error on the central value of the mass distribution Δm_ρ, we have decided to do the following: Where a disagreement has to be established (as in the calculated $\Gamma_{(a)}$ and $\Gamma_{(b)}$) we take the largest possible error (20%); where an agreement has to be established (as in the calculated Γ) we take the smallest possible error (10%).

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Optimal Bounds on Cross-Section Moments in π-π and π-N Scattering

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Froissart-like bounds are derived for a selection of moments of the total cross section in π-π and π-N scattering. These bounds stem from an economical set of constraints – unitarity and the Froissart-Gribov representation of the D-wave scattering length in the crossed channel α_2^t – which are rigorous to within the value assigned to this scattering length. The results are optimally tight for the given constraints and represent a substantial improvement on the bounds of Yndurain and Common, from whose work this paper has been developed. In the case of π-N scattering, it is possible to invert the argument by using experimental data on total cross sections to impose a lower bound on α_2^t.

I. INTRODUCTION

Yndurain and Common[1-3] have originated a method of clarifying the role of the Froissart bound in inhibiting the growth of total cross sections with energy. They define moments of the total cross section by

$$\bar{\sigma}_T(s_1, s_2) \equiv \int_{s_1}^{s_2} q(s')\sigma_T(s')ds', \tag{1}$$

where $q(s')$ is a positive weight function normalized over the energy range (s_1, s_2), and bound this quantity. Their results are explicit for finite s_2 (in their work, s_1 is invariably fixed at threshold), and the energy scale in the $\ln^2(s/s_0)$ is defined.

These bounds are rigorous to within a value assigned to a D-wave scattering length and have been evaluated both here and in the work cited above for π-π and π-N scattering.

In this paper, we employ a variational technique described by Einhorn and Blankenbecler[4] to improve and generalize the results obtained by Yndurain and Common. The main aspect of improvement is that general theorems of optimization theory show that the bounds derived here are maximally tight for the given set of constraints. The principal sources of generalization are that we consider a wider class of weight functions $q(s)$, and that in the method applied here it is not necessary to keep s_1 at threshold: Any energy interval in the physical region can be considered.

The rest of the paper is organized as follows: Section II contains a resumé of earlier work in the field. In Sec. III, we give an account of the method employed here. The numerical outcome of this analysis is presented in Sec. IV. Finally, conclusions and further applications are summarized in Sec. V.

II. CASE HISTORY

From an input of partial-wave unitarity and sufficient analyticity to guarantee the Froissart-Gribov representation of the t-channel, D-wave scattering length α_2^t, Yndurain manipulated the following bound from the partial-wave expansion of α_2^t, for π^0-π^0 scattering:

$$\bar{\sigma}_T^{(m)}(s) \leq \left(\frac{n+1}{n}\right)^2 \left(\frac{m+1}{m}\right) \frac{\pi}{\mu^2} \ln^2\left(\frac{s}{\mu^2}\right) + \left(\frac{n+1}{n}\right)\left(\frac{m+1}{m-\frac{1}{2}}\right) \frac{4\pi}{\mu\sqrt{s}} \ln\left(\frac{s}{\mu^2}\right) + (m+1)30\pi^2 M_n^{-1}\alpha_2^t, \tag{2}$$

where

$$M_n = \frac{(2n)!}{2^n(n!)^2[(2n+1)/(n+1)]^n}, \quad n = 1, 2, 3 \ldots \tag{3}$$

and we have defined

$$\bar{\sigma}_T^{(m)}(s) = \frac{m+1}{(s-4\mu^2)^{m+1}} \int_{4\mu^2}^{s} \sigma_T(s')(s'-4\mu^2)^m ds' . \tag{4}$$

The same premises, subjected to a variational method derived by Lukascuk and Martin, led Common to a bound which betters that of Yndurain only by a factor of 2 in the low-intermediate energy range after the latter has been minimized with respect to n – a free integer parameter which enters the calculation in an asymptotic approximation to the Legendre polynomial [cf. Fig. 1(b)]. Common's bound must be evaluated numerically at low energy, but ultimately assumes the form

$$\bar{\sigma}_T^{(m)}(s) \lesssim \left(\frac{m+1}{m}\right) \frac{\pi}{\mu^2} \ln^2\left(\frac{s}{s_0}\right), \tag{5}$$

where

$$s_0 = \frac{s_1^{3/2}}{8\mu} \ln\left[\frac{s}{\frac{3}{8}(s_1^2/4\mu^2)^{1/2}\ln(s/s_1)}\right]; \quad s_1^{3/2} = \frac{4e}{15\pi^2\mu^2\alpha_2^t} . \tag{6}$$

An asymptotic bound, virtually identical except in the definition of the energy scale s_1, was also given for the case of crossing-symmetric π-p scattering. Quantitatively, these bounds are of the order of thousands of millibarns, thereby massively exceeding known hadronic total cross sections and, hence, anticipated pion-pion ones.

