present, we prove that Eq. (13) must be valid. Thus we shall use it in the next step.

The spin-spin forces given in Eq. (17) are here

$$\vec{F}^{ss}_{(i)}=\frac{1}{2m_i}\vec{\nabla}\sum_{j\neq i}\vec{S}_i\cdot(g_i'q_i\vec{B}_{(j)}+m_i\vec{b}_{(j)}). \tag{27}$$

Using (1), (9), and (13), we find

$$F^{ss}_{(i)}=0\ .$$

Therefore, even if we have not proved that the expressions of forces and torques here obtained are applicable to the PIW solutions, we may assume that they are correct at least for the lowest multipoles present. Thus we may say that it is the fact that the gyromagnetic factors are equal to 2 (both active and passive) which is responsible for the balancing of electromagnetic and gravitational forces, in the PIW solution.

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Time Operators, Partial Stationarity, and the Energy-Time Uncertainty Relation*

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The reciprocal time operator which is suggested by the notion of partial stationarity is shown to permit an unambiguous and nonsingular statement of the energy-time uncertainty relation.

The recurring debates[1] over the formulation and meaning of the Heisenberg uncertainty relation for energy and time make clear its unsatisfactory features. Some of these features have counterparts in the ambiguous phase-number and angle–angular-momentum uncertainty relations which have only recently been investigated thoroughly.[2]

The most serious questions, however, accompany the notion of a time operator. Such an operator, $\mathcal{T}$ say, may satisfy $[\mathcal{T},\mathcal{H}]=i\hbar$ and thereby imply the canonical uncertainty relation; but it *cannot simultaneously* be an operator with a continuous, unbounded and real spectrum (which one would identify with the values taken by the parameter t, physical time).

This has not prevented the introduction[1,3-5] of various operators which in restricted circumstances behave very much as a time operator "should" behave. Often subtle arguments are necessary to show that these operators are well enough behaved, when applied properly in a limited domain. The central unsatisfactory feature of these operators, though, apart from their singularities and ambiguities, is their *ad hoc* character. Operators designed for free-particle wave packets rarely have relevance to the decay of bound systems, and conversely. At the most heuristic level this appears to be widely recognized, but is never remarked upon explicitly.[6]

Thus it is remarkable that there already exists one operator which is unambiguous and nonsingular and which serves to define an energy-uncertainty time in agreement with the familiar decay lifetimes and packet spreading times. As it happens, this is not a "time" operator but a "reciprocal time" operator. It exists for all quantum systems with a density matrix and a Hamiltonian.

In the following paragraphs we show how the concept of partial stationarity leads to a consideration of this operator. Then the notion of a stationarity time for a quantum-statistical system follows naturally. We show that this stationarity time is precisely the energy-uncertainty time of the system. When applied to a free wave packet the stationarity time is identical with the packet-spreading time; and when applied to a decaying excited state the stationarity time is found to be equal to the usual lifetime.

It is well known that the dynamical variables of a system whose density matrix commutes with the Hamiltonian are statistically stationary, and conversely.[7] Although stationarity, or translation invariance, is also useful as a spatial or angular concept,[8] we are here concerned with it in the sense of time translations alone. For example, if A and B are two Heisenberg operators, stationarity implies that the expectation $\langle A(t)B(t+\tau)\rangle$ depends on the time argument difference τ and not on the origin of time implied by t.

Since $i\hbar\, d\rho(t)/dt=[\mathcal{H},\rho(t)]$ (in the Schrödinger picture), we see that stationarity implies $d\rho/dt=0$, and conversely.[7] It is always true that $\langle\dot\rho\rangle=0$ so it is natural to adopt the dispersion of $\dot\rho$ as a measure of the *degree of stationarity* which pertains in any given physical situation. Clearly $\dot\rho$ is, in some sense, a reciprocal time operator. It is Hermitian. Let us define a stationarity time T_s by the relation

$$1/T_s{}^2\equiv\Delta\dot\rho^{\,2}. \tag{1}$$

We prove below that any quantum system,

$$T_s{}^2\Delta H^2\geq\hbar^2, \tag{2}$$

where the mean-square dispersion of an operator O is computed as usual: $\Delta O^2=\langle(O-\langle O\rangle)^2\rangle$, and $\langle(\cdots)\rangle$ means $\mathrm{Tr}[\rho(\cdots)]$. In other words, we prove that the stationarity time T_s can be invoked unambiguously as the true energy uncertainty time of the quantum system.

