PHYSICAL REVIEW D VOLUME 7, NUMBER 2 15 JANUARY 1973

Consequences of t-Channel Unitarization in the Strong Regge-Cut Model*

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(Received 26 June 1972)

Using the K-matrix method the amplitude predicted by the strong Regge-cut model is made consistent with t-channel unitarity. The resulting amplitude is found to have complex-conjugate Regge poles, $\alpha_R \pm i\alpha_I$, for $t \le 0$ on the physical sheet of the j plane. Its j-plane discontinuity has a sharp peak at $j=\alpha_R$. The contribution of the discontinuity near the tip of the cut, $j=\alpha_c$, as well as the problem of approximating the entire discontinuity integral by complex poles are discussed.

I. INTRODUCTION

The basic idea inherent in the absorption picture is that at sufficiently high energies the low impact-parameter values of a given S-matrix element are strongly suppressed due to competition with other channels. The scattering amplitude for a nondiffractive process is, therefore, negligible at low impact parameters. It is well known that a single Regge pole, with some exceptions, does *not* have this property. Since the experimental evidence for nondiffractive processes is overwhelmingly in support of the absorption picture the single-Regge-pole term clearly needs correction.

The strong Regge-cut model (SCRAM) describes the absorption picture by treating the Regge pole as a "Born" term of the scattering amplitude, $T(s,t)$, with a Regge cut, obtained through a specific prescription, as the correction term.[1] The prescription is such that the cut term is *negative* relative to the pole and has a slower falloff in t. Writing it explicitly for the imaginary part, where SCRAM works best experimentally, we have

$$\mathrm{Im}T_{sc}(s,t)=\beta_0 s^{\alpha_0}-\beta_c\frac{s^{\alpha_c}}{\ln s}, \qquad \beta_0,\ \beta_c>0 \quad (1)$$

where the subscript "sc" corresponds to the strong-cut model.

Let us define the following quantities which involve $\mathrm{Im}T(s,t)$ and its Mellin transform $A(j,t)$:

$$\begin{aligned}A(s,t)&=\mathrm{Im}T(s,t)\\&=\sum_j A(j,t)s^j\\&=\frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma}dj\,s^jA(j,t)\end{aligned} \quad (2)$$

and

$$A(j,t)=\int_{s_0}^{\infty}ds\,s^{-j-1}A(s,t), \quad (3)$$

where we note that $A(j,t)$ is proportional to the partial-wave amplitude in the t channel, γ is to the right of all the singularities of $A(j,t)$ in the complex j plane, and s_0 is the threshold value for the amplitude $T(s,t)$. We have assumed throughout that the Regge scale factor is 1 BeV^2.

For $A_{sc}(s,t)$ defined by (1) we obtain from (3)

$$A_{sc}(j,t)=\frac{\beta_0}{j-\alpha_0}+\beta_c\ln(j-\alpha_c). \quad (4)$$

Thus the t-channel partial-wave amplitude predicted by SCRAM has a pole at $j=\alpha_0$, the "unperturbed" pole, and it has a cut from $j=\alpha_c$ to $-\infty$. We assume that α_0, β_0, and α_c are *real* functions for all values of t.

The partial-wave amplitude above, in general, is not consistent with unitarity in the t channel. Our purpose is to investigate the consequences of unitarization of $A_{sc}(j,t)$. In many of the reactions the physical region for s-channel scattering corresponds to $t \le 0$. Since the t-channel threshold for those reactions is normally at $t=4m_\pi^2$, a value very close to $t=0$, the t-channel unitarity condition should play an important role. We shall use the K-matrix procedure to unitarize the SCRAM amplitude given by (4).

As we shall see in Secs. II and III, one of the consequences of unitarization is that there are complex conjugate poles, $\alpha_R \pm i\alpha_I$, in the physical sheet of the j plane for $t<0$, and, furthermore, the discontinuity across the cut is sharply peaked at $j=\alpha_R$. The fact that, as a result of unitarity, there would be complex Regge poles for $t<0$ has been noted by several authors.[2] Moreover, a peak in the discontinuity function at $j=\alpha_R$ has already been predicted by Ball, Marchesini, and Zachariasen (hereafter called BMZ) on general grounds.[3]

We now have an explicit model on the basis of which we can discuss the effect of complex Regge poles. One of our chief interests is to investigate the validity of approximating the entire discontinuity function by the peak due to complex poles, the so-called BMZ approximation.[3] We find that whereas it is entirely possible to approximate the discontinuity integral by complex poles, the "residue" of the poles is *not* s-independent. There is a slow s dependence due to $\ln s$ factors. This is discussed in Sec. III.

In Sec. IV we try to isolate the s-dependent term and find that it is due to the contribution from the term s^j which is maximum at $j=\alpha_c$. Thus the discontinuity function can be roughly divided into two parts: one due to the peak at $j=\alpha_R$, and the rest which peaks at $j=\alpha_c$ and which extends over an infinite interval. The integral of the former satisfies the BMZ approximation whereas the integral of the latter does not satisfy the approximation and is comparable to the former.

In Appendix A we discuss the poles on the real axis, whose residues are found to be negligible. In Appendix B we discuss the square-root cut in the j plane instead of the logarithmic cut found in SCRAM.

II. CONSEQUENCES OF K-MATRIX UNITARIZATION

Let us consider two-body scattering of spinless particles with equal mass, m. The K-matrix procedure corresponds to writing for the unitarized amplitude $A(j,t)$ the following form:

$$A(j,t)=\frac{A_{sc}(j,t)}{1-i\rho A_{sc}(j,t)}=\left(\frac{4m^2}{t-4m^2}\right)^{1/2}e^{i\delta_j}\sin\delta_j\,,\qquad t>4m^2$$

where $\rho=[(t-4m^2)/4m^2]^{1/2}$. Now for $t\lesssim 0$, one can, to a good approximation, write $-i\rho\approx 1$. Thus

$$A(j,t)\simeq\frac{A_{sc}(j,t)}{1+A_{sc}(j,t)}\,,\qquad t\lesssim 0.$$

Substituting, $A_{sc}(j,t)$ from (4) we obtain

$$A(j,t)=\frac{\beta_0+\beta_c(j-\alpha_0)\ln(j-\alpha_c)}{j-\alpha_0+\beta_0+\beta_c(j-\alpha_0)\ln(j-\alpha_c)}\,. \tag{5}$$

The partial-wave amplitude, $A(j,t)$, above has a cut in the j plane and it has poles at the positions where the denominator in (5) vanishes. These poles are, however, complex when $j<\alpha_c$ because, for $j<\alpha_c$, $\ln(j-\alpha_c)$ is complex and therefore so are the roots of the denominator. The poles are complex conjugates of each other because the imaginary part of $\ln(j-\alpha_c)$ can be $\pm i\phi$, depending on whether one is above the cut or below the cut. As pointed out earlier, the existence of complex conjugate poles is a general consequence of unitarity. Note that in the absence of cuts ($\beta_c=0$) the denominator corresponds to a single zero, $j=\alpha_0-\beta_0$. The term, $-\beta_0$, added to α_0 simply corresponds to the continuation to $t\leqslant 0$ of the imaginary part, $i\rho\beta_0$, *above* the threshold $t=4m^2$.

