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Low-Energy Limit of the σ Model

G. S. Guralnik*
Department of Physics, Brown University, Providence, Rhode Island 02912

and

Hung-sheng Tsao†
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790

and

T. F. Wong‡
Department of Physics, Rutgers, The State University, New Brunswick, New Jersey 08903
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Recently, Li and Pagels indicated that there might be nonanalytic corrections as bad as $m_\pi^2 \ln m_\pi^2$ in the matrix elements. We study the nonanalytic corrections in a renormalized linear σ model in the one-loop approximation. Although we observe these corrections [of the form $\ln m_\pi^2$, $m_\pi^2 \ln m_\pi^2$, $(\not{p}-m)\ln m_\pi^2$, $(\not{p}_1-m)\times(\not{p}_2-m)\ln m_\pi^2$] in individual diagrams, they sum up to zero in the physically interesting amplitudes (pion and nucleon propagators, $\pi\pi$ and πN scattering amplitudes). We conclude that the nonanalytic correction is of higher order and thus does not invalidate the low-energy results derived from a phenomenological Lagrangian for these amplitudes.

I. INTRODUCTION

Current algebra together with soft pions produces some very interesting theoretical results.[1] It is well known that these results can be reproduced by an appropriate phenomenological Lagrangian (PhL) model[2] in the tree approximation. Dashen and Weinstein,[3] and Lee[4] have shown, by somewhat different methods, that the PhL is the generating functional of pion irreducible vertices correct to the order ζ^2 and m_π^2. Let us denote the pion irreducible vertex with n external pion lines by $\pi_n(k_1\cdots k_n)$. The low-energy limit is characterized by a parameter ζ, where $k_i=\zeta p_i$ and the p_i's are a set of finite momenta. Dashen and Weinstein's theorem shows that the soft-pion ($\zeta\to 0$) and weak-chiral-symmetry-breaking limit ($m_\pi^2\to 0$) of π_n is of the form

$$\lim_{\zeta\to 0;\, m_\pi^2\to 0} \pi_n(k_1\cdots k_n)=\alpha\zeta^2+\beta m_\pi^2 + \text{higher terms in } \zeta^2, m_\pi^2, \tag{1}$$

where α and β are finite and that the correct expressions for π_n in this limit are given by the PhL,

$$\pi_n=\frac{\delta^n \mathcal{L}}{\delta\phi_1\cdots\delta\phi_n}. \tag{2}$$

The validity of the expansion in Eq. (1) was challenged recently by Li and Pagels.[5] They showed that π_n cannot be analytic in m_π^2 at the origin. Let

$$\mathcal{L}=\mathcal{L}_0+c\mathcal{L}'.$$

If $\mathcal{L}_0$ is realized by Goldstone bosons (pions) and $\mathcal{L}'$ breaks the symmetry with characteristic strength c, then the pions acquire a squared mass of $m_\pi^2\sim c$. Since a matrix element has diagrams containing an internal pion loop, by studying the behavior of the loop momentum near the threshold, they argue that the matrix element may contain terms of the form $m_\pi^2 \ln m_\pi^2$, i.e.,

$$S(m_\pi^2)=S(0)+am_\pi^2\ln m_\pi^2+b(m_\pi^2)^2\ln m_\pi^2+\cdots. \tag{3}$$

Therefore, the assumed form in Eq. (1) is questionable.

In addition, Carruthers and Haymaker[6] pointed out that the radius of convergence of the SU(3) $\otimes$SU(3) perturbation limit is much smaller than the experimental value of the symmetry breaking parameter, even in the tree approximation.

In view of this, it is desirable to use a renormalizable chiral Lagrangian model as a guide to investigate the contribution of the Goldstone boson loop. If we do not interpret Eq. (1) as a double power-series expansion in ξ^2 and m_π^2, but rather that $\pi_n - \alpha\xi^2 - \beta m_\pi^2$ goes to zero faster than ξ^2 or m_π^2 as ξ^2, $m_\pi^2 \to 0$ (e.g., terms like $\xi^4 \ln m_\pi^2$, $m_\pi^4 \ln m_\pi^2$, etc.), then the usual PhL still gives the correct low-energy limit. It will be very interesting to find a model to study this hypothesis.