In some respects our uncertainty time is similar to a class of uncertainty times described first, apparently, by Mandelstam and Tamm.[3,9] However, T_s has several distinctions not enjoyed by the Mandelstam-Tamm set of times, the most important of which are:

(1) T_s refers to the quantum system as whole, through $\dot\rho$, rather than to a single dynamical variable of the system.

(2) While *not* a member of the Mandelstam-Tamm set of times, T_s appears to bound all of those times from below, thus providing a sharper statement of the energy-time uncertainty relation. This is easily proved for pure-state expectations.

(3) T_s has a physical meaning which is independent of its appearance in the energy uncertainty relation.

As far as the formal uncertainty principle goes, it is of course irrelevant what this physical meaning is and what considerations motivated the study of the parameter T_s. Nevertheless, it seems worth a further brief digression to establish the sense in which T_s is a stationarity time, or an indicator of partial stationarity. Consider two simple examples. The first is a free single-mode radiation field, with Hamiltonian $\mathcal{H}=\hbar\,\omega a^\dagger a$. Assume the system is in the coherent state $|v\rangle$, so that $\rho=|v\rangle\langle v|$, where $v=|v|\exp(i\phi)$; and compute the autocorrelation of the real "field" $A(t)=a(t)+a^\dagger(t)$:

$$\langle A(t)A(t+\tau)\rangle = e^{i\omega\tau}+(2/\omega T_s)^2\cos[\omega(t+\tau)+\phi]\cos(\omega t+\phi)\,. \tag{3}$$

As expected, since $[\rho,\mathcal{H}]\neq0$, the correlation depends explicitly on t as well as on τ. However an apparently general feature is evident here. The importance of the nonstationary t-dependent part of the correlation function is directly controlled by T_s. That is, if the stationarity time T_s is long, the t dependence is unimportant, and we can consider the system to be stationary.

As a second simple example, consider a free particle of mass m moving in one dimension and described at time $t=0$ by the wave function $\psi(x,0)=Ce^{ikx}\exp(-\frac{1}{2}\lambda^2x^2)$ where C is the normalization constant. In this case a simple calculation shows that the stationarity time is inversely proportional to λ^2; specifically, $T_s = \frac{8}{5}(m/\hbar\lambda^2)$. This is natural since when $\lambda \to 0$ the state is a free-particle energy eigenstate and thus completely stationary, requiring an infinite stationarity time. The position variance $F(t, \tau) = \langle \Delta x(t) \Delta x(t+\tau)\rangle$ is worth examining. In this case we find

$$F(t, \tau) = \langle \Delta x(0)^2 \rangle + \frac{i\hbar\tau}{2m} + \frac{8}{5}\langle \Delta x(0)^2\rangle \frac{t(t+\tau)}{T_s^{\,2}}. \qquad (4)$$

Thus, again, for times t short compared with the stationarity time T_s the correlation function effectively depends only on τ and the system is essentially stationary.

We now prove the uncertainty relation $T_s^{\,2}\Delta\mathcal{H}^2 \geq \hbar^2$. We work in the (necessarily discrete) basis of eigenstates of ρ, and write those eigenstates $|m\rangle$, so that $\rho = \sum_m p_m |m\rangle\langle m|$. We have $1 \geq p_m \geq 0$ in general, and in the special case of a pure state all of the p's are zero except one which is unity.

First we calculate the dispersion in energy. A short calculation leads to

$$\Delta\mathcal{H}^2 = \sum_m p_m (\Delta\mathcal{H}^2)_m + \sum_m p_m \left| \mathcal{H}_{mm} - \sum_l p_l \mathcal{H}_{ll}\right|^2, \qquad (5)$$

where $(\Delta\mathcal{H}^2)_m$ is the dispersion calculated in the pure state $|m\rangle\langle m|$ and $\mathcal{H}_{kk} = \langle k|\mathcal{H}|k\rangle$. Clearly then,

$$\Delta\mathcal{H}^2 \geq \sum_m p_m (\Delta\mathcal{H}^2)_m = \sum_{m,k\neq m} p_m \mathcal{H}_{mk}\mathcal{H}_{km}. \qquad (6)$$

Next consider $(1/T_s^2) = \Delta\dot{\rho}^2$. One finds the result:

$$\hbar^2 \Delta\dot{\rho}^2 = \sum_{k,m} p_m (p_m - p_k)^2 \mathcal{H}_{mk}\mathcal{H}_{km}. \qquad (7)$$

Since $1 \geq (p_m - p_k)^2$, a comparison of Eqs. (6) and (7) proves the inequality $\Delta\mathcal{H}^2 \geq \hbar^2\Delta\dot{\rho}^2$ or, what is is the same thing, $T_s^2\Delta\mathcal{H}^2 \geq \hbar^2$, the energy-time uncertainty relation given in (2).