We consider the following specific case:

$$\alpha_0=a+bt\,,\qquad \alpha_c=a\,;$$

$$\beta_0=\text{constant }(>0),\qquad \beta_c=\text{constant }(>0).$$

Thus we are considering a fixed cut.[4] For a typical example of ρ exchange, $a\approx\frac{1}{2}$ and $b\approx 1$ BeV^{-2}. We can also estimate β_0 and β_c. From the resonance region one can determine β_0 because at the ρ mass, $t=m_\rho^2$, we know for the $\pi\pi$ case, for instance, that

$$\left(\frac{m_\rho^2-4m_\pi^2}{4m_\pi^2}\right)\beta_0\approx m_\rho\Gamma_\rho\alpha'\,,$$

where $\alpha'=b\approx 1$ BeV^{-2}. Therefore,

$$\beta_0\approx 0.05. \tag{6}$$

We can estimate β_c from SCRAM. One of the significant facts about SCRAM is that it is capable of producing zeros in the scattering amplitude. To obtain the experimentally observed zero in the imaginary part of the scattering amplitude at $t\simeq -0.2$ BeV^2 for $s\sim 10$ BeV^2, we must have, from (1),

$$\beta_0 s^{\alpha_0}-\beta_c\frac{s^{\alpha_c}}{\ln s}=0$$

for the above s and t values. Thus we obtain[4]

$$\beta_c\approx\beta_0\,. \tag{7}$$

With the estimates that we now have of α_0, α_c, β_0, and β_c let us return to Eq. (5) and let

$$\lambda=j-a\,. \tag{8}$$

In the λ plane, (5) becomes

$$A(j,t)=\frac{\beta_0+\beta_c(\lambda-bt)\ln\lambda}{\lambda-bt+\beta_0+\beta_c(\lambda-bt)\ln\lambda}\,.$$

The Regge poles are given by the roots of the following equation:

$$\lambda-bt+\beta_0+\beta_c(\lambda-bt)\ln\lambda=0. \tag{9}$$

Let us first give an *approximate* solution of the above equation.

In view of the small magnitudes involved, an *approximate* solution to (9) is given by using iteration procedure,

$$\begin{aligned}\lambda&=bt-\beta_0-\beta_c(\lambda-bt)\ln\lambda\\&\approx bt-\beta_0-\beta_c(-\beta_0)\ln(bt-\beta_0).\end{aligned} \tag{10}$$

For $t\leqslant 0$, the complex conjugate solutions, therefore, are

$$\lambda_\pm = bt - \beta_0 + \beta_0\beta_c[\ln(\beta_0 - bt) \pm i\pi]$$
$$\approx bt - \beta_0 \pm i\pi\beta_0\beta_c \tag{11}$$

and

$$\lambda_\pm - bt \approx \beta_0 e^{\pm i\pi(1-\beta_c)}, \quad t \approx 0. \tag{12}$$

In (11) we neglected $\beta_0\beta_c \ln(\beta_0 - bt)$ in view of the smallness of β_0 and relation (7). In (12) we note that since the branch point is at $\lambda = 0$, and since $1-\beta_c > 0$, therefore, the poles $\lambda_\pm$ are on the *physical* sheet. Thus one of the interesting consequences of unitarization of the absorption model as given by SCRAM is that the complex poles are on the physical sheet. There are infinitely many roots of (9); we consider only the most important ones which in our model are on the physical sheet.

The poles in the j plane, $\alpha_\pm$, are given by

$$\alpha_\pm = a + \lambda_\pm$$
$$= \alpha_R \pm i\alpha_I \ .$$

From (12), then

$$\alpha_R \approx a + bt - \beta_0 ,$$
$$\alpha_I \approx \pi\beta_0\beta_c \ .$$

The residues of these physical-sheet poles are given from (10) to be

$$\beta_\pm = \left.\frac{\beta_0 + \beta_c(\lambda - bt)\ln\lambda}{1 + \beta_c \ln\lambda + \beta_c(\lambda - bt)/\lambda}\right|_{\lambda=\lambda_\pm} ,$$
$$= \left.\frac{(\lambda - bt)}{1 + \beta_c \ln\lambda + \beta_c(\lambda - bt)/\lambda}\right|_{\lambda=\lambda_\pm} . \tag{13}$$

An *approximate* value of $\beta_\pm$ can be obtained if we ignore β_c in (13) except in obtaining the phase. Therefore,

$$\beta_\pm \approx \frac{-\beta_0 e^{\pm i\pi(1-\beta_c)}}{1 \pm i\pi\beta_c} \ .$$

Therefore,

$$\beta_\pm \approx \beta_0 e^{\mp 2i\pi\beta_c} \ . \tag{14}$$

There is also an additional pole to the *right* of $\lambda = 0$ which remains essentially fixed for all values of t. This pole is found to have a negligible residue. We will discuss it in Appendix A. However, we will ignore it temporarily.

Let us now calculate the discontinuity across the cut in λ. From (5), it is given by

$$\mathrm{disc}A(j,t) = \frac{\mathrm{disc}A_{sc}(j,t)}{|1 + A_{sc}(j,t)|^2}$$
$$= \frac{\pi\beta_c(\lambda - bt)^2}{|\lambda - bt + \beta_0 + \beta_c(\lambda - bt)\ln\lambda|^2} \ . \tag{15}$$

One of the interesting points to note is that disc$A$ vanishes at the tip of the branch cut, i.e., at $\lambda = 0$. This should be compared with the discontinuity prior to unitarization [i.e., discA_{sc} given by (4)], which is simply β_c and does not vanish at $\lambda = 0$. The vanishing of the discontinuity function at the tip of the branch cut was shown by Bronzan and Jones[5] to be a general consequence of unitarity. Our special model confirms their result.