These bounds are based purely on properties with immaculate backgrounds in field theory (viz., analyticity in the appropriate domain and unitarity), except for the value attributed to the scattering length α_2^t. To evaluate this quantity for π-N scattering, Yndurain and Common performed a phenomenological calculation,[3] corrected in a later publication.[5] In the work presented here, we use the latter value, although the results that one obtains for the bound depend monotonically on the value assigned to α_2^t. By performing a quadrature on the experimental total cross sections for π-p scattering tabulated in Giacomelli's compilation,[6] we have also reversed the argument to obtain a semiphenomenological lower bound on this scattering length.

As for π-π, we feel that the most reputable values for the D-wave scattering lengths currently available are those calculated by Morgan and Shaw[7] using dispersion relations which depend on the unambiguous mass and width parameters of the ρ meson and on the existence, but not the "vital statistics," of the controversial[8,9] σ meson (cf. Table I).

III. DERIVATION OF THE BOUND

We give here a detailed account of the work for π^0-π^0 scattering. To handle the case of $\pi^\pm$-π^0, it is sufficient merely to change to the appropriate combination of isospin amplitudes, both for the direct channel partial-wave expansions and for α_2^t. A parallel skeleton derivation for the amplitude corresponding to $\frac{1}{2}(\pi^+$-$p) + \frac{1}{2}(\pi^-$-$p)$ appears in Appendix B.

The variational technique of Langrange multipliers, generalized to handle inequality constraints, has been described in Ref. 4. We appropriate that formalism, more or less intact. In this section, we shall set up the variational equations and describe the resulting extremal partial-wave spectrum. The iterative solution scheme is given in Appendix A. In common with Common's bound, our exact solution must be evaluated numerically.

Define a Lagrangian

$$\mathcal{L} = \bar{\Sigma}_T + \tfrac{1}{2}D(s_1, s_2)\left[a - \int_{4\mu^2}^{\infty} \frac{\sum_0^\infty (2l+1)a_l(s')P_l(x')ds'}{k'(s')^{5/2}}\right] + \sum_0^\infty (2l+1) \int_{4\mu^2}^{\infty} K(s')\lambda_l(s', s_1, s_2)(a_l - a_l^2 - r_l^2)ds' , \tag{7}$$

where

$$a \equiv \frac{15\pi\mu}{8}\alpha_2^t = \int_{4\mu^2}^{\infty} \frac{\operatorname{Im} F(s', 4\mu^2)}{(s')^3}, \tag{8}$$

$$\overline{\Sigma}_T = \frac{1}{32\pi}\overline{\sigma}_T(s_1, s_2) = \int_{s_1}^{s_2} \sum_0^{\infty}(2l+1)a_l(s')\frac{q(s')}{(s'-4\mu^2)}\,ds'. \tag{9}$$

With this normalization, the optical theorem takes the form

$$\sigma_T(s) = \frac{8\pi}{k\sqrt{s}} \operatorname{Im} F(s,0). \tag{10}$$

Partial-wave unitarity implies

$$a_l(s) - a_l^2(s) - r_l^2(s) \geq 0 \ \ \forall l, \quad \forall s > 4\mu^2. \tag{11}$$

We have used the definitions (x' is obtained by replacing s by s')

$$x = 1 + \frac{8\mu^2}{s-4\mu^2}, \qquad s = 4(k^2+\mu^2). \tag{12}$$

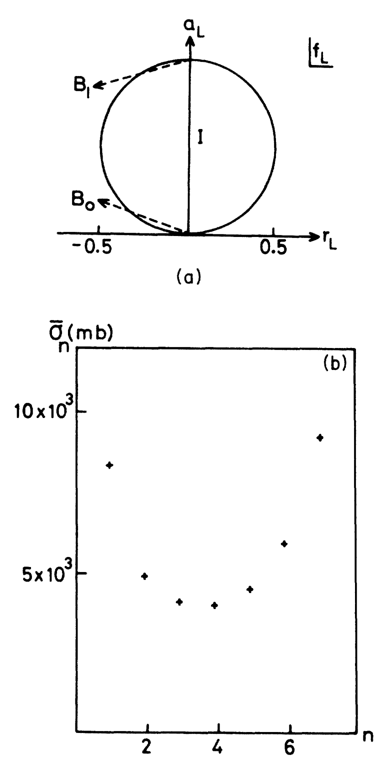

FIG. 1. (a) The partial-wave unitarity circle, showing the locations of the sets B_0, B_1, I. (b) Minimization with respect to n of the bound achieved by Yndurain for π^0-π^0 scattering at $p_{\mathrm{lab}} = 3$ GeV/c, with the weight function $q(s) \sim (s-4\mu^2)$.

