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Comments on a Model of Spontaneously Broken Scale Invariance

N. J. Papastamatiou and H. Umezawa

Physics Department, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

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Some interesting features of a model of spontaneously broken scale invariance examined previously by us are discussed and elaborated upon. In particular, we present a concrete illustration of the inequivalence of the commutator of the scale generator D with the weak limit of the Hamiltonian and the weak limit of the commutator of D and the Heisenberg Hamiltonian.

In a previous paper[1] we tried to elucidate several features of the mechanism of spontaneous breakdown of scale invariance by examining a specific field-theoretical model. The purpose of this note is to clarify some results obtained in Ref. 1 and to elaborate on an unusual feature of the model.

Let us first summarize very briefly the results of Ref. 1. The basic Lagrangian of the model was that of a massless self-interacting scalar field:

$$\mathfrak{L} = -\tfrac{1}{2}(\partial_\mu\phi\partial_\mu\phi + \tfrac{1}{2}\lambda\phi^4) \ . \tag{1}$$

This Lagrangian is manifestly scale-invariant; spontaneous breakdown of the symmetry was imposed through the condition

$$\langle 0|\phi^2(x)|0\rangle \equiv C \neq 0 \ . \tag{2}$$

Condition (2) was preferred over the seemingly simpler condition $\langle 0|\phi(x)|0\rangle \neq 0$ because the latter implies spontaneous breaking of yet another invariance of (1), namely $\phi(x) \to -\phi(x)$, and it would have been difficult to separate the consequences of the two effects in the final results.

It is easy to see that (2) implies spontaneous breaking of scale invariance. If we define a dilatation generator D satisfying

$$i[D, \phi(x)] = (1 + x\cdot\partial)\phi(x) \ , \tag{3}$$

we find

$$i\langle 0|[D, \phi^2(x)]|0\rangle = 2C \neq 0 \tag{4}$$

and therefore

$$D|0\rangle \neq 0 .$$

Applying the Goldstone theorem[2] to (4), we see that there must appear a zero-mass state, whose asymptotic field we denote by $B(x)$. Simultaneously, the field ϕ will acquire mass because of the condition (2) whose value in the "pair approximation"[3] will be

$$m^2 = 3\lambda C \ . \tag{5}$$

We therefore expect that the Hilbert space corresponding to the solution of (1) (referred to as the "physical Hilbert space") will be the Fock space of asymptotic fields $\phi^{\text{in}}(x)$, $B(x)$ satisfying the equations of motion:

$$(\Box^2 - m^2)\phi^{\text{in}}(x) = 0 \ , \tag{6}$$

$$\Box^2 B(x) = 0 \ . \tag{7}$$

It is clearly impossible for $\phi^{\text{in}}(x)$ to satisfy the same commutation relations with D as $\phi(x)$ does, since this would be incompatible with (6) and the time-independence of D. Therefore D does not act as a dilatation generator in the space of the asymptotic fields; the symmetry has been "rearranged" by the dynamics. The purpose of Ref. 1 was to investigate the nature of this rearrangement, i.e., to establish the action of D on the physical Hilbert space. The main results are as follows:

Starting from the Bethe-Salpeter equation for

$$G(x, y) \equiv \langle 0|T(\phi(x)\phi(y))|q\rangle \ ,$$

we solve it in the pair approximation with the constraint (2), for both a bound state and a two-particle scattering state. In the former case we obtain an eigenvalue equation which yields a bound state of zero mass, thus verifying the Goldstone theorem in this model. After we obtain an expression for $G(x, y)$ for the scattering case as well, we can identify the corresponding matrix elements of the dilatation generator D by taking appropriate derivatives; this in turn yields an expansion of D up to second order in the asymptotic field $\phi^{\text{in}}(x)$. In this argument, the problem of divergences has to be handled with special care: We must eliminate them without introducing any dimensional constant such as a cutoff energy, because the presence of such cutoffs would violate scale invariance and prevents D from being conserved, as it must in the case of purely spontaneous symmetry breaking. Our prescription is to let the cutoff go to infinity at the same time as the bare coupling constant λ goes to zero; this eliminates the divergence and preserves the time independence of D. Normally, one would expect that the S matrix would become trivial in this limit; it turns out however that $G(x, y)$ with q a two-particle scattering state *does not vanish in the limit*, thus yielding nontrivial scattering. This comes through the contribution of the bound state; explicitly, the matrix element in question is proportional to

$$\frac{\lambda}{1 - 3i\lambda Q(k^2)} , \tag{8}$$

where k is the momentum transfer and $Q(k^2)$ is a logarithmically divergent expression which by virtue of (5) satisfies the condition

