Dynamical Model for an Inelastic Resonance*

Philip W. Coulter
Department of Physics, The University of Alabama, University, Alabama 35486
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A dynamical model of a resonance interfering with an inelastic background is presented. The scattering amplitude is written as a resonant term plus a unitary background. The width of the resonance is found to be a function of the background scattering amplitude. The case of two resonances interfering with each other and with an inelastic background is also studied. Implications of the model with regard to determining masses and widths of resonances and coupling constants in unitary symmetry schemes are discussed.

I. INTRODUCTION

Obtaining reliable values for the masses and widths of resonances is of great importance in high-energy physics. Before predictions of SU_3 can be subjected to experimental tests[1] we must be able to determine the relevant masses and coupling constants from experimental data. Resonance parameters are sometimes guessed at from cross-section data, but the only way of obtaining accurate values of masses, widths, and branching ratios[2] is to use partial-wave phase-shift analyses.[3]

One usually determines the masses and widths of resonances by fitting phase-shift analyses with an amplitude of the form[4]

$$T_{ij}=T_{ij}^{B}+e^{i\alpha_{ij}}\frac{\frac{1}{2}(\Gamma_i\Gamma_j)^{1/2}}{m-E-\frac{1}{2}i\Gamma}\ , \tag{1}$$

where T_{ij}^{B} is a unitary background amplitude and α_{ij} is the phase angle connecting the background with a usual Breit-Wigner amplitude. m is the resonant mass, Γ_i is the partial width for decay into the ith channel, and Γ is the total width. In practice all we have available are the phase shift and inelasticity in the elastic channel. Thus we must try to determine the elastic width, the total width, and the mass by using[5,6]

$$T_{11}=\frac{\eta_B e^{2i\delta_B}-1}{2i}+e^{2i\alpha_{11}}\frac{\frac{1}{2}\Gamma_1}{m-E-\frac{1}{2}i\Gamma}\ , \tag{2}$$

where T_{11} is determined by the phase-shift analysis. We assume some standard energy-dependent form for the widths and a simple energy-dependent form for η_B and δ_B consistent with unitarity. We then try to determine the resonance parameters by obtaining the best fit to T_{11}. α_{11} is usually set equal to $(\delta_B+\theta)$, where θ is either set equal to zero[5] or treated as a free parameter[6] ($\theta=0$ if $\eta_B=1$). The widths which are obtained in this manner are often assumed to be proportional to squares of coupling constants given by unitary symmetry schemes.

In this paper we present a dynamical model for a scattering amplitude with two open channels and one or two resonances. We can write the scattering amplitude in the form of Eq. (1). Within the context of the model it is possible to talk about two kinds of widths. We can pretend that the background scattering amplitude is zero and determine the "bare" width of the resonance. This "bare" width is what is predicted by SU_3 since SU_3 calculations do not take background effects into account. We can then switch on the background interaction and obtain the widths Γ_i in Eq. (1). It is the Γ_i which are determined from the phase-shift analysis. We will see that these two widths can be distinctly different. We obtain the result that the masses and widths of resonances are strongly dependent on the background amplitude. We obtain explicit expressions for this dependence in the model calculation.

The model for a single resonance interfering with an inelastic background is presented in the next section. A model for two resonances in the same partial wave is discussed in Sec. III. A certain amount of detail is unavoidable here to understand the model. Some of the nonessential details have been deferred to an appendix. We discuss some of the implications of the model in Sec. IV.

II. ONE RESONANCE WITH INELASTIC BACKGROUND

We consider a model in which there are two open channels and a third high-mass closed channel. We assume that in the absence of coupling to the two open channels there is a low-mass bound state in the third channel. The bound state then becomes a resonance once it is coupled to the open channels.