Kinematic weight functions $K(s)$, $h(s)$ are defined from the basic normalized distribution $q(s)$ by

$$\frac{1}{K(s)} = (s-4\mu^2)^{1/2}(s)^{5/2} = \frac{h(s)(s-4\mu^2)}{q(s)}. \tag{13}$$

For π^0-π^0, only even partial waves enter, as a consequence of Bose statistics. In an orthodox way, the multiplier $D(s_1, s_2)$ corresponds to the rate of change of the bound with respect to the value of the parameter "a". This formalism is particularly suited to imposing the constraint of unitarity (frequently awkward owing to its nonlinearity) in the strongest sense. The multipliers $\lambda_l(s', s_1, s_2)$ are energy-dependent, which reflects on the way in which unitarity is fulfilled at each point on the energy range over which the cross-section moment is defined.

Variational Equations

$$\frac{\partial \mathcal{L}}{\partial r_l} = 0 = K(s')\lambda_l(s', s_1, s_2)r_l(s'), \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial a_l} = 0 = \theta(s'-s_1)\theta(s_2-s')h(s') - D(s_1,s_2)P_l(x') + \lambda_l(s')[1-2a_l(s')], \tag{15}$$

$$\frac{\partial^2 \mathcal{L}}{\partial a_l \partial a_m} \text{ negative definite for maximum} \rightarrow \lambda_l(s', s_1, s_2) \geq 0. \tag{16}$$

TABLE I. Results obtained by Morgan and Shaw for the D-wave π-π scattering lengths $\alpha_2^{(I=0,2)}$, with sensitivity to the mass-width parameters of the σ meson.

M_σ (MeV)	Γ_σ (MeV)	$\alpha_2^{(I=0)}$	$\alpha_2^{(I=2)}$
600	208	0.0015	0.0002
	500	0.0020	0.0009
765	206	0.0015	0.0
	519	0.0015	0.0002
	835	0.0020	0.0008
900	386	0.0014	0.0
	800	0.0016	0.0003

In I, the inelastic interior of the unitarity circle [cf. Fig. 1(a)] where $\lambda_l = 0$, the solution is indeterminate. However, for given l, it is apparent that the variational equation cannot be satisfied at more than l distinct values of s' in I. Since $a_l(s') < 1$ in I, such contributions to either integral would be zero.

In B_1,

$$\begin{aligned} &a_l(s') = 1\,, \\ &r_l(s') = 0\,, \\ &\lambda_l(s', s_1, s_2) > 0\,, \\ &\lambda_l(s', s_1, s_2) = h(s') - D(s_1, s_2)P_l(x') > 0 \end{aligned} \tag{17}$$

provided $s' \in [s_1, s_2]$. Furthermore, $D(s_1, s_2) > 0$ from the interpretation of this multiplier given above. In B_0,

$$\begin{aligned} &a_l(s') = 0\,, \\ &r_l(s') = 0\,, \\ &\lambda_l(s', s_1, s_2) > 0\,, \\ &\lambda_l(s', s_1, s_2) = D(s_1, s_2)P_l(x') - h(s') > 0\,. \end{aligned} \tag{18}$$

(a) For given (l, s') satisfying (17), the effect of decreasing s' will be to saturate the inequality (17) at $s' = s_l$ ($s_l \in I$) with $a_l(s') \in B_0$ for $s' \in [4\mu^2, s_l[$. This observation depends critically on $h(s')$ being a strictly increasing function.

(b) The monotonically increasing property of Legendre polynomials with arguments greater than unity indicates a cutoff for the set of partial waves with nonempty B_1 at L such that

$$P_L(x_2) < \frac{h(s_2)}{D(s_1, s_2)} < P_{L+2}(x_2)\,. \tag{19}$$

A scheme designed to evaluate $D(s_1, s_2)$ for a given value of "a" and hence to calculate

$$\bar{\sigma}_T(\max; s_1, s_2) = 32\pi \sum_0^L (2l+1) \int_{s_l}^{s} \frac{q(s')}{(s' - 4\mu^2)}\, ds' \tag{20}$$

is given in Appendix A. The resulting partial-wave distribution is given and compared to the one emerging from Common's analysis in Fig. 2.