A linear SU(2)$\otimes$SU(2) σ model is a simple model to work with. On the one hand, it has all the symmetry properties assumed in the argument of Li and Pagels and it can be solved exactly in c to any given number of loops. On the other hand, it is renormalizable, with a simple symmetry-breaking term that gives the partially conserved axial-vector current relation (PCAC). Also, in tree approximation, it reproduces the current-algebra results. The σ model has been intensively studied in the recent years. Lee and Symanzik[7] gave a procedure to renormalize the linear σ model, and attempts were made[8] to use the one-loop result in conjunction with the Padé approximation as an explicit unitarization of the current-algebra result. All the applications to date are concentrated on numerical calculations.

In this paper, we evaluate the explicit low-energy form of π, N two-point functions, $\pi\pi$ scattering amplitude, and πN scattering amplitude. In all the matrix elements we have studied, the PhL does give the correct low-energy limit with all the nonanalytic corrections vanishing faster than ξ^2 or m_π^2.

This paper is organized as follows. In Sec. II we briefly review the σ model, the Lagrangian, and the Feynman rules. In Sec. III we describe the explicit renormalization at one loop, and give the two-point, three-point, and four-point functions. Detailed calculations will be found in Appendix B. In Sec. IV the nonanalytic correction is discussed. The summary and conclusions are in Sec. V.

II. σ MODEL

In this section, we summarize some σ-model renormalization results. They will also serve as definitions of our notations. The Lagrangian density of the σ model, in terms of bare fields, is

$$\mathcal{L}(x) = \tfrac{1}{2}[(\partial_\mu\sigma_0)^2 + (\partial_\mu\vec{\pi}_0)^2] - \tfrac{1}{2}\mu_0^2(\sigma^2 + \vec{\pi}_0^2) - \tfrac{1}{4}\lambda_0^2(\sigma^2+\vec{\pi}_0^2)^2 + \bar{\psi}_0[i\gamma\cdot\partial - g_0(\sigma_0 + i\vec{\pi}_0\cdot\vec{\tau}\gamma_5)]\psi_0 + c_0\sigma_0, \tag{4}$$

where $(\sigma_0, \vec{\pi}_0)$ are the isoscalar-scalar and isovector-pseudoscalar mesons, respectively; ψ_0 is an isodoublet fermion field. This Lagrangian is SU(2)$\otimes$SU(2)-symmetric with a linear SU(2) symmetry-breaking term.

We assume that vacuum is only SU(2)-symmetric, thus $\langle\sigma_0\rangle_0 \equiv v_0 \neq 0$. A new field S_0 is introduced,

$$S_0 = \sigma_0 - \langle\sigma_0\rangle_0 .$$

The translated Lagrangian is

$$\begin{aligned}\mathcal{L}(x) = {} & \tfrac{1}{2}[(\partial_\mu S_0)^2 - (m_{0\pi}^2 + 2\lambda_0^2 v_0^2)S_0^2] + \tfrac{1}{2}[(\partial_\mu\vec{\pi}_0)^2 - m_{0\pi}^2\vec{\pi}^2] + \bar{\psi}_0(i\gamma\cdot\partial - m_0)\psi_0 \\ & - \tfrac{1}{4}\lambda_0^2(S_0^2+\vec{\pi}_0^2)^2 - \lambda_0^2 v_0 S_0(S_0^2+\vec{\pi}_0^2) - g_0\bar{\psi}_0(S_0 + i\vec{\pi}_0\cdot\vec{\tau}\gamma_5)\psi_0 + (c_0 - v_0 m_{0\pi}^2)S_0,\end{aligned} \tag{5a}$$

where

$$m_{0\pi}^2 = \mu_0^2 + (\lambda_0 v_0)^2 \quad \text{and} \quad m_0 = g_0 v_0 .$$

Note that we have omitted a c-number constant in Eq. (5a).