We use the multichannel ND^{-1} formalism[7] and assume a Born matrix of the form

$$B=\begin{pmatrix} B_{11} & \dfrac{g_{12}}{E+E_0} & \dfrac{g_{13}}{E+E_0} \\ \dfrac{g_{12}}{E+E_0} & B_{22} & \dfrac{g_{23}}{E+E_0} \\ \dfrac{g_{13}}{E+E_0} & \dfrac{g_{23}}{E+E_0} & B_{33} \end{pmatrix}. \tag{3}$$

A pole approximation is used for the off-diagonal terms to make the equations tractable. There are no restrictions placed on the diagonal terms. The scattering amplitude may be written as

$$T_{ij}=(ND^{-1})_{ij}(\rho_i\rho_j)^{1/2}\ . \tag{4}$$

If we make a subtraction in D at $E=-E_0$ the diagonal elements of N can be obtained by solving the integral equations

$$N_{ii}(E)=B_{ii}(E)+\frac{1}{\pi}\int_{E_i}^{\infty}\left(B_{ii}(E')-\frac{E+E_0}{E'+E_0}B_{ii}(E)\right)\frac{\rho_i(E')N_{ii}(E')}{E'-E}\,dE'\ , \tag{5}$$

where E_i is the threshold for the ith channel and ρ_i is a kinematical factor. The off-diagonal N term may be written as

$$N_{ij}=g_{ij}F_i\ , \tag{6}$$

where

$$F_i(E)=\frac{1}{E+E_0}+\frac{1}{\pi}\int_{E_i}^{\infty}\left(B_{ii}(E')-\frac{E+E_0}{E'+E_0}B_{ii}(E)\right)\frac{\rho_i(E')F_i(E')}{E'-E}\,dE'\ . \tag{7}$$

The D functions are found from the equations

$$D_{ii}(E)=1-\frac{E+E_0}{\pi}\int_{E_i}^{\infty}\frac{\rho_i(E')N_{ii}(E')}{E'+E_0}\,\frac{dE'}{E'-E-i\epsilon}\ , \tag{8}$$

and if $i\neq j$ we write

$$D_{ij}(E)=-g_{ij}\phi_i(E) \tag{9}$$

when

$$\phi_i(E)=\frac{E+E_0}{\pi}\int_{E_i}^{\infty}\frac{\rho_i(E')F_i(E')}{E'+E_0}\,\frac{dE'}{E'-E-i\epsilon}\ . \tag{10}$$

We define the background to be the amplitude which would have been observed in the absence of coupling to the bound state. Thus the background N and D functions are

$$N^B=\begin{pmatrix} N_{11} & g_{12}F_1 \\ g_{12}F_2 & N_{22}\end{pmatrix} \tag{11}$$

and

$$D^B=\begin{pmatrix} D_{11} & -g_{12}\phi_1 \\ -g_{12}\phi_2 & D_{22}\end{pmatrix}\ . \tag{12}$$

The background scattering amplitude is

$$T^B_{ij}=[N^B(D^B)^{-1}]_{ij}(\rho_i\rho_j)^{1/2}\ . \tag{13}$$

We define

$$\Delta^B=\det(D^B)\ . \tag{14}$$

We assume that in the region of the resonance it is a good approximation to write

$$D_{33}(E)=d(m_B-E)\ , \tag{15}$$

where m_B is the mass that the bound state would have in the absence of coupling to the other channels. We find that

$$\det(D)=(d\Delta^B)\Lambda\ , \tag{16}$$

where

$$\Lambda\equiv m_B-E-\frac{g_{23}{}^2\phi_2\phi_3}{d\Delta^B}D_{11}-\frac{g_{13}{}^2\phi_1\phi_3}{d\Delta^B}D_{22} -2\,\frac{g_{12}g_{13}g_{23}}{d\Delta^B}\,\phi_1\phi_2\phi_3\ . \tag{17}$$

If we now solve for the scattering amplitudes we find that they can be written as

$$T_{11}=T^B_{11}+\frac{R_1{}^2}{m-E-\frac{1}{2}i\Gamma}\ , \tag{18}$$

$$T_{12}=T^B_{12}+\frac{R_1R_2}{m-E-\frac{1}{2}i\Gamma}\ , \tag{19}$$

$$T_{22}=T^B_{22}+\frac{R_2{}^2}{m-E-\frac{1}{2}i\Gamma}\ , \tag{20}$$

where

$$\Lambda\equiv m-E-\tfrac{1}{2}i\Gamma \tag{21}$$

and

$$R_1=\left(\frac{\rho_1\phi_3}{d(E+E_0)}\right)^{1/2}\frac{g_{12}g_{23}\phi_2+g_{13}D_{22}}{\Delta^B}\ , \tag{22}$$

$$R_2=\left(\frac{\rho_2\phi_3}{d(E+E_0)}\right)^{1/2}\frac{g_{12}g_{13}\phi_1+g_{23}D_{11}}{\Delta^B}\ . \tag{23}$$

We will now rewrite Eqs. (18)–(20) in a more useful form.