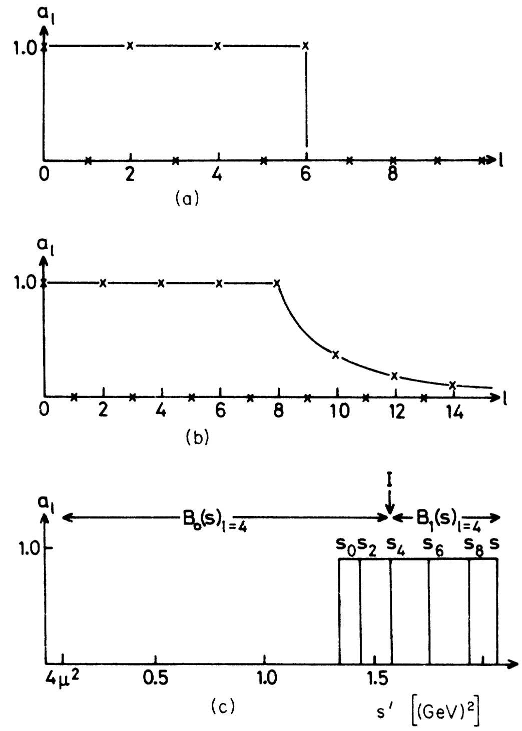

FIG. 2. (a) Truncation of extremal partial-wave distribution in π^0-π^0 scattering with $q(s) \sim (s - 4\mu^2)^3$, $s = 1.00$ GeV^2. (b) Partial-wave spectrum emerging from analysis given by Common.[2] (c) Details of bounding partial-wave distribution for π^0-π^0 scattering at $s = 2.06$ GeV^2 with $q(s) \sim (s - 4\mu^2)^3$.

IV. RESULTS

First of all, we present the bound calculated for π^0-π^0 scattering, with s_1 at threshold and s_2 varying over a considerable range, for the linearly increasing weight function $q(s) \sim (s - 4\mu^2)$. This "sighting shot" is to test whether the improvement obtained by this method over previous results is significant. Here we use the value for α_2^t taken by Yndurain and Common (~80% higher than the Morgan and Shaw results) to ensure that any improvement does not just come from a reduction in the value of α_2^t. To check compatibility of normalization, Yndurain's results were retrieved. The verdict, summarized in Table II and Fig. 3, indicates a substantial tightening of the bound by a factor of the order of 5.

Table III and Fig. 4 describe the bounds for π^0-π^0 scattering with a selection of five different weight functions whose behaviors vary from decreasing as $1/s$ to increasing as $(s - 4\mu^2)^3$. Again $(s_1, s_2) = (4\mu^2, s)$. The energy dependence of the bound is similar in each case: It decreases to a minimum in the vicinity of $p_{\rm lab} \sim 1$ GeV/c and then rises monotonically. This minimum is more pronounced for rapidly increasing weight functions, but the values achieved by the bound tend to be lower when the low-energy contributions to the cross-section moment are emphasized. These numbers are not unduly sensitive to the value of α_2^t and represent a worthwhile bound on the strength of a very economical set of input constraints.

Such bounds may also be derived for the cross-section moments of π^+-π^0. In this case, there are

TABLE II. Comparison of bounds obtained on cross-section moments in π^0-π^0 scattering.

p_{lab} (GeV/c)	Yndurain's bound at $n=1$ (mb)	Yndurain's bound minimization with respect to n' (mb)	Common's bound (mb)	This bound (mb)
1	4.84×10^3	2.52×10^3	7.18×10^2	1.82×10^2
3	8.37×10^3	3.98×10^3	1.60×10^3	2.71×10^2
5	10.50×10^3	4.79×10^3	2.20×10^3	3.40×10^2
7	12.00×10^3	5.39×10^3	2.66×10^3	4.01×10^2
9	13.30×10^3	5.87×10^3	3.04×10^3	4.50×10^2
11	14.30×10^3	6.28×10^3	3.36×10^3	4.97×10^2
13	15.20×10^3	6.75×10^3	3.66×10^3	5.37×10^2

also contributions from the odd angular momentum channels. Numerically, the results, displayed in Table IV and Fig. 5, are slightly higher than their counterparts for π^0-π^0 scattering over the same energy range.

On taking an energy interval of fixed length and moving it away from threshold, one finds that the bound on the moment of the cross section over that range increases rather rapidly, as depicted in Fig. 6(b). Alternatively, one can progressively reduce the length of some interval centered on a particular energy (e.g., the ρ-resonance mass for π^+-π^0 scattering). One finds that a limit is not reached, but that the cross-section moment diverges in the limit of zero interval [cf. Fig. 6(a)].