We can obtain an *approximate* expression for disc$A(j,t)$ by expressing the denominator in (15) by its roots given in (11). Thus

$$\mathrm{disc}A(j,t) \approx \frac{\pi\beta_c(\lambda - bt)^2}{(\lambda - bt + \beta_0 - i\pi\beta_0\beta_c)(\lambda - bt + \beta_0 + i\pi\beta_0\beta_c)}$$
$$\approx \frac{\pi\beta_c(\lambda - bt)^2}{(\lambda - \lambda_+)(\lambda - \lambda_-)} = \frac{\pi\beta_c(\lambda - bt)^2}{(\lambda - \lambda_R)^2 + \lambda_I^{\,2}}$$
$$= \frac{\pi\beta_c(j - \alpha_0)^2}{(j - \alpha_R)^2 + \alpha_I^{\,2}} \ . \tag{16}$$

It is clear, therefore, that disc$A$ has a peak at $\lambda = \lambda_R = -\beta_0 + bt$. It has also a (quadratic) zero at $\lambda = bt$ to the right of this peak for $t \leqslant 0$. In Fig. 1, we have plotted disc$A(j,t)$ vs j for $t = 0$ and $t = -0.5$ BeV^2.

We can now write down the expression for $A(s,t)$ using the Mellin transform (2) for $t \leqslant 0$

$$A(s,t) = \beta_+ s^{\alpha_+} + \beta_- s^{\alpha_-} - \frac{1}{\pi}\int_{-\infty}^{\alpha_c} dj\, s^j\, \mathrm{disc}A(j,t), \tag{17}$$

where we have ignored the fixed pole with negligible residue discussed earlier. The approximate expressions for $\beta_\pm$, $\alpha_\pm$, and disc$A$ have been given earlier.

In Sec. III we obtain exact expressions for $\alpha_\pm$

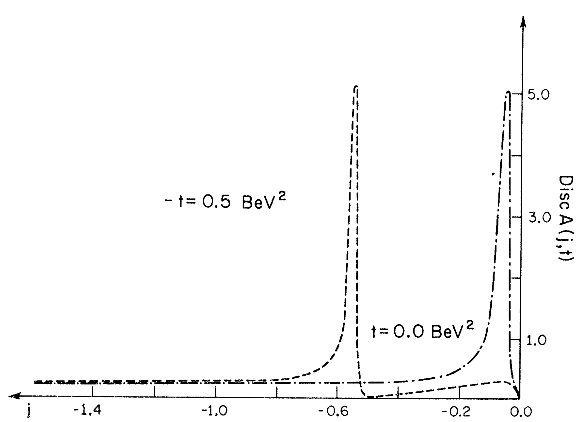

FIG. 1. The discontinuity disc$A(j,t)$ plotted vs j for $t=0$ and $t=-0.5$ BeV^2, with $\beta_0 = 0.05$ and $\beta_c = 0.07$. The numerical value of β_c is chosen so that the SCRAM zero is at $t = -0.2$ BeV^2 for $s = 20$ BeV^2.

and $\beta_\pm$. We will also discuss whether in (17) the integral over the discontinuity function can be expressed in terms of the complex conjugate poles which give rise to the peak in disc$A$.

III. DETERMINATION OF THE COMPLEX POLE PARAMETERS AND THE BMZ APPROXIMATION

We will now attempt to calculate the exact values of $\alpha_\pm$ and $\beta_\pm$. The roots of (9) are $\lambda_\pm$ which are related to $\alpha_\pm$ by (8). In view of the fact that β_c is small, one can expand the solution of (9) in a power series in β_c. The coefficients in the power series can be obtained by comparing appropriate terms in (9).

If we write for $t \leqslant 0$

$$\lambda_+ = \mathrm{Re}^{i\theta} = \lambda_-^* \tag{18}$$

and expand in power series,

$$R = (\beta_0 - bt) + a_1\beta_c + a_2\beta_c^2 + \cdots ,$$

$$\theta = \pi + c_1\beta_c + c_2\beta_c^2 + \cdots ,$$

then from (9) we obtain (again for $t \leqslant 0$) for the first few terms

$$a_1 = -\beta_0 \ln(\beta_0 - bt),$$

$$a_2 = -\pi^2\beta_0 + \frac{1}{2}\frac{\pi^2\beta_0^2}{\beta_0 - bt} + \beta_0 \ln^2(\beta_0 - bt),$$

$$c_1 = \frac{-\pi\beta_0}{\beta_0 - bt} ,$$

$$c_2 = \frac{2\pi\beta_0 \ln(\beta_0 - bt)}{\beta_0 - bt} + \frac{\pi\beta_0^2}{(\beta_0 - bt)^2}[1 - \ln(\beta_0 - bt)] .$$

Similarly for the residues one can write

$$\beta_+ = \beta_-^* = x_0 + x_1\beta_c + x_2\beta_c^2 + \cdots + i(y_0 + y_1\beta_c + y_2\beta_c^2 + \cdots) .$$

From (13) we can obtain the above coefficients in terms of those of λ_+. The result for the first few terms is (again for $t \leqslant 0$)

$$x_0 = \beta_0 ,$$

$$x_1 = -2\beta_0 \ln(\beta_0 - bt) - \frac{\beta_0^2}{\beta_0 - bt} ,$$

$$y_0 = 0 ,$$

$$y_1 = -2\pi\beta_0 ,$$

$$y_2 = \frac{-\pi\beta_0^3}{(\beta_0 - bt)^2} + \frac{6\pi\beta_0^2}{\beta_0 - bt} + 6\pi\beta_0 \ln(\beta_0 - bt).$$

The above results for the trajectory and residue functions confirm our approximations (12) and (14) for small β_c and for $t \approx 0$. As anticipated, the trajectory and residue functions have a logarithmic branch cut in t with a branch point at $t = \beta_0/b$.

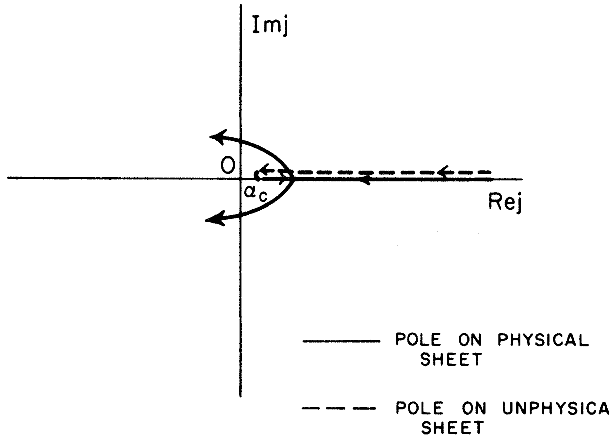

FIG. 2. The motion of the poles in the complex j plane.