Since the background amplitudes satisfy unitarity we can write them as[8]

$$T_{11}^B = \frac{\eta_B e^{2i\delta_{1B}} - 1}{2i}\,, \tag{24}$$

$$T_{12}^B = \pm\tfrac{1}{2}(1-\eta_B{}^2)^{1/2} e^{i(\delta_{1B}+\delta_{2B})}\,, \tag{25}$$

$$T_{22}^B = \frac{\eta_B e^{2i\delta_{2B}} - 1}{2i}\,. \tag{26}$$

δ_{1B} and δ_{2B} are the background phase shifts in the first and second channels, respectively, while η_B is the background inelasticity. The sign of T_{12}^B is determined by the sign of g_{12}.

If $B_{11}=B_{22}=g_{12}=0$, then we obtain the usual Breit-Wigner amplitudes with partial widths for decay into the ith channel given by

$$\gamma_i = g_{i3}{}^2 \rho_i \frac{\phi_3}{d(E+E_0)}\,. \tag{27}$$

If we now introduce two more parameters

$$C_i = \frac{\mathrm{Re}\,\phi_i}{\rho_i F_i} \tag{28}$$

for $i=1,2$, we can rewrite R_1 and R_2 in terms of the partial widths, the (complex) background phase shifts, and C_1 and C_2. We obtain the result (see the Appendix)

$$\begin{aligned} R_1 = e^{i\delta_{1B}}\{ &[\xi_1\tfrac{1}{2}(1+\eta_B)(C_1\sin\delta_{1B}+\cos\delta_{1B}) \pm \tfrac{1}{2}\xi_2(1-\eta_B{}^2)^{1/2}(C_2\cos\delta_{2B}-\sin\delta_{2B})] \\ &+ i[\xi_1\tfrac{1}{2}(1-\eta_B)(C_1\cos\delta_{1B}-\sin\delta_{1B}) \pm \tfrac{1}{2}\xi_2(1-\eta_B{}^2)^{1/2}(C_2\sin\delta_{2B}+\cos\delta_{2B})]\}\,, \end{aligned} \tag{29}$$

where

$$\xi_i = \gamma_i{}^{1/2} F_i(E+E_0) \tag{30}$$

and the $\pm$ sign is chosen to be the same as the sign of g_{12}. A similar expression is obtained for R_2. We can simplify Eqs. (16)–(18) considerably by writing

$$R_1 = e^{i(\delta_{1B}+\theta_1)}\Gamma_1{}^{1/2}\,, \tag{31}$$

where

$$\begin{aligned} \Gamma_1 = &[\xi_1\tfrac{1}{2}(1+\eta_B)(C_1\sin\delta_{1B}+\cos\delta_{1B}) \pm \tfrac{1}{2}\xi_2(1-\eta_B{}^2)^{1/2}(C_2\cos\delta_{2B}-\sin\delta_{2B})]^2 \\ &+ [\xi_1\tfrac{1}{2}(1-\eta_B)(C_1\cos\delta_{1B}-\sin\delta_{1B}) \pm \tfrac{1}{2}\xi_2(1-\eta_B{}^2)^{1/2}(C_2\sin\delta_{2B}+\cos\delta_{2B})]^2\,. \end{aligned} \tag{32}$$