We define the renormalized fields and renormalization constants as follows:

$$\begin{aligned} & \psi_0 = Z_F^{1/2}\psi, \\ & (\vec{\pi}_0, S_0, v_0) = Z_M^{1/2}(\vec{\pi}, S, v), \\ & \lambda_0^2 = \lambda^2 Z_\lambda / Z_M^2, \\ & g_0 = g Z_g/(Z_F Z_M^{1/2}), \\ & m_{0\pi}^2 = m_\pi^2 + \delta\mu_\pi^2/Z_M, \\ & m_0 = g v Z_g/Z_F = m + \delta m/Z_F, \\ & M^2 \equiv 2(\lambda v)^2 . \end{aligned} \tag{5b}$$

Notice that we have introduced the renormalization constants and counter terms in a SU(2)$\otimes$SU(2)-symmetric way. If these constants are sufficient to make all matrix elements finite (as we shall show in the next section), the renormalized matrix elements will then satisfy the current-algebra constraints and PCAC. The renormalized Lagrangian is

$$
\begin{aligned}
\mathcal{L}(x) &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{counter}} + \mathcal{L}_S\,,\\
\mathcal{L}_0 &= \bar\psi(i\gamma\cdot\partial - m)\psi + \tfrac12\left[(\partial_\mu\vec\pi)^2 - m_\pi{}^2\vec\pi^{\,2}\right] + \tfrac12\left[(\partial_\mu S)^2 - (M^2+m_\pi{}^2)S^2\right],\\
\mathcal{L}_{\text{int}} &= -g\bar\psi(S + i\vec\pi\cdot\vec\tau\gamma_5)\psi - \tfrac14\lambda^2(S^2+\vec\pi^{\,2})^2 - \lambda^2 v S(S^2+\vec\pi^{\,2})\,,\\
\mathcal{L}_{\text{counter}} &= (Z_F-1)\bar\psi(i\gamma\cdot\partial - m)\psi - \delta m\bar\psi\psi - g(Z_g-1)\bar\psi(S+i\vec\pi\cdot\vec\tau\gamma_5)\psi + \tfrac12(Z_M-1)\left[(\partial_\mu S)^2 + (\partial_\mu\vec\pi)^2 - m_\pi{}^2(\vec\pi^{\,2}+S^2)\right]\\
&\quad - \tfrac12 M^2(Z_M-1)S^2 - \tfrac12\delta\mu_\pi{}^2(S^2+\vec\pi^{\,2}) - \tfrac14\lambda^2(Z_\lambda-1)(S^2+\vec\pi^{\,2})^2 - \lambda^2 v(Z_\lambda-1)S(S^2+\vec\pi^{\,2})\,,\\
\mathcal{L}_S &= \left[c_0 Z_M{}^{1/2} - v(Z_M m_\pi{}^2 + \delta\mu_\pi{}^2)\right]S\,.
\end{aligned}
\tag{6}
$$

There are four independent parameters in our model, they are v, $m_\pi{}^2$, m, and λ.

The summed effect of $\mathcal{L}_S$ should be zero. It can be shown that this leads to a relation

$$c = v[-\Delta^\pi(0)]^{-1}\,, \tag{7}$$

where $c = c_0 Z_M{}^{1/2}$ and $\Delta^\pi(s)$ is the renormalized pion propagator.

Equation (7) states the Goldstone theorem. In the limit of spontaneous symmetry breaking, where $c\to0$ and $v\neq0$, the physical pion mass should be zero.

We end this section by listing the Feynman rules for $\mathcal{L}_{\text{int}}$. See Fig. 1.

We use solid, waved, and dashed lines to represent nucleon, σ, and pions, respectively.

III. RENORMALIZATION OF ONE-LOOP AMPLITUDES

We list the contribution of various quantities, keeping the counterterms where necessary. The inverse pion propagator is (Fig. 2)

$$\Delta_\pi{}^{-1}(p^2) = p^2 - m_\pi{}^2 + \lambda^2[B_\sigma(0) + 5B_\pi(0)] + 4\lambda^2(\lambda v)^2 B_{\pi\sigma}(p^2) + 2g^2 B^\pi_{NN}(p^2) + (Z_M-1)(p^2-m_\pi{}^2) - \delta\mu_\pi{}^2\,. \tag{8a}$$