$$3i\lambda Q(0) = 1 . \tag{9}$$

This is the bound-state condition referred to above. Substituting (9) into (8), we get

$$\frac{1}{3i[Q(0) - Q(k^2)]}$$

which is independent of λ. Furthermore, $Q(0) - Q(k^2)$ does not contain any divergence. This is a very intriguing result, raising the possibility of a new possibility for finite results in field-theoretic calculations. It should be remarked that this mechanism is very closely related to the existence of bound states; in the second model considered in Ref. 1, that of a massless fermion interacting with a massless scalar boson, there was no bound state and the limit of vanishing bare coupling constant gave a trivial S matrix (no scattering). It would be interesting to examine a soluble model in this light, to see if this phenomenon persists to all orders. Since in this limit the scattering is mediated only through the bound state, there seems also to be a connection with the bootstrap idea.

After the cutoff is eliminated by the above prescription, the expression for D becomes time-independent and its action on the physical Hilbert space can be determined.[4] It turns out that D acts as a generator of *dimensional* transformations of the asymptotic fields, according to the relation

$$i[D, \phi^{\text{in}}(x)] = \left(1 + x\cdot\partial - m\frac{\partial}{\partial m}\right)\phi^{\text{in}}(x) , \tag{10}$$

where the mass derivative acts only on the wave function but not on the creation and annihilation operators. The commutation relation of B is anomalous:

$$i[D, B(x)] = (1 + x\cdot\partial)B(x) + 2\frac{m^2}{g_B} , \tag{11}$$

where g_B, the coupling constant of the bound state, is given by

$$g_B{}^2 = 192\pi^2 m^2 . \tag{12}$$

This result gives us a very simple way of understanding the mechanism of spontaneous breakdown of scale invariance. It has been known[5] that there is a special transformation under which any field theory is invariant; this invariance reflects the ordinary dimensional consistency of field equations and was therefore called the dimensional transformation. When there is no dimensional constant in the theory (we use $\hbar = c = 1$), the dimensional transformation becomes the scale transformation. When the mass is the only dimensional constant, the generator of dimensional transformations acts as D in Eq. (10). It is easy to verify that the equation of motion (6) is invariant under D. Equations (3) and (10) show that D, which induces scale transformations of the Heisenberg field ϕ, generates the dimensional transformations of the asymptotic field ϕ^{in}. The scale invariance in terms of the original field ϕ is not lost; it is rearranged into the dimensional invariance in terms of ϕ^{in}. We would also like to point out that this rearranging mechanism does not fix the value of the mass, as can be seen from Eq. (5) and our limiting prescription. This is a manifestation of the original scale invariance: Two solutions with different values of the mass give the same physical results.[6] Though the value of m is not fixed, we can use m to determine the unit of mass.

We also find that the free Hamiltonian of $\phi^{\text{in}}(x)$ and $B(x)$, which we shall denote by H^{in}, has the following commutator with D:

$$i[D, H^{\text{in}}] = H^{\text{in}} - m\frac{\partial}{\partial m}H^{\text{in}} . \tag{13}$$

Equation (13) presents a problem that we would like to clarify here. It differs from the expected commutation relation

$$i[D, H] = H \tag{13'}$$

by the term $m\,\partial H/\partial m$. It was mentioned in Ref. 1 that the discrepancy between (13) and (13′) is due to the fact that the asymptotic limit was taken before the calculation of the commutator rather than after it. The main purpose of this note is to demonstrate a concrete mechanism for this.

The main point of the argument is that the Heisenberg Hamiltonian H corresponding to (1) goes over to the (free) Hamiltonian of the asymptotic fields (which we denoted by H^{in}) only in the weak sense. To make the situation more explicit, we simulate the weak limiting process by an adiabatic switching on and off of the interaction, i.e., replace λ in the Lagrangian (1) by

$$\lambda \to \lambda(t) = \lambda e^{-\epsilon|t|},$$

where ϵ is a positive quantity tending to zero. Then the Heisenberg Hamiltonian is connected to the free Hamiltonian of the asymptotic fields through the relation[7]

$$H = H^{\text{in}} + K\ . \tag{14}$$

The operator K has the following basic properties: (i) It has only energy-conserving matrix elements, in the sense that the energy transfers in these matrix elements are of order ϵ. (ii) All its matrix elements are of order ϵ, thus vanishing in the limit $\epsilon \to 0$; this simulates the fact that the Heisenberg Hamiltonian H coincides with H^{in} in the weak sense. Due to the existence of massless bosons B, K contains terms in which ϕ^{in} is accompanied by many massless bosons of infinitesimal energy. What we want to show is that, although the contribution of these bosons of infinitesimal energy does not survive in the limit $\epsilon \to 0$ of Eq. (14), it is nonvanishing when the commutator of H with D is calculated, and is precisely responsible for the discrepancy between (13) and (13′).