We can similarly define θ_2 and Γ_2 for the second channel. Then we find that the amplitudes can be rewritten as

$$T_{11} = \frac{\eta_B e^{2i\delta_{1B}} - 1}{2i} + e^{2i(\delta_{1B}+\theta_1)}\frac{\Gamma_1}{m-E-\frac{1}{2}i\Gamma}\,, \tag{18$'$}$$

$$T_{12} = \pm\tfrac{1}{2}(1-\eta_B{}^2)^{1/2} e^{i(\delta_{1B}+\delta_{2B})} + e^{i(\delta_{1B}+\delta_{2B}+\theta_1+\theta_2)}\frac{(\Gamma_1\Gamma_2)^{1/2}}{m-E-\frac{1}{2}i\Gamma}\,, \tag{19$'$}$$

$$T_{22} = \frac{\eta_B e^{2i\delta_{2B}} - 1}{2i} + e^{2i(\delta_{2B}+\theta_2)}\frac{\Gamma_2}{m-E-\frac{1}{2}i\Gamma}\,, \tag{20$'$}$$

where[9]

$$\cos 2\theta_1 = \frac{\Gamma_1(1+\eta_B{}^2) - \Gamma_2(1-\eta_B{}^2)}{2\eta_B\Gamma_1}\,, \tag{33}$$

$$\cos 2\theta_2 = \frac{\Gamma_2(1+\eta_B{}^2) - \Gamma_1(1-\eta_B{}^2)}{2\eta_B\Gamma_2}\,, \tag{34}$$

and it can be shown by using Eqs. (15) and (19) that (see the Appendix)

$$\Gamma = \Gamma_1 + \Gamma_2\,. \tag{35}$$

We can also write down an expression for the resonant mass m in terms of the bound-state mass and the other parameters. We find

$$\begin{aligned} m = m_B &- \tfrac{1}{2}\eta_B\xi_1{}^2[C_1\cos 2\delta_{1B} + \tfrac{1}{2}(C_1{}^2-1)\sin 2\delta_{1B}] - \tfrac{1}{2}\eta_B\xi_2{}^2[C_2\cos 2\delta_{2B} + \tfrac{1}{2}(C_2{}^2-1)\sin 2\delta_{2B}] \\ &\pm \tfrac{1}{2}\xi_1\xi_2(1-\eta_B{}^2)^{1/2}[(C_1C_2-1)\cos(\delta_1+\delta_2) - (C_1+C_2)\sin(\delta_1+\delta_2)]\,. \end{aligned} \tag{36}$$

Thus the resonant mass has an energy dependence which is also determined by the background amplitudes.

III. TWO RESONANCES WITH INELASTIC BACKGROUND

In order to produce two resonances interfering with an inelastic background we add a second high-mass channel with a bound state which we couple to the two open channels. The Born matrix becomes

$$B=\begin{pmatrix} B_{11} & \frac{g_{12}}{E+E_0} & \frac{g_{13}}{E+E_0} & \frac{g_{14}}{E+E_0} \\ \frac{g_{12}}{E+E_0} & B_{22} & \frac{g_{23}}{E+E_0} & \frac{g_{24}}{E+E_0} \\ \frac{g_{13}}{E+E_0} & \frac{g_{23}}{E+E_0} & B_{33} & 0 \\ \frac{g_{14}}{E+E_0} & \frac{g_{24}}{E+E_0} & 0 & B_{44} \end{pmatrix}. \qquad (37)$$

The N and D matrices are determined from Eqs. (3)–(8) as before. The only difference is that the matrices are 4×4 matrices. We denote the resonances due to the third and fourth channels by α and β, respectively.

We can again follow the procedure of separating out the resonant terms. First let us single out the resonance β due to the fourth channel. Then the "background" amplitude becomes

$$T_{ij}^{B\alpha}=[N^{B\alpha}(D^{B\alpha})^{-1}]_{ij}(\rho_i\rho_j)^{1/2}\,, \qquad (38)$$

where

$$N^{B\alpha}=\begin{pmatrix} N_{11} & g_{12}F_1 & g_{13}F_1 \\ g_{12}F_2 & N_{22} & g_{23}F_2 \\ g_{13}F_3 & g_{23}F_3 & N_{33} \end{pmatrix} \qquad (39)$$

with an obvious expression for $D^{B\alpha}$. Thus the "background" includes the resonance α due to the third channel and is the same as the amplitude determined in the previous section. The α is used in $T^{B\alpha}$ to remind us that the background includes this resonance. Again we find that

$$T_{11}=T_{11}^{B\alpha}+e^{2i(\delta_{1B\alpha}+\theta_{1\beta})}\frac{\frac{1}{2}\Gamma_{1\beta}}{m_\beta-E-\frac{1}{2}i\Gamma_\beta}\,, \qquad (40)$$

where

$$T_{11}^{B\alpha}=\frac{\eta_{B\alpha}e^{2i\delta_{1B\alpha}}-1}{2i}\,, \qquad (41)$$