In Fig. 2 the motion of $\alpha_t(t)$ in the complex j plane is described. In Figs. 3(a) and 3(b), α_+ vs t and β_+ vs t are plotted. The important thing to note is that no drastic structure in β_+ (especially Reβ_+) is introduced by our model. In other

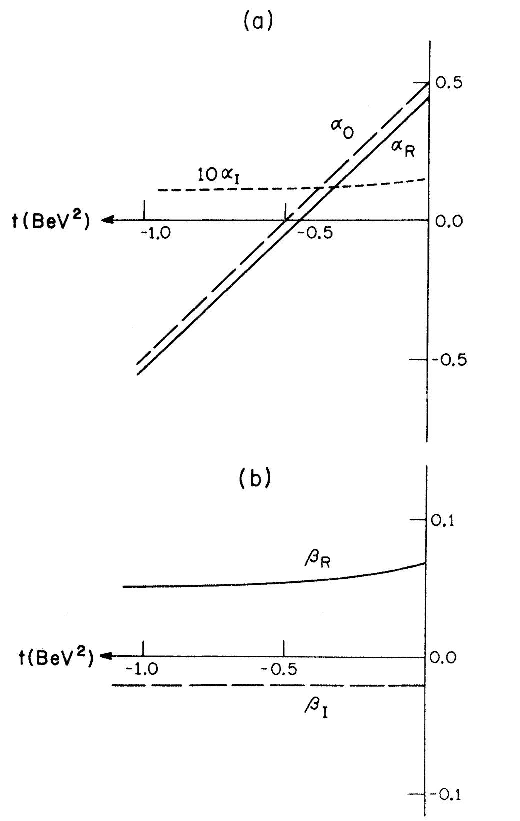

FIG. 3. (a) The input trajectory $\alpha_0(t)$ and the real and imaginary parts of α_+ $(= \alpha_R + i\alpha_I)$ plotted vs t. (b) The real and imaginary parts of the residue $\beta(t)$ $(= \beta_R + i\beta_I)$ of the complex pole, plotted vs t.

words, Reβ_+, which corresponds to β_0 in the SCRAM limit, is a slowly varying function of t.

Let us now consider the integral over the discontinuity function

$$I_c = \frac{1}{\pi}\int_{-\infty}^{\alpha_c} dj\, s^j \operatorname{disc}A(j,t)$$

$$= \beta_c \int_{-\infty}^{0} d\lambda \frac{s^{\lambda+a}(\lambda - bt)^2}{(\lambda-\lambda_+)(\lambda-\lambda_-)} , \qquad (19)$$

where we have used the approximation (16). The error is found to be extremely small. In order to obtain I_c we first calculate the quantity I_0 defined by

$$I_0 = \beta_c \int_{-\infty}^{0} \frac{d\lambda\, s^\lambda}{(\lambda-\lambda_+)(\lambda-\lambda_-)}$$

$$= \frac{\beta_c}{2i\alpha_I}\int_{-\infty}^{0}\frac{d\lambda\, s^\lambda}{\lambda-\lambda_+} + \text{c.c.},$$

where

$$\lambda_+ - \lambda_- = \alpha_+ - \alpha_-$$

$$= 2i\alpha_I$$

and c.c. means complex conjugate.

It can be easily shown that

$$I_c = s^{\alpha_c}\left(\frac{d^2}{d\ln s^2} - 2bt\frac{d}{d\ln s} + b^2t^2\right) I_0 . \qquad (20)$$

Now I_0 can be obtained exactly,

$$I_0 = \frac{\beta_c}{2i\alpha_I}\left\{s^{\lambda_+}\left[i\theta + (\gamma + \ln R + \ln\ln s) + \sum_{n=1}^{\infty}\frac{(-R\ln s)^n e^{in\theta}}{n\,n!}\right]\right\} + \text{c.c.}, \qquad (21)$$

where γ is Euler's constant, and R and θ are defined in (18). From (20) we can then write

$$I_c = \bar\beta_+(t,s)s^{\alpha_+} + \bar\beta_-(t,s)s^{\alpha_-},$$

where $\bar\beta_+ (=\bar\beta_-^*)$ is given by (for $t \le 0$)

$$\bar\beta_+(t,s) = \frac{\beta_c}{2\alpha_I}\left\{(\alpha_+ - \alpha_0)^2\left[\theta - i(\gamma + \ln|\alpha_c - \alpha_+| + \ln\ln s) - i\sum_{n=1}^{\infty}\frac{(\alpha_c-\alpha_+)^n \ln^n s}{n\,n!}\right]\right.$$
$$\left. + 2(\alpha_+ - \alpha_0)\left[\frac{-i}{\ln s} - i\sum_{n=1}^{\infty}\frac{(\alpha_c - \alpha_+)^n \ln^{n-1}s}{n!}\right] + \left[\frac{i}{\ln^2 s} - i\sum_{n=1}^{\infty}\frac{(\alpha_c-\alpha_+)^n(n-1)\ln^{n-2}s}{n!}\right]\right\}, \qquad (22)$$

where we note once again that $\alpha_0 = a + bt$ and $\alpha_c = a$, and $\theta = \pi - \tan^{-1}(\alpha_I/|\alpha_R - \alpha_c|)$. This expression is quite complicated and it depends on both s and t. Let us look at it in the limit when β_c and t are small. In this limit

$$\alpha_I \sim \pi\beta_0\beta_c ,$$

$$\theta \sim \pi ,$$

$$\alpha_+ - \alpha_0 \sim -(\alpha_c - \alpha_+) \sim -\beta_0 + i\alpha_I .$$

Furthermore, we write

$$\bar\beta_+ = \bar\beta_R + i\bar\beta_I .$$

Instead of considering $\bar\beta_R$ and $\bar\beta_I$ in great detail, we will consider only a few terms of $\bar\beta_R$. Our main purpose is to demonstrate that $\bar\beta_+$ depends on t as well as s. The contribution of the first bracket in (22) to $\bar\beta_R$ is dominated by the following term:

$$\frac{\beta_c}{2\alpha_I}(\alpha_+ - \alpha_0)^2\theta \sim \frac{\beta_c}{2(\pi\beta_0\beta_c)}\beta_0^2\pi = \tfrac{1}{2}\beta_0 .$$

This term by itself is independent of s and satisfies the BMZ approximation. However, let us consider the second bracket in (22) and find its dominant contribution to $\bar\beta_R$. The relevant term is

$$\frac{\beta_c}{2\alpha_I}2(\alpha_+ - \alpha_0)\frac{-i}{\ln s} \sim \frac{\beta_c}{2\alpha_I}(i2\alpha_I)\frac{-i}{\ln s} = \frac{\beta_c}{\ln s} .$$

This term, indeed, depends on s and, furthermore, it is comparable to the term $\frac{1}{2}\beta_0$ above [see relation (7)]. There are other terms which also contribute to $\bar\beta_R$ (and to $\bar\beta_I$). The above example is sufficient, however, to demonstrate the fact that $\bar\beta_+$ depends on both s and t such that the s-dependent term is not negligible. In Figs. 4(a) and 4(b) we have plotted $\bar\beta_R$ and $\bar\beta_I$ vs t for different values of s. These graphs make the s dependence of $\bar\beta_+$ quite explicit.