Let us first notice that insertion of a zero-energy boson B into any Feynman diagram can be effected by operating with $g_B B\,\partial/\partial m^2$ on all the ϕ^{in} lines in the diagram. Starting from this observation, we shall modify H^{in} to take into account the effect of bosons of infinitesimally small energies as follows:

$$H' = \exp\left(C B_\epsilon \frac{\partial}{\partial m}\right) H^{\text{in}}\ , \tag{15}$$

where

$$C = \frac{g_B}{2m} \tag{16}$$

and

$$B_\epsilon \equiv \tfrac{1}{2}\epsilon \int_{-\infty}^{+\infty} dt\, B(\vec{x}, t) e^{-\epsilon|t|}\ . \tag{17}$$

Notice that C is independent of m, because of (12).

In terms of the creation and annihilation operators of B, Eq. (17) takes the form

$$B_\epsilon = \frac{\epsilon^2}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2k}} \frac{1}{k^2+\epsilon^2} \left(b_{\vec{k}} e^{+i\vec{k}\cdot\vec{x}} + b_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}}\right),$$

and, using the property

$$\lim_{\epsilon\to 0} \frac{\epsilon}{\epsilon^2 + k^2} = \pi\delta(k)\ ,$$

we have

$$B_\epsilon = \frac{\epsilon}{2(2\pi)^{1/2}} \int \frac{d\vec{k}}{\sqrt{2k}} \delta(k) \left(b_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + b_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}}\right)\ . \tag{18}$$

It is obvious from (18) that B_ϵ gets contributions only from zero-frequency Goldstone bosons, and goes to zero in the limit $\epsilon \to 0$. Going back to (15), we see indeed it is of the form (14), with K vanishing in the limit $\epsilon \to 0$ and having only energy-conserving matrix elements. Notice also that B_ϵ is really independent of the space coordinate $\vec{x}$, thus allowing us to operate with the exponential in (15) directly on the fields $\phi^{\text{in}}(x)$ appearing in the Hamiltonian density.

It is easy to see that the commutators of H' with the fields agree with those of H^{in} in the limit $\epsilon \to 0$. Indeed, since $B(x)$ is mass-independent, we have

$$[H', B(x)] = [H^{\text{in}}, B(x)] + O(\epsilon)\ .$$

Also, working to first order in ϵ,

$$[H', \phi^{\text{in}}(x)] = [H^{\text{in}}, \phi^{\text{in}}(x)] + C B_\epsilon [\partial H^{\text{in}}/\partial m, \phi^{\text{in}}]\ .$$

The commutator in the second term is independent of ϵ, and therefore

$$[H', \phi^{\text{in}}(x)] = [H^{\text{in}}, \phi^{\text{in}}(x)] + O(\epsilon)\ .$$

A different situation occurs in the commutator of H' with the dilatation generator D. From (11) we have

$$i[D, B_\epsilon] = (x\cdot\partial + 1)B_\epsilon - \epsilon\frac{\partial}{\partial\epsilon}B_\epsilon + 2\frac{m^2}{g_B}$$

$$= 2\frac{m^2}{g_B} + O(\epsilon)\ .$$

Noting that $[D, \partial H^{\text{in}}/\partial m]$ is independent of ϵ and using (13), we find in the limit of small ϵ

$$i[D,H'] = i[D,H^{\rm in}] + iC[D, B_\epsilon\,\partial H^{\rm in}/\partial m]$$

$$= H^{\rm in} - m\frac{\partial}{\partial m}H^{\rm in} + 2C\frac{m^2}{g_B}\frac{\partial}{\partial m}H^{\rm in} + O(\epsilon)\ ;$$

so that, if we use the value of C in Eq. (16), we find

$$i[D,H'] = H^{\rm in} = H' + O(\epsilon) \tag{19}$$

which in the limit $\epsilon \to 0$ coincides with (13′).

We have thus shown that the limiting procedure and the commutator cannot be interchanged. In fact, $[D, \lim_{\epsilon\to 0} H']$ gives (13), while $\lim_{\epsilon\to 0}[D,H']$ gives (13′). It should be noted that, since the physical mass is determined by $p_\mu^{\rm in} p_\mu^{\rm in}$ and not by $p_\mu p_\mu$, the commutator (13) is the one that corresponds to the observable symmetry.

In conclusion, we have demonstrated in a concrete example the inequivalence of the commutator of the weak limit and the weak limit of the commutator for the dilatation generator in our model, and showed how Eq. (13) can be reconciled with (13′) when proper care is taken in the transition to the limit.

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