Γ_β is the total width of resonance β, and $\Gamma_{1\beta}$ is the partial width for the decay of that resonance into the first channel. We again find that $\Gamma_{1\beta}$ is given by an expression like Eq. (32) where we make the substitutions $\eta_B\to\eta_{B\alpha}$, $\delta_{1B}\to\delta_{1B\alpha}$, and $\delta_{2B}\to\delta_{2B\alpha}$. $\theta_{1\beta}$ is determined from Eq. (33) by making the substitutions $\Gamma_1\to\Gamma_{1\beta}$, $\Gamma_2\to\Gamma_{2\beta}$, and $\eta_B\to\eta_{B\alpha}$. Thus the formalism is essentially unchanged by adding another resonance. We only need to remember that the "background" now includes a resonant term. This resonance may again be singled out by repeating the process. Thus we can write

$$T_{11}^{B\alpha}=T_{11}^{B}+e^{2i(\delta_{1B}+\theta_{1\alpha})}\frac{\frac{1}{2}\Gamma_{1\alpha}}{m_\alpha-E-\frac{1}{2}i\Gamma_\alpha}\,. \qquad (42)$$

We note that channels 3 and 4 are completely equivalent and we can just as easily reverse the procedure and separate out the resonance α first. We would obtain the result

$$T_{11}=T_{11}^{B\beta}+e^{2i(\delta_{1B\beta}+\theta'_{1\alpha})}\frac{\frac{1}{2}\Gamma'_{1\alpha}}{m'_\alpha-E-\frac{1}{2}i\Gamma'_\alpha}\,. \qquad (43)$$

We see immediately that the widths and masses obtained for resonance α in Eqs. (42) and (43) are *not* the same because the background phase shifts are not the same in the two equations and hence we obtain different widths by using Eq. (32). Of course if we continue to complex E to obtain the pole position we will obtain the same result in each case. Rewriting the amplitude cannot change the pole position, but it can change the parameters used along the real energy axis.

IV. DISCUSSION

As an example of how we propose to use the formalism presented here let us consider the $\Delta(1236)$ resonance which appears in the P_{33} partial wave in πN scattering. The background in this partial wave is small and has often been neglected in computing the resonance parameters of the $\Delta(1236)$. If the background is taken into account the mass and width of the resonance are shifted by a few MeV.[2]

First consider the energy range below 1300 MeV where the scattering is elastic and we expect $\eta_B=1$ and $\theta_1=0$. The width is then a function of the background phase shift δ_{B1} and the parameters ξ_1 and C_1. Since the background phase shift is small we expect that B_{11} will be small and $F_1\approx(E+E_0)^{-1}$. This means that $\xi_1^{\,2}\approx\gamma_1$, where γ_1 is the width which the $\Delta(1236)$ would have in the absence of background effects. We expect that γ_1 would correspond to the SU_3 prediction for the $\Delta(1236)$ width since no background effects are taken into account in SU_3 calculations. A careful examination of the energy dependence of γ_1 in Eq. (27) shows that it has the usual centrifugal-barrier factor and it can be given a standard resonance parametrization (we neglect the energy dependence of ϕ_3 and d). We also find that C_1 can be parametrized as $C_1=\text{const}/\rho_1$. We can then choose a standard parametrization for δ_{1B} and adjust the parameters to obtain the best fit to the P_{33} phase shift. The value we obtain for γ_1 would be interpreted as the SU_3 width.