Our results show that whereas complex poles exist and dominate the discontinuity function, the BMZ approximation is not valid. The residues are not s-independent. The function $A(s,t)$ has the form given by

$$A(s,t) = \beta_+(t)s^{\alpha_+} + \beta_-(t)s^{\alpha_-}$$
$$-[\bar\beta_+(t,s)s^{\alpha_+} + \bar\beta_-(t,s)s^{\alpha}]$$
$$= \gamma_+(t,s)s^{\alpha_+} + \gamma_-(t,s)s^{\alpha_-} .$$

In Fig. 5 we have compared the above $A(s,t)$ with

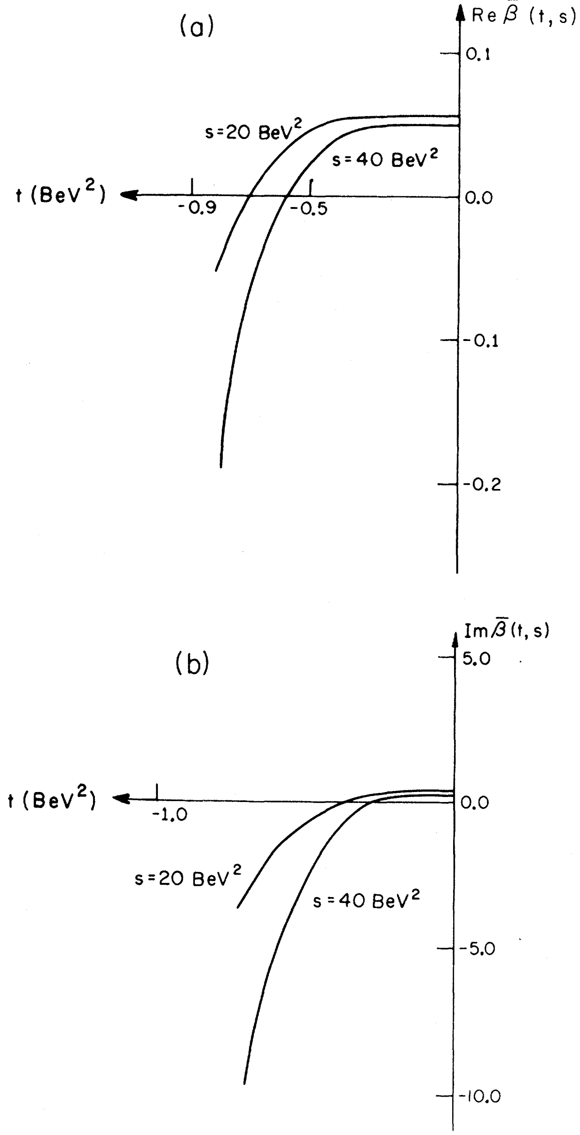

FIG. 4. The contribution $\bar{\beta}(t, s)$ arising from the integral I_c, plotted vs t for $s=20$ and $s=40$ BeV^2. (a) Re$\bar{\beta}(t, s)$, (b) Im$\bar{\beta}(t, s)$.

the original SCRAM version given by (1).[6]

Even though $\bar{\beta}_{\pm}(t, s)$ and $\gamma_{\pm}(t, s)$ depend on t as well as s, it is important to emphasize that the discontinuity function *does* have a peak at the position of the complex poles. The question then to ask is whether this peak itself can be approximated by complex poles with *s-independent* residues. The rest of the discontinuity may then be described by a suitable function which in turn can be approximated by complex poles with *s-dependent* residues. In Sec. IV we will show that this is entirely possible with the s-dependent part coming from the contribution of the tip of the cut.

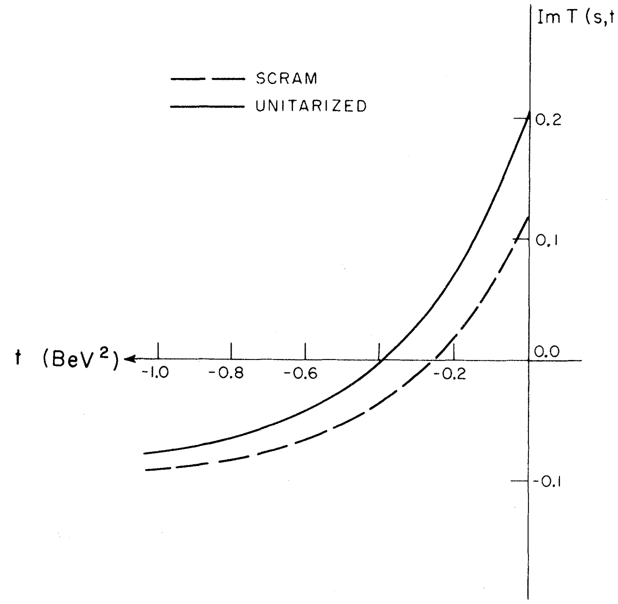

FIG. 5. The amplitude $A(s, t)$ [$=\mathrm{Im}\,T(s, t)$] plotted vs t, for SCRAM and the unitarized model, with $\beta_0=0.05$, $\beta_c=0.07$, and $s=20$ BeV^2.

IV. THE CONTRIBUTION FROM THE TIP OF THE CUT

In this section we show that the discontinuity function can be separated into two parts: one being the peak at the position of the complex conjugate poles and the other being the contribution near the tip of the cut. The latter discontinuity function, which is maximum near $j=\alpha_c$, extends over a much larger j interval than the former.