Of considerably greater interest are the results obtained for π-p scattering. Here, there are ample experimental data with which to confront the bound, as in Table V and Fig. 7. For the unbiased cross-section moment, the bound, at its tightest, exceeds the experimental quantity by a factor of only 6. The smoothing effect on the data of taking such a moment is remarkable. Resonance structure is almost completely suppressed, except in the

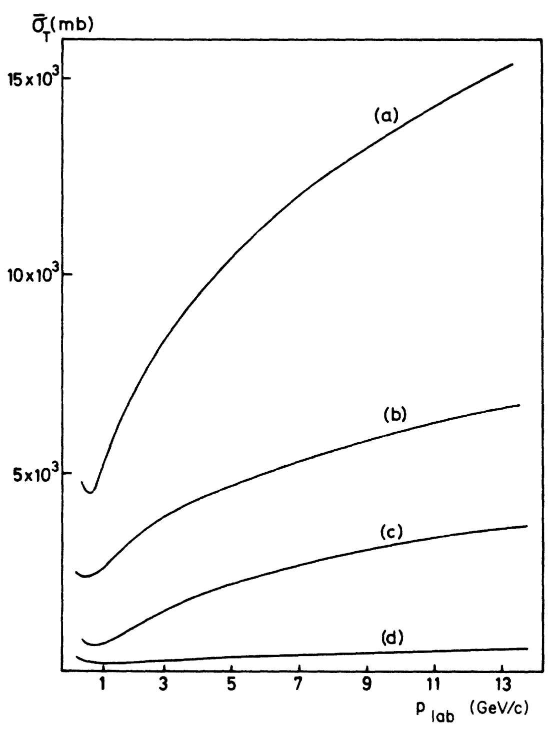

FIG. 3. Comparison of bounds obtained for π^0-π^0 scattering with $q(s)\sim(s-\mu^2)$. (a) Yndurain's bound at $n=1$. (b) Yndurain's bound minimized with respect to n. (c) Common's bound. (d) This bound.

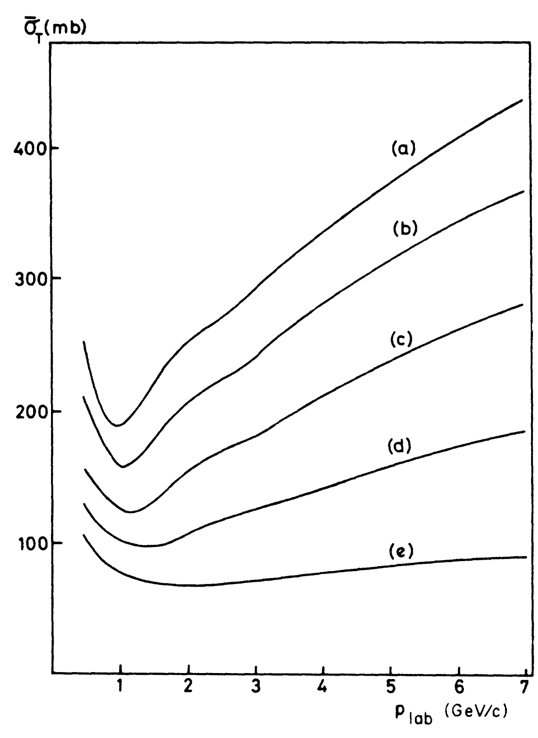

FIG. 4. Set of best bounds on π^0-π^0 cross-section moments for the following weight functions. (a) $q(s)\sim(s-4\mu^2)^3$. (b) $q(s)\sim(s-4\mu^2)^2$. (c) $q(s)\sim(s-4\mu^2)$. (d) $q(s)\sim(s-4\mu^2)/s$. (e) $q(s)\sim(s-4\mu^2)/s^2$.

TABLE III. Tabulation of bounds derived for π^0-π^0 scattering with a variety of normalized weight functions. The letters (a)–(e) correspond to the weight functions of Fig. 4.

$\bar{\sigma}_T^{\max}$ (mb) (e) $q(s)\sim\frac{s-4\mu^2}{s^2}$	(d) $q(s)\sim\frac{s-4\mu^2}{s}$	(c) $q(s)\sim(s-4\mu^2)$	(b) $q(s)\sim(s-4\mu^2)^2$	(a) $q(s)\sim(s-4\mu^2)^3$	p_{lab} (GeV/c)
101.1	127.0	154.0	207	249	0.50
84.4	111.0	137.0	175	201	0.75
77.0	104.0	127.0	158	188	1.00
72.8	98.1	124.0	166	202	1.25
69.9	97.9	132.0	179	222	1.50
66.9	108.0	154.0	207	252	2.00
68.9	117.0	170.0	223	269	2.50
71.4	127.0	181.0	240	293	3.00
76.8	141.0	212.0	281	335	4.00
80.6	158.0	237.0	310	374	5.00
84.9	172.0	260.0	343	408	6.00
88.5	185.0	282.0	366	438	7.00

neighborhood of the (3, 3) resonance. At the point of closest approach, one can reverse the argument to obtain a lower bound on the scattering length by progressively reducing α_2^t until the curve of the upper bound crosses the experimental curve [cf. (b)→(e)→(c) in Fig. 7]. In this way, one can assert that the actual value of α_2^t must be at least 15% of that assessed by Yndurain and Common.