At higher energies, where $\eta_B < 1$, the analysis becomes more complicated. We must now introduce a phenomenological higher-mass channel to represent the $\pi\pi N$ channel and the widths and mass become dependent on δ_{2B}. This clearly introduces too much freedom in the fit, and we would be reluctant to take the results seriously. If we assume that the bound state is not coupled directly to the $\pi\pi N$ channel, then $\xi_2 = 0$ and the expressions for the mass and widths are greatly simplified. We then find that

$$\Gamma_1 = \gamma_1\{[\tfrac{1}{2}(1+\eta_B)(C_1 \sin\delta_{1B} + \cos\delta_{1B})]^2 + [\tfrac{1}{2}(1-\eta_B)(C_1 \cos\delta_{1B} - \sin\delta_{1B})]^2\} . \quad (44)$$

The inelastic width, Γ_2, is

$$\Gamma_2 = \gamma_1 \tfrac{1}{4}(1-\eta_B{}^2)(1+C_1{}^2) \quad (45)$$

and

$$m = m_B - \tfrac{1}{2}\eta_B\gamma_1[C_1 \cos 2\delta_{1B} + \tfrac{1}{2}(C_1{}^2 - 1)\sin 2\delta_{1B}] . \quad (46)$$

The dependence on δ_{2B} drops out in this simplified version of the model. In the absence of the background term, the mass would be $m_B - \frac{1}{2}C_1\gamma_1$. We would interpret this mass as the appropriate mass to be used in SU_3 mass formulas. It is obvious that the energy dependence of Γ_1 can be expected to be quite different from the standard energy dependence of widths because of the dependence on the background amplitude.[10] We have done a detailed analysis of the P_{33} partial wave using this model. The results will be presented in a future publication.[11]

The model discussed here has interesting implications for partial waves containing two or more resonances. It was shown in Sec. III that the parameters of the two resonances can vary depending on the order in which the separation is performed. In fitting the S_{11} or P_{11} partial waves in πN scattering we cannot attach any theoretical significance to the resonance parameters without relying on a model to interpret the results. It is clear that Γ_{1B} in Eq. (40) may be changing rapidly in the vicinity of resonance α ($E \approx m_\alpha$). This is true because Γ_{1B} is dependent on the "background" amplitude which contains resonance α and is changing rapidly for $E \approx m_\alpha$. θ_{1B} is also changing rapidly in this energy region. We must use a model which includes the effects of the background in order to untangle this complicated energy behavior.

Although the model presented here has obvious limitations, we feel that many of the qualitative features which we found will also be present in other models which take background effects into account. In this paper we have pointed out some difficulties that are encountered in determining masses and widths of resonances from partial-wave phase-shift analyses. We hope that models of the type presented here will help resolve these difficulties.

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APPENDIX

We will show here how Γ, defined by Eqs. (17) and (21), can be rewritten in terms of the background phase shifts.

We write the background amplitude

$$T_{11}^B = \rho_1 \frac{N_{11}D_{22} + g_{12}{}^2 F_1 \phi_2}{\Delta^B} \quad (A1)$$

as

$$T_{11}^B = \frac{1}{C_1 + i}\left(\frac{D_{22}}{F_1(E+E_0)\Delta^B} - 1\right) , \quad (A2)$$

where

$$\phi_1 = \rho_1 F_1 (C_1 + i) , \quad (A3)$$

and we use the fact that

$$N_{11}\phi_1 + F_1 D_{11} = \frac{1}{E+E_0} . \quad (A4)$$

If we use Eq. (24) for T_{11}^B, we obtain

$$\frac{D_{22}}{F_1(E+E_0)\Delta^B} = (C_1 + i)\frac{\eta_B e^{2i\delta_{1B}} - 1}{2i} + 1$$

and

$$\operatorname{Im}\frac{g_{13}{}^2\phi_3\phi_1 D_{22}}{d\Delta^B} = g_{13}{}^2\rho_1 \frac{\phi_3 F_1{}^2(E+E_0)}{d}\,[\eta_B(C_1 \sin\delta_{1B} + \cos\delta_{1B})^2 + \tfrac{1}{2}(C_1{}^2+1)(1-\eta_B)] . \quad (A5)$$