The circumstances under which the equality holds are easily deduced. In the case of a pure-state density matrix one always has the minimum energy-time uncertainty product $T_s^{\,2}\Delta\mathcal{H}^2 = \hbar^2$, and conversely.

In this brief note we have not explored all of the implications of the ideas presented here. There seems to be nothing especially quantum mechanical about the underlying notion of stationarity time. Perhaps all statistical theories in which the statistical distribution and the random variables both obey a dynamical law of motion should be expected to have sensible stationarity times. As a final step here, however, we should make clear that the stationarity time T_s is intimately related to all of the commonly accepted measures of an energy-uncertainty time.

First, let us show how T_s is related to the lifetime of an unstable excited state. Consider the well-known Weisskopf-Wigner model[10] for excited-state decay. In this model the lifetime of the excited state τ_0 is given by

$$\frac{1}{\tau_0} = (2\pi/\hbar^2)|\langle f|H_I|i\rangle|^2\rho(\omega_f), \qquad (8)$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states, H_I is the interaction Hamiltonian, and $\rho(\omega_f)$ is the usual density of final states. For this same excited system it is also possible to compute T_s. Initially $\rho = |i\rangle\langle i|$, and one finds

$$1/T_s^{\,2} = \hbar^{-2}\sum_\chi{}' |\langle i|H_I|\chi\rangle|^2, \qquad (9)$$

where the primed sum means that one includes only those intermediate states which H_I connects with the initial state. In the Weisskopf-Wigner spirit, these are of course just the states designated $|f\rangle$ above. The sum includes all of them in a narrow range of final energies $\delta E = \hbar\delta\omega$. Thus we find $1/T_s^{\,2} = \delta\omega/2\pi\tau_0$. However, in the Weisskopf-Wigner approximation the range of final-state frequencies which occur with appreciable probability is simply related to the state lifetime, $\delta\omega \sim 2\pi/\tau_0$. Thus we find the canonical result $T_s \sim \tau_0$.

As a second, much simpler, example let us look at the free particle discussed earlier. In that case we find, up to an unimportant numerical factor, essentially the conventional relation between T_s and the uncertainties in position and velocity: $T_s^{\,2} = (\frac{8}{5})(\Delta x^2)/(\Delta v^2)$. So, in these two very different contexts of free particle and excited bound state, we find that T_s reduces in each case to an appropriate familiar heuristic "uncertainty time."

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Baryon-Antibaryon Phase Transition at High Temperature*

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Present experimental data on nucleon-antinucleon scattering allow a study of the possibility of a phase transition in a nucleon-antinucleon gas at high temperature. Estimates can be made of the general behavior of the elastic phase shifts without resorting to theoretical derivation. A phase transition which separates nucleons from antinucleons is found at about 280 MeV in the approximation of the second virial coefficient to the free energy of the gas. A rigorous treatment of the contribution of the inelastic channels remains, however, a difficulty.

I. INTRODUCTION

Harrison[1] has suggested that if baryon-antibaryon inhomogeneities existed in the early universe, several problems of galaxy formation could be solved. Harrison's suggestion and the earlier conjecture[2] of a charge-symmetrical boundary condition between baryons and antibaryons has led to the proposal[3] for mechanisms to separate baryons and antibaryons at high temperature. Statistical fluctuations in the baryon number density are not adequate to explain the present baryon density in the universe. Dynamical mechanisms are therefore required to separate baryons and antibaryons if a symmetrical boundary condition is assumed. Apart from the necessity of finding a separation mechanism, symmetrical models must explain various observational data such as the present ratio of the number of photons to baryons and the absence of any appreciable mixture of matter and antimatter[4] in interstellar gas.

A model has been proposed by Omnès[3] which gives baryon-antibaryon separation in the black-body radiation at a temperature of 350 MeV. The system under consideration is a gas of pions, nucleons, and antinucleons at constant volume and temperature. To obtain the equilibrium configuration, the free energy is minimized with respect to variations in the numbers of nucleons and antinucleons. The free energy is expanded in powers of the numbers of nucleons and antinucleons. It is found in minimizing the free energy that if the second virial coefficient has a large enough positive value (corresponding to an effective repulsion between nucleons and antinucleons) separation is possible.

An effective repulsion between nucleons and antinucleons arises in the Omnès model from the assumption of validity of Levinson's theorem, and considering that the corresponding bound states