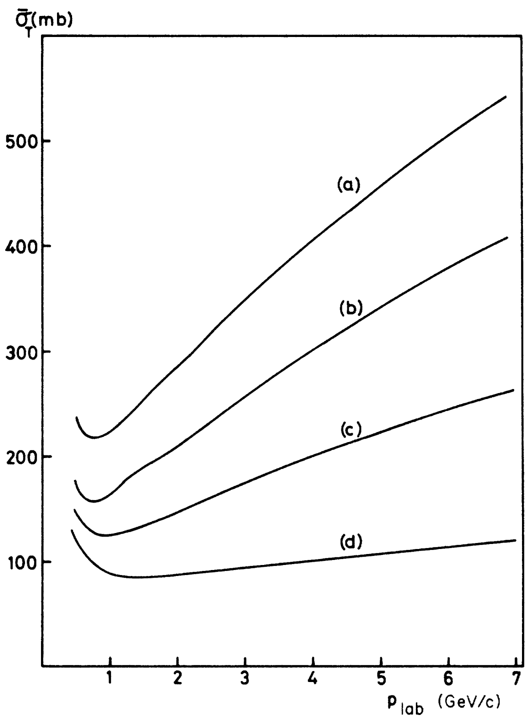

FIG. 5. Set of best bounds on $\pi^\pm$-π^0 cross-section moments for the following weight functions. (a) $q(s)\sim(s-4\mu^2)^2$. (b) $q(s)\sim(s-4\mu^2)$. (c) $q(s)\sim(s-4\mu^2)/s$. (d) $q(s)\sim(s-4\mu^2)/s^2$.

V. DISCUSSION

Evidently, the low-energy range is where the results achieved by this bound are the most remarkable. On extending the energy interval, its power

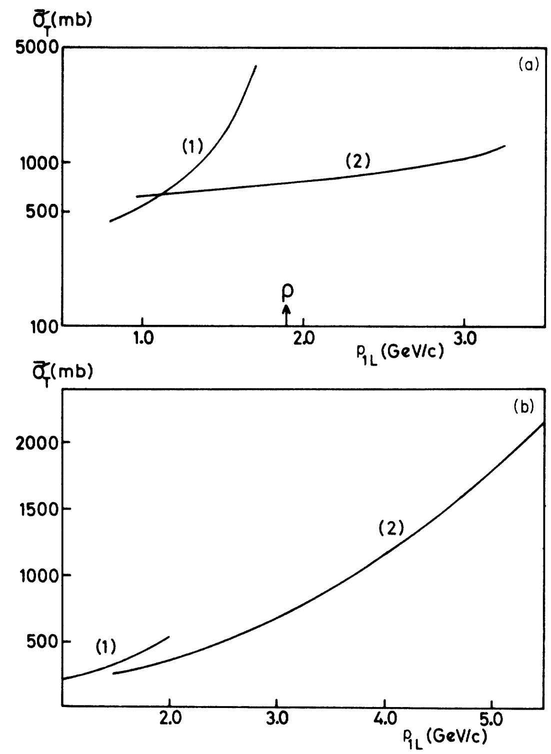

FIG. 6. (a) Plot for π^+-π^0 cross-section moment with $q(s)\sim(s-4\mu^2)$ for an energy interval $[s_1(p_{1l}),\ s_2(p_{2l})]$ centered on (1) the ρ mass ($p_{1l}=1.95$ GeV/c); (2) the f^0 mass ($p_{1l}=6.00$ GeV/c). (b) Plot for π^0-π^0 cross-section moment with $q(s)\sim(s-4\mu^2)$ for an energy interval of constant length: (1) $\Delta p_l=2$ GeV/c, centered on p_{1l}; (2) $\Delta_{pl}=3$ GeV/c centered on p_{1l}.

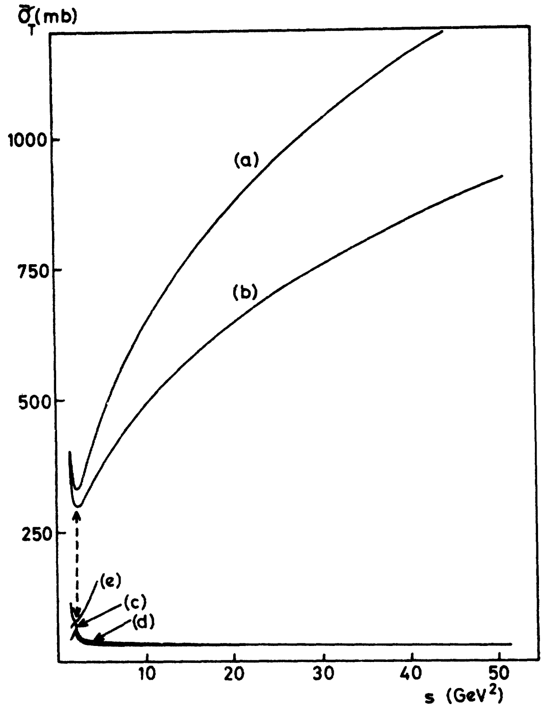

FIG. 7. Best bounds for crossing-symmetric π-p scattering with (a) $q(s) \sim (s-s_T)$ and (b) $q(s) \sim (s-s_T)/s$, for comparison with their "experimental" counterparts, (c) and (d) respectively. Reducing the scattering length to 15% of its estimated value (b)→ (e) imposes a phenomenological lower bound on this quantity.