The discontinuity integral is

$$\beta_c \int_{-\infty}^{\alpha_c} dj \frac{s^j (j-\alpha_0)^2}{(j-\alpha_R)^2+\alpha_I{}^2} \quad .$$

Let us write

$$\begin{aligned}(j-\alpha_0)^2 &= (j-\alpha_R+\beta_0)^2 \\ &= (j-\alpha_R)^2 + 2\beta_0(j-\alpha_R) + \beta_0{}^2 . \end{aligned} \tag{23}$$

We consider the last term first. We have

$$\beta_c \beta_0{}^2 \int_{-\infty}^{\alpha_c} dj \frac{s^j}{(j-\alpha_R)^2+\alpha_I{}^2} \quad .$$

This integral is proportional to I_0 defined earlier [see (21)]. It is the integral which was considered by BMZ and shown to be well represented by a pair of complex conjugate poles with s-independent residues.[3] The point is that in (21) the term $\ln\ln s$ as well as the sum involving $(\ln s)^n$ are very small compared to the s-independent terms. We can write the approximate relation

$$\beta_c\beta_0{}^2\int_{-\infty}^{\alpha_c} dj\,\frac{s^j}{(j-\alpha_R)^2+\alpha_I{}^2} \approx \frac{\beta_c\beta_0{}^2}{2\alpha_I}[\theta - i(\gamma+\ln R)]s^{\alpha_+}+\text{c.c.}$$

We note in passing that in the limit when $\beta_c\to 0$ we have, using $\alpha_I\approx\pi\beta_0\beta_c$ and $\theta\to\pi$, $\alpha_+\to\alpha_R$, simply the expected result

$$\beta_c\beta_0{}^2\int_{-\infty}^{\alpha_c} dj\,\frac{s^j}{(j-\alpha_R)^2+\alpha_I{}^2}\xrightarrow[\alpha_I\to 0]{}\beta_0 s^{\alpha_0}\,.$$

Now let us look at the contribution of the second term in (23). The integral then involves $(j-\alpha_R)$ in the numerator. This integral should be very small because of two reasons. One is that because of the form of the denominator, the major contribution comes near $j=\alpha_R$ where the numerator vanishes; the other is that the numerator changes sign at $j=\alpha_R$, so that the contributions above and below this point tend to cancel. We, therefore, can write a *very crude* approximation,

$$\int_{-\infty}^{\alpha_c} dj\,\frac{s^j(j-\alpha_R)}{(j-\alpha_R)^2+\alpha_I{}^2}\approx 0.$$

The first term in (23) gives $(j-\alpha_R)^2$ to the numerator. Even though, as mentioned above, the denominator gives a maximum contribution at $j=\alpha_R$ where the numerator vanishes, the quantity $(j-\alpha_R)^2$ is *positive definite* and the integral extends up to $-\infty$. Consequently the contribution of $(j-\alpha_R)^2$ cannot be negligible. We can, in fact, write the following approximate form for small α_I:

$$\beta_c\int_{-\infty}^{\alpha_c} dj\,\frac{s^j(j-\alpha_R)^2}{(j-\alpha_R)^2+\alpha_I{}^2}\approx\beta_c\int_{-\infty}^{\alpha_c} dj\,\frac{s^j[(j-\alpha_R)^2+\alpha_I{}^2]}{(j-\alpha_R)^2+\alpha_I{}^2} =\beta_c\int_{-\infty}^{\alpha_c} dj\; s^j \approx\beta_c\,\frac{s^{\alpha_c}}{\ln s}\;.$$

This is totally s-dependent and it is not of the complex conjugate form. This is what we call the tip of the cut contribution because the integrand is simply s^j which is maximum at $j=\alpha_c$. It so happens that in the example considered here, namely the unitarized SCRAM model, the function s^j is integrated over an *infinite* interval so that this integral is as important as the complex-pole term even at moderate energies (see Fig. 6). In other models it may well happen that the function s^j is valid only in the neighborhood of $j=\alpha_c$, the rest of the discontinuity function being well approximated by complex poles. If this interval is $-\alpha_c-\epsilon$ to $-\alpha_c$, then we have

$$\beta_c\int_{-\alpha_c-\epsilon}^{-\alpha_c} dj\, s^j\sim\beta_c\frac{s^{\alpha_c}}{\ln s}(1-s^{-\epsilon})$$

as the contribution of the tip of the cut. This contribution will be quite negligible unless s is *extremely* large.

In summary then, the discontinuity integral can be divided roughly into two parts, one coming from the peak due to complex poles and the other due to the tip of the cut. The former satisfies the BMZ approximation and can be represented by complex conjugate poles with s-independent residues. The latter cannot. In the unitarized SCRAM model considered here, the latter is comparable to the former. However, this need not be the case for other models.

V. CONCLUSION

We have found that the SCRAM amplitude after K-matrix unitarization has complex conjugate poles, for $t\le 0$, at $j=\alpha_R\pm i\alpha_I$ on the physical sheet of the j plane. Its j-plane discontinuity has a sharp peak at $j=\alpha_R$. The integral over the discontinuity function can be expressed in terms of complex poles but the residue function depends on s. However, the discontinuity function can be divided roughly into two parts. One part is due to the peak at $j=\alpha_R$ and its integral can be expressed in terms of complex poles with s-independent residues. The other part has a peak at $j=\alpha_c$ and its integral corresponds to the s-dependent residue. In SCRAM the latter integral is comparable to the former. We have also considered, in Appendix B, the square-root cut and the results are the same.

APPENDIX A

The partial-wave amplitude $A(j,t)$ as defined in (5) has a real pole, which remains very close to $j=\alpha_c$ for almost all values of t. The poles are given by the roots of the denominator of $A(j,t)$. In the λ plane, the denominator is

$$D(\lambda,t)=\lambda-bt+\beta_0+\beta_c(\lambda-bt)\ln\lambda\;. \qquad \text{(A1)}$$

For the scattering region $t\le 0$, and $\beta_0>0,\beta_c>0$, we find

$$D(\lambda,t)=\lambda+b|t|+\beta_0+\beta_c(\lambda+b|t|)\ln\lambda \qquad \text{(A2)}$$

will vanish for $0<\lambda<1$.

To find the position of the real pole approximately, we write the denominator as

$$D(\lambda_0,t)\simeq b|t|+\beta_0+\beta_c\,b|t|\ln\lambda_0=0,$$
$$\lambda_0=\exp\left[-\frac{1}{\beta_c}-\frac{\beta_0}{\beta_c|t|b}\right]. \qquad \text{(A3)}$$

