angular momentum even in the most favorable case of uniform distribution on the surface (we should not expect the mass to be concentrated as a ring along the equator):

$$\begin{aligned}\vec{a}_{\rm orb}&=\vec{L}/m\\&=\tfrac{2}{3}\vec{\omega}_0R^2.\end{aligned} \tag{50a}$$

But the Kerr-Newman solution (1) gives a larger value:

$$\vec{J}/m=\vec{a}\cong\vec{\omega}_0R^2, \tag{50b}$$

which compared to (49b) leads to $g=2$. Thus in such cases the difference must also be made up by spin polarization. The same problem arises in case (a) where a conductive surface current plays the role of the magnetization here.

(ii) A completely different approach to this problem comes from the analysis of the problem of a rotating charged shell for small q and ω_0, which shall be done elsewhere.[9] Here we do not impose the Kerr-Newman solution for weak fields and have in such case (49a) and (50a) for $\vec{\mu}$ and $\vec{a}$, with $g=1$. If the radius of the shell is made to converge to the radius of the horizon the strong gravitational field induces via the nonlinear terms in Maxwell's and Einstein's equations additional effective currents,[9] which lead to expressions (49b), (50b), with $g=2$ in the limiting case. Of course in cases (a) and (b) the additional contribution to $\vec{a}_{\rm orb}$ may also arise only in the limit $(R\to r_H)$ of strong gravitational fields.

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APPENDIX: Generalized Deutsch Solution of Maxwell's Equations

Deutsch's exact solution[8] for a rotating neutral nonmagnetic conducting sphere with nonvanishing magnetic moment $\vec{\mu}$ produced by internal currents ($\vec{\mu}$ parallel to $\vec{\omega}_0$) is given for the external region by expressions (47)–(48) with the substitutions:

$q \to 0$ in the Coulomb term,
$qa \to \mu$ in the magnetic dipole term,
$qa^2 \to \mu\omega_0 R^{-2}$ in the electric quadrupole term.

(r, θ, ϕ) are now spherical coordinates. In the internal region the simplest Deutsch solution is given by[10]

$$\vec{B} = 2\vec{\mu}/R^3, \quad \vec{D} = -\vec{v}\times\vec{B}. \tag{A1}$$

The electric field $\vec{D}$ can be split into a monopole plus a quadrupole field

$$\vec{D} = -\frac{4}{3R^3}\,\omega_0\mu\vec{r} + \frac{2}{R^3}\left[\vec{\mu}(\vec{\omega}_0\cdot\vec{r}) - \tfrac{1}{3}\,\omega_0\mu\vec{r}\right]. \tag{A2}$$

The volume charge and current densities are

$$\rho' = -\omega_0\mu/(\pi R^3), \tag{A3}$$

$$\vec{j} = 0. \tag{A4}$$

Thus a charge $q' = -4\omega_0\mu/3$ is uniformly distributed on the volume and its convective current $\rho'\vec{v}$ is exactly cancelled by a conduction current. The surface charges and current densities are

$$\sigma = -\frac{q'}{4\pi R^2} + \frac{\omega_0 R\mu(3\cos^2\theta - 1)}{12\pi}, \tag{A5}$$

$$\vec{j}_S = \frac{3\mu\vec{v}}{4\pi R^3 v}. \tag{A6}$$

Again write

$$\vec{j}_S = \sigma\vec{v} + \vec{j}'_S \tag{A7}$$

and find that the conduction current $\vec{j}'_S$ must cancel exactly the convection currents $\sigma\vec{v}$, leaving the net result (A6).

To extend Deutsch's solution to the case of arbitrary charge on the conductor it is only necessary to add to the previous solutions the fields:

$$\begin{aligned} \delta\vec{D} &= q\vec{r}/r^3 \quad (r>R), \\ \delta\vec{D} &= 0 \quad (r<0), \\ \delta\vec{B} &= 0 \quad (0<r<\infty), \end{aligned} \tag{A8}$$

or the additional charge-current distribution

$$\begin{aligned} \delta\rho &= \delta\vec{j} = 0, \\ \delta\sigma &= q/(4\pi R^2), \\ \delta\vec{j}_S &= \vec{v}\delta\sigma + \delta\vec{j}'_S = 0. \end{aligned} \tag{A9}$$

The particular case mentioned in the text corresponds to

$$\vec{\mu} = q\vec{\omega}_0 R^2. \tag{A10}$$

This solution is very similar to that of case (a) for the ellipsoidal conductor. In both cases (small ω_0) the conduction surface current contribution to the magnetic moment is double that of the convective current of the charge.

The total energy of our generalized Deutsch solution is

$$W = W_D + q^2/(2R), \tag{A11}$$

where W_D is the (positive) energy corresponding to Deutsch's solution ($q = 0$).

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*Part of this work was done while the author was at Princeton University.

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