declines. This is presumably attributable mainly to the fact that the extremal partial-wave spectrum is totally elastic except on a set of measure zero, and inelastic processes are of increasing importance at higher energies. Furthermore, specification of a scattering length is a low-energy constraint. At present, we are not aware of any reputable quantitative statement of inelasticity whose inclusion in the constraint set might rectify this problem.

There exists in the literature a prolific family of constraints on π-π amplitudes (e.g., Refs. 10–12) derived from properties such as crossing, but many of these refer to the partial-wave amplitudes in the unphysical Euclidean region of phase space, which disqualifies them as prospective additions to the set of constraints employed here. Several authors[13,14] have derived sets of sum rules over the physical region, but a tentative investigation has suggested to us that they are not sufficiently powerful to impose bounds of the type described here.

ACKNOWLEDGMENTS

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APPENDIX A: CALCULATION PROCEDURE

This solution scheme is based on determining the multiplier $D(s_1, s_2)$. Let $D(s_1, s_2) = h(s_\alpha(s_1, s_2))$ with $s_\alpha \in [4\mu^2, s_2]$. Each choice of s_α will generate a cutoff value for the partial-wave series at L, and a sequence of threshold energies $s_0, s_2, s_4, \ldots, s_L$ given by

$$P_l(x_l) = \frac{h(s_l)}{h(s_\alpha)} \quad \text{if } s_l > s_1$$

and by (A1)

$$s_l = s_1 \quad \text{if } s_l < s_1 \,.$$

TABLE IV. Tabulation of bounds derived for π^+-π^0 with a variety of normalized weight functions. The letters (a)–(d) correspond to the weight functions of Fig. 5.

	$\bar{\sigma}_T^{\max}$ (mb)			
p_{lab} (GeV/c)	(d) $q(s) \sim \frac{s-4\mu^2}{s^2}$	(c) $q(s) \sim \frac{s-4\mu^2}{s}$	(b) $q(s) \sim (s-4\mu^2)$	(a) $q(s) \sim (s-4\mu^2)^2$
0.50	119	148	177	235
0.75	98.9	128	158	217
1.00	88.6	125	166	230
1.25	86.5	129	178	242
1.50	86.7	135	188	255
2.00	89.2	147	212	287
2.50	91.8	161	234	319
3.00	95.8	173	259	349
4.00	102	200	300	406
5.00	109	224	341	454
6.00	115	244	379	502
7.00	122	266	411	545

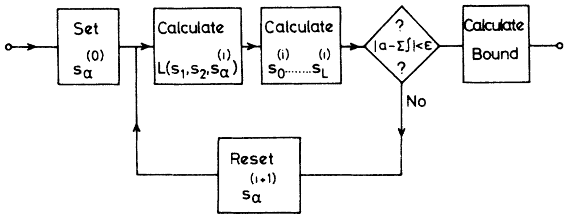

FIG. 8. Flow diagram indicating how bound is calculated to within a predetermined accuracy.

Thus we essentially have to determine

$$\sum_0^L (2l+1)\int_{s_l}^{s} \frac{q(s')}{(s'-4\mu^2)}\,ds' \tag{A2}$$

subject to

$$\tfrac{1}{2}a = \sum_0^L (2l+1)\int_{s_l}^{s} P_l(x')K(s')ds' . \tag{A3}$$

From the monotonicity properties of Legendre polynomials with arguments >1 and of our kinematic weight functions, one can locate the correct value of s_α to within a predetermined level of accuracy and hence evaluate the bound by a continued bisection method, which is outlined schematically in Fig. 8.