We find the numerical solution of $D(\lambda,t)$ to be

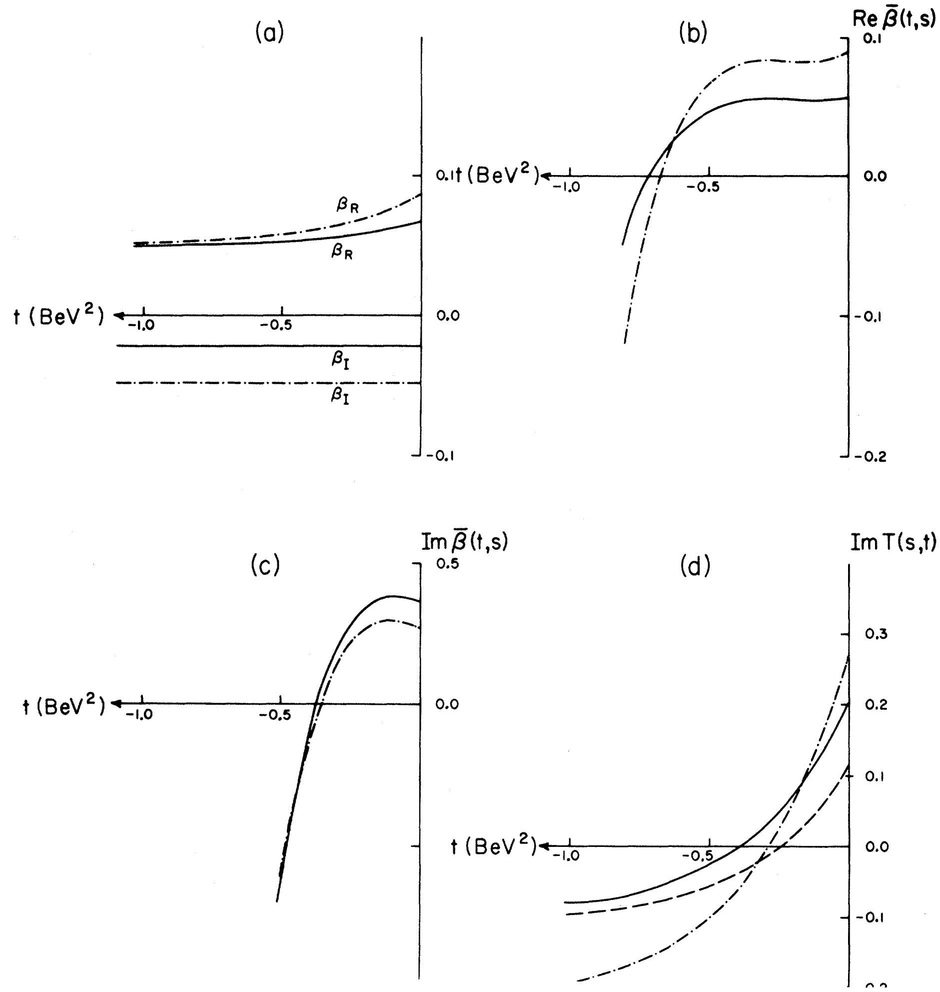

FIG. 6. For $\beta_c = 0.07$ (solid line) and $\beta_c = 0.15$ (dot-dashed line) the following quantities for unitarized SCRAM are plotted vs t, with $\beta_0 = 0.05$ and $s = 20$ BeV^2: (a) β_R, β_I; (b) Re$\bar{\beta}(t, s)$; (c) Im$\bar{\beta}(t, s)$; and (d) Im$T(s, t)$. The dashed line in (d) corresponds to the SCRAM with $\beta_c = 0.07$.

well approximated by this functional form. As β_0 and β_c are small in magnitude (≈ 0.05), we find the real pole to lie very close to the cut, for $t < 0$.

The residue $\hat{\beta}_0$ of the real pole is given from (13) as

$$\hat{\beta}_0 = \frac{-\lambda_0 - b|t|}{\beta_c - \beta_0/b|t| + \beta_c b|t|/\lambda_0}$$

$$\simeq -\frac{\lambda_0}{\beta_c} \ . \tag{A4}$$

For $\beta_0 \approx \beta_c \approx 0.05$, the residue of the real pole is quite small, in fact negligible compared to the residues of the complex conjugate poles.

APPENDIX B

Proceeding exactly along the same line as in the text, we can look at the results obtained by unitarizing the square-root-cut model. The t-channel partial-wave amplitude is

$$A_{\text{sq. rt.}}(j, t) = \frac{\beta_0}{j - \alpha_0(t)} + \beta_c \sqrt{j - \alpha_c} \ , \tag{B1}$$

where we again assume α_0, β_0, and α_c to be real functions.

The scattering amplitude $A(s, t)$ is given by

$$A_{\text{sq. rt.}}(s, t) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} dj\, s^j A_{\text{sq. rt.}}(j, t)$$

$$= \beta_0 s^{\alpha_0} - \frac{\beta_c}{2\sqrt{\pi}} \frac{s^{\alpha_c}}{(\ln s)^{3/2}} \ , \tag{B2}$$

where γ is to the right of all singularities in the t channel. The constant β_c is determined from the similar argument as in the text. For Im$T(s, t)$ [$=A(s, t)$] to vanish at $t = -0.2$, the so-called crossover point, we find

$$\beta_c \approx 10\beta_0 .$$

The unitarized amplitude $A(j,t)$ obtained by the K-matrix method is then

$$A(j,t)=\frac{A_{\text{sq. rt.}}(j,t)}{1+A_{\text{sq. rt.}}(j,t)} ,$$
$$A(j,t)=\frac{\beta_0+\beta_c(j-\alpha_0)(j-\alpha_c)^{1/2}}{j-\alpha_0+\beta_0+\beta_c(j-\alpha_0)(j-\alpha_c)^{1/2}} . \tag{B3}$$

The amplitude has a square-root cut in the j plane, from $-\infty$ to α_c, and the poles of the amplitude are given by the solutions of the denominator of (B3). Considering the simplified case with a fixed cut,

$$\alpha_0=a+bt,$$

$$\alpha_c=a,$$

and

$$\lambda=j-\alpha_c ,$$

$$A(j,t)=\frac{\beta_0+\beta_c(\lambda-bt)\sqrt{\lambda}}{\lambda-bt+\beta_0+\beta_c(\lambda-bt)\sqrt{\lambda}} . \tag{B4}$$

In the $\sqrt{\lambda}$ plane, the zeros of the denominator are given by the solutions of

$$\lambda-bt+\beta_0+\beta_c(\lambda-bt)\sqrt{\lambda}=0.$$

This equation has three roots, one real and two complex conjugate ones, given by

$$\sqrt{\lambda_1}=-\frac{1}{\beta_c}-\beta_c(\beta_0+\tfrac{1}{9}bt),$$
$$\sqrt{\lambda_\pm}=\tfrac{1}{2}\beta_c(\beta_0+\tfrac{1}{9}bt)\pm i(\beta_0-bt+\tfrac{1}{9}bt)^{1/2}$$
$$\simeq\tfrac{1}{2}\beta_0\beta_c\pm i(\beta_0-bt)^{1/2} .$$

In the λ plane, the real solution λ_1 thus goes into the unphysical sheet and so does not contribute to the scattering amplitude $A(s,t)$.

In the λ plane, the solutions are

$$\lambda_1\simeq\frac{1}{\beta_c^{\,2}} ,$$
$$\lambda_\pm\simeq-(\beta_0-bt)+\tfrac{1}{4}\beta_0^{\,2}\beta_c^{\,2}\pm i\beta_0\beta_c(\beta_0-bt)^{1/2} . \tag{B5}$$

Thus we are left with a real pole in the unphysical sheet and a pair of complex conjugate poles in the physical sheet, as a result of unitarization.