Similarly, using

$$T_{12}^B=\frac{g_{12}(\rho_1\rho_2)^{1/2}}{(E+E_0)\Delta^B}$$
$$=\pm\tfrac{1}{2}(1-\eta_B{}^2)^{1/2}e^{i(\delta_{1B}+\delta_{2B})}\ , \qquad \text{(A6)}$$

we obtain

$$\operatorname{Im}\frac{g_{12}g_{13}g_{23}\phi_1\phi_2\phi_3}{d\Delta^B}=\pm g_{13}g_{23}(\rho_1\rho_2)^{1/2}\frac{\phi_3F_1F_2(E+E_0)}{2d}(1-\eta_B{}^2)^{1/2}[C_1\cos\delta_{1B}-\sin\delta_{1B})(\cos\delta_{2B}+C_2\sin\delta_{2B})$$
$$+(C_2\cos\delta_{2B}-\sin\delta_{2B})(\cos\delta_{1B}+C_1\sin\delta_{1B})]\ . \qquad \text{(A7)}$$

If we now combine Eqs. (A5) and (A7) along with an obvious result for $\operatorname{Im}g_{23}{}^2\phi_2\phi_3D_{11}/(d\Delta^B)$ we can write down an expression for the total width Γ. Using Eqs. (17), (21), (A5), and (A7), we obtain

$$\Gamma=\lambda_1{}^2+\lambda_2{}^2\ , \qquad \text{(A8)}$$

where

$$\lambda_1=\frac{1}{\sqrt{2}}\,[\xi_1(1+\eta_B)^{1/2}(C_1\sin\delta_{1B}+\cos\delta_{1B})\pm(1-\eta_B)^{1/2}\xi_2(C_2\cos\delta_{2B}-\sin\delta_{2B})]\ , \qquad \text{(A9)}$$

$$\lambda_2=\frac{1}{\sqrt{2}}\,[\xi_2(1+\eta_B)^{1/2}(C_2\sin\delta_{2B}+\cos\delta_{2B})\pm(1-\eta_B)^{1/2}\xi_1(C_1\cos\delta_{1B}-\sin\delta_{1B})]\ . \qquad \text{(A10)}$$

From Eq. (29) we see that

$$R_1=e^{i\delta_{1B}}\left[\left(\frac{1+\eta_B}{2}\right)^{1/2}\lambda_1+i\left(\frac{1-\eta_B}{2}\right)^{1/2}\lambda_2\right] \qquad \text{(A11)}$$

and also

$$R_2=e^{i\delta_{2B}}\left[\left(\frac{1+\eta_B}{2}\right)^{1/2}\lambda_2+i\left(\frac{1-\eta_B}{2}\right)^{1/2}\lambda_1\right]\ . \qquad \text{(A12)}$$

We can easily show that $\Gamma=\Gamma_1+\Gamma_2$.

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[1]For example see Frank von Hippel and C. Quigg, Phys. Rev. D 5, 624 (1972).

[2]Particle Data Group, Rev. Mod. Phys. 43, S1 (1970).

[3]For πN phase shifts see: P. Bareyre, C. Bricman, and G. Villet, Phys. Rev. 165, 1730 (1968); A. Donnachie, R. G. Kirsopp, and C. Lovelace, Phys. Letters 26B, 161 (1968); D. J. Herndon, A. Barbaro-Galtieri, and A. H. Rosenfeld, Lawrence Radiation Laboratory Report No. UCRL-20030, 1970 (unpublished); R. Ayed, P. Bareyre, and G. Villet, Phys. Letters 31B, 598 (1970).

[4]K. T. R. Davies and M. Baranger, Ann. Phys. (N.Y.) 19, 383 (1962).

[5]A. T. Davies, Nucl. Phys. B21, 359 (1970).

[6]R. Ayed, P. Bareyre, and G. Villet, in *Proceedings of the Fifteenth International Conference on High Energy Physics, Kiev, U.S.S.R. 1970* (Atomizdat, Moscow, 1971).

[7]For a discussion of the multichannel ND^{-1} equations see J. Fulco, G. Shaw, and D. Wong, Phys. Rev. 137, B1242 (1965).

[8]C. G. Goebel and K. W. McVoy, Phys. Rev. 164, 1932 (1967).

[9]This result is not dependent on the model; it follows from unitarity for two open channels. P. Coulter, Phys. Rev. Letters 29, 450 (1970).

[10]P. Coulter, B. Harms, and S. Jones (unpublished).

[11]P. Coulter, B. Harms, and S. T. Jones (unpublished).