APPENDIX B: PION-NUCLEON SCATTERING

The notation derives from that used in Ref. 5. The increased complication of the formalism arises from unequal masses and nonzero spin, but the resolution of the problem is formally identical to the π-π case.

$$\mathcal{L} = \overline{\Sigma}_T^{(+)} + D(s_1, s_2)\left\{\frac{\hat a}{4\pi} - \int_{s_T}^{\infty} \frac{\sqrt{s'}(1+e^{-|\phi'|})}{k'(s'-M^2+\mu^2)^3}\left[\sum_0^{\infty}(l+1)b_l a_{l^+}(s') + \sum_1^{\infty} l b_{l-1} a_{l^-}(s')\right]ds'\right\}$$
$$+\int_{s_T}^{\infty}\left[\sum_0^{\infty}(l+1)K(s')\lambda_l(s')(a_{l^+} - a_{l^+}{}^2 - r_{l^+}{}^2) + \sum_1^{\infty} lK(s')\mu_l(s')(a_{l^-} - a_{l^-}{}^2 - r_{l^-}{}^2)\right]ds' , \tag{B1}$$

where

$$\overline{\Sigma}_T^{(+)} = \frac{1}{8\pi}[\bar\sigma_T(\pi^+p) + \bar\sigma_T(\pi^-p)]$$
$$= \int_{s_1}^{s_2} \frac{q(s')}{(k')^2}\left\{\sum_0^{\infty}[(l+1)a_{l^+} + l a_{l^-}]\right\}ds' \tag{B2}$$

and

TABLE V. Bounds obtained in crossing-symmetric π-N scattering. The letters (a)–(d) correspond to Fig. 7.

s (GeV^2)	$\bar\sigma_T^{\max}$ (mb) (b) $q(s)\sim(s-s_T)/s$	$\bar\sigma_T$ (exp) (c)	$\bar\sigma_T^{\max}$ (mb) (a) $q(s)\sim(s-s_T)$	$\bar\sigma_T$ (exp) (mb) (d)
1.50	394	48.6	405	72.7
1.75	317	62.5	337	75.8
2.00	298	53.3	325	53.1
2.50	301	43.8	337	37.8
3.75	335	40.1	401	36.7
5.00	374	38.0	462	35.0
7.50	437	35.4	559	32.7
10.0	489	33.8	643	31.2
15.0	575	31.8	766	29.4
20.0	647	30.6	869	28.4
25.0	703	29.6	958	27.9
30.0	755	28.9	1028	27.8
40.0	839	27.9	1152	27.5
50.0	917	27.3	1252	27.2

$$k^2=\frac{(s-s_A)(s-s_T)}{4s}\,,\quad s_T=(M+\mu)^2,\quad s_A=(M-\mu)^2,\quad x=1+\frac{2\mu^2}{k^2}\,,$$

$$b_l(s')=\frac{P_{l+1}'(x')-P_l'(x')}{l+1}\,,\quad h(s')=\frac{q(s')(s'-M^2+\mu^2)^3}{k'\sqrt{s'}}\,, \tag{B3}$$

$$\sin\phi'=\frac{-2M\{t[su-(M^2-\mu^2)^2]\}^{1/2}}{[(s-s_T)(s-s_A)(u-s_T)(u-s_A)]^{1/2}}\,,\quad \hat{a}=\frac{15\pi M a_2^t}{2(M^2-\mu^2)}-\frac{G_\pi^{\;2}}{\mu^4}=0.177\mu^{-4}\,.$$

The expression for $\hat{a}$ is not quite as simple as that for the corresponding quantity in π-π scattering, owing to the presence of a term originating in the nucleon pole in the Froissart-Gribov representation of the D wave prior to taking the scattering-length limit. The value which we quote for $\hat{a}$ was calculated by Yndurain and Common.[5]

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Saturation of the Drell-Hearn-Gerasimov Sum Rule*

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The Drell-Hearn-Gerasimov sum rule for the forward spin-flip amplitude of nucleon Compton scattering is decomposed into separate sum rules originating from different isospin components of the electromagnetic current. The resulting sum rules are reexamined using recently available analyses of single-pion photoproduction in the region up to photon laboratory energies of 1.2 GeV. All three sum rules receive important nonresonant as well as resonant contributions. The isovector sum rule whose contributions are known best is found to be nearly saturated, lending support to the assumptions underlying the sum rules. The failure of the isoscalar-isovector sum rule to be saturated is then presumably to be blamed on inadequate data for inelastic contributions.

I. INTRODUCTION

The Drell-Hearn-Gerasimov[1] sum rule for the spin-flip amplitude in forward Compton scattering rests on two assumptions. These are the low-energy theorem[2] for the spin-flip amplitude and the validity of an unsubtracted dispersion relation. Since these are simple and relatively well-accepted assumptions, and are often used together with additional stronger assumptions in deriving other sum rules, it is of interest to look into the validity of the Drell-Hearn-Gerasimov sum rule in the light of the present experimental data.

In their original paper, Drell and Hearn[1] did attempt to investigate the validity of the sum rule for a proton target by using an isobar model of single-pion photoproduction. Their results were generally encouraging, but some important contribution from high energy (greater than 1 GeV) seemed to be likely. Somewhat later, Chau *et al.*[3]