The positions of complex conjugate poles in the j plane are

$$\alpha_\pm=\alpha_c-(\beta_0-bt)+\tfrac{1}{4}\beta_0\beta_c^{\,2}\pm i\beta_0\beta_c(\beta_0-bt)^{1/2},$$

$$\alpha_R=\alpha_c-(\beta_0-bt)+\tfrac{1}{4}\beta_0^{\,2}\beta_c^{\,2},$$

$$\alpha_I=\beta_0\beta_c(\beta_0-bt)^{1/2} .$$

The residues for these complex conjugate poles are

$$\beta_\pm(t)=\beta_0[1-(\beta_0-bt)\beta_c^{\,2}-2\beta_0\beta_c^{\,2}]$$
$$\times[1\pm i\tfrac{1}{2}\beta_c(\beta_0-bt)^{1/2}] . \tag{B6}$$

If we look at the discontinuity across the cut in the λ plane we find

$$\text{disc}A(j,t)=\frac{\beta_c}{[1+\beta_c^{\,2}(\beta_0-bt)+\beta_0\beta_c^{\,2}]}\,\frac{\sqrt{-\lambda}\,(\lambda-bt)^2}{(\lambda-\lambda_+)(\lambda-\lambda_-)} .$$

The discontinuity function has the same characteristics as in the logarithmic-cut case; it has a peak at $\lambda=\text{Re}(\lambda_+)$ and a quadratic zero at $\lambda=bt$, very close to the peak.

On taking the Mellin transform, we get the scattering amplitude

$$A(s,t)=\beta_+s^{\alpha_+}+\beta_-s^{\alpha_-}-\frac{1}{\pi}\int_{-\infty}^{\alpha_c}dj\,s^j\text{disc}A(j,t). \tag{B7}$$

The integral over the cut is

$$I_c=\frac{\beta_c}{\pi[1+\beta_c^{\,2}(\beta_0-bt)+2\beta_0\beta_c^{\,2}]}\int_{-\infty}^{\alpha_c}dj\,\frac{s^j(j-\alpha_0)^2(\alpha_c-j)^{1/2}}{(j-\alpha_+)(j-\alpha_-)}$$
$$=K\int_{-\infty}^{0}d\lambda\,\frac{s^\lambda(\lambda-bt)^2\sqrt{-\lambda}}{(\lambda-\lambda_+)(\lambda-\lambda_-)} , \tag{B8}$$

with

$$K=\frac{\beta_c}{\pi[(1+\beta_c^{\,2}(\beta_0-bt)+2\beta_0\beta_c^{\,2}]}$$

$$I_c=K\int_{-\infty}^{0}d\lambda\,\frac{s^\lambda(-\lambda)^{5/2}}{(\lambda-\lambda_+)(\lambda-\lambda_-)}+2btK\int_{-\infty}^{0}d\lambda\,\frac{s^\lambda(-\lambda)^{3/2}}{(\lambda-\lambda_+)(\lambda-\lambda_-)}+b^2t^2K\int_{-\infty}^{0}d\lambda\,\frac{s^\lambda\sqrt{-\lambda}}{(\lambda-\lambda_+)(\lambda-\lambda_-)} .$$

Using

$$\Gamma(a,x)=\frac{e^{-x}x^{-a}}{\Gamma(1-a)}\int_{-\infty}^{0}du\,\frac{e^{-u}u^{-a}}{u+x}$$

and

$$\Gamma(a,x)=\Gamma(a)-x^a\sum_{n=0}^{\infty}\frac{(-x)^n}{(n+a)n!}\ ,$$

we have

$$I_c=\bar\beta_+(t,s)s^{\alpha_+}+\bar\beta_-(t,s)s^{\alpha_-}\ ,$$

where

$$\bar\beta_+(t,s)=\frac{K\pi\sqrt{\alpha_+}}{2i\alpha_I}(\alpha_+-\alpha_0)^2+\frac{K}{2i\alpha_I}\left\{\frac{\Gamma(\frac{7}{2})}{(\ln s)^{5/2}}\sum_{n=0}^{\infty}\frac{[(\alpha_c-\alpha_+)\ln s]^n}{(n-\frac{5}{2})n!}+\frac{2bt\Gamma(\frac{5}{2})}{(\ln s)^{3/2}}\sum_{n=0}^{\infty}\frac{[(\alpha_c-\alpha_+)\ln s]^n}{(n-\frac{3}{2})n!}+\frac{b^2t^2\Gamma(\frac{3}{2})}{(\ln s)^{1/2}}\sum_{n=0}^{\infty}\frac{[(\alpha_c-\alpha_+)\ln s]^n}{(n-\frac{1}{2})n!}\right\}\ . \tag{B9}$$

The first term of $\bar\beta_+(t,s)$ is

$$\bar\beta_+'(t,s)=\frac{\beta_0}{2[1+2\beta_0\beta_c^{\,2}+\beta_c^{\,2}(\beta_0-bt)]}[1+i2\beta_c(\beta_0-bt)^{1/2}]\ ,$$

$$\bar\beta_+'\xrightarrow[\beta_c\to 0]{}\tfrac{1}{2}\beta_0\ .$$

The scattering amplitude $A(s,t)$ is given by

$$A(s,t)=\beta_+(t)s^{\alpha_+}+\beta_-(t)s^{\alpha_-}-[\bar\beta_+(t,s)s^{\alpha_+}+\bar\beta_-(t,s)s^{\alpha_-}]$$

$$=\gamma_+(t,s)s^{\alpha_+}+\gamma_-(t,s)s^{\alpha_-}\ , \tag{B10}$$

where $\beta_\pm$ and $\bar\beta_\pm$ are given by Eqs. (B6) and (B9), respectively.

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*Supported in part by the National Science Foundation.

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[4]Normally, if the Pomeranchukon has a nonzero slope, then α_c will depend on t. The slope of α_c in that case would be $b\alpha_P'/(b+\alpha_P')$ and some of our results [e.g., (7)] will be slightly altered. However, we will assume throughout that $\alpha_P'\approx 0$.

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[6]We note that the SCRAM zero at $t\simeq -0.2$ BeV^2 is shifted to $t\simeq -0.4$. In order to recover this zero in the unitarized model, we need to start with a larger β_c. It is found that $\beta_c=0.15$ brings the zero in ImT back to $t\simeq -0.2$. In Figs. 6(a)–6(d) the different quantities corresponding to this new value of β_c are plotted.