The inverse σ propagator is (Fig. 3)

$$
\begin{aligned}
\Delta_\sigma{}^{-1}(p^2) &= p^2 - (m_\pi{}^2 + M^2) + 3\lambda^2[B_\sigma(0) + B_\pi(0)] + 6\lambda^2(\lambda v)^2[3B_{\sigma\sigma}(p^2) + B_{\pi\pi}(p^2)]\\
&\quad + 2g^2 B^\sigma_{NN}(p^2) + (Z_M-1)(p^2 - m_\pi{}^2) - \delta\mu_\pi{}^2 - 2(\lambda v)^2(Z_\lambda - 1)
\end{aligned}
\tag{8b}
$$

and the $\sigma\pi^2$ vertex is (Fig. 4)

$$
\begin{aligned}
T^{\alpha\beta}_{\pi\pi\sigma}(k_1k_2k_3) &= -2i\lambda^2 v\Gamma_\sigma(k_1k_2k_3)\,,\\
\Gamma_\sigma(k_1k_2k_3) &= Z_\lambda - 5\lambda^2 B_{\pi\pi}(k_3{}^2) - 3\lambda^2 B_{\sigma\sigma}(k_3{}^2) - 2\lambda^2 B_{\pi\sigma}(k_1{}^2) - 2\lambda^2 B_{\pi\sigma}(k_2{}^2)\\
&\quad - 4\lambda^2(\lambda v)^2[C_{\pi\pi\sigma}(k_1k_2) + 3C_{\sigma\sigma\pi}(k_1k_2)] - 2\frac{g^3}{\lambda^2 v}E_{NNN}(k_1k_2k_3)\,,
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
&B_x(p^2) = \frac1i\int_l \frac{1}{(p-l)^2 - m_x{}^2}\,, \qquad B_{xy}(p^2) = \frac1i\int_l\frac{1}{(l^2-m_x{}^2)[(l-p)^2 - m_y{}^2]}\,,\\
&B^\pi_{NN}(p^2) = \frac1i\int_l \mathrm{Tr}\left(\gamma_5\frac{1}{\not l - m}\gamma_5\frac{1}{\not l - \not p - m}\right), \qquad B^\sigma_{NN}(p^2) = -\frac1i\int_l\mathrm{Tr}\left(\frac{1}{\not l - m}\,\frac{1}{\not l - \not p - m}\right),\\
&C_{xyz}(k_1k_2k_3) = \frac1i\int_l\frac{1}{[(l-k_1)^2 - m_x{}^2][(l+k_2)^2 - m_y{}^2](l^2 - m_z{}^2)}\,,\\
&E_{NNN}(k_1k_2k_3) = \frac1i\int_l\mathrm{Tr}\left(\frac{1}{\not l + \not k_2 - m}\gamma_5\frac{1}{\not l - m}\gamma_5\frac{1}{\not l - \not k_1 - m}\right),
\end{aligned}
\tag{10}
$$

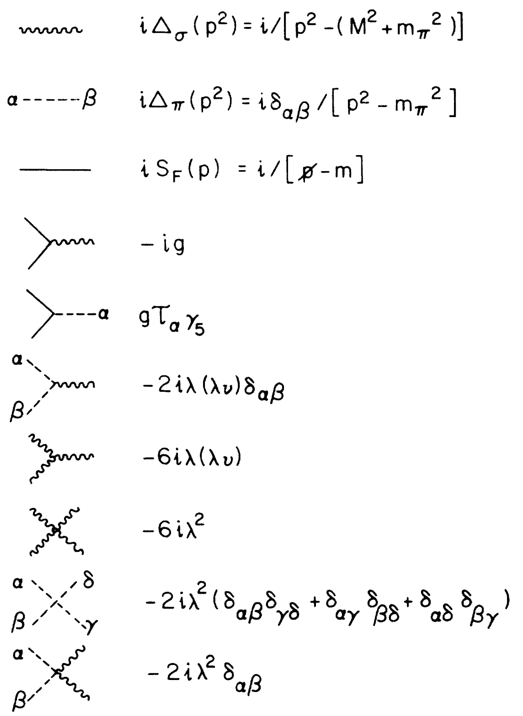

FIG. 1. Feynman rules for the tree quantities.

where

$$\int_l \equiv \int \frac{d^4l}{(2\pi)^4} \quad \text{and} \quad k_3 = k_1 + k_2 .$$

The following relations hold:

$$\begin{aligned} B_{NN}^{\pi}(p^2) &= -2[B_N(p^2)+B_N(0)-p^2B_{NN}(p^2)], \\ B_{NN}^{\sigma}(p^2) &= B_{NN}^{\pi}(p^2) - 8m^2B_{NN}(p^2), \\ E_{NNN}(k_1k_2k_3) &= -4mB_{NN}(k_3^2) \\ &\quad + 2m(k_1^2+k_2^2-k_3^2)C_{NNN}(k_1k_2k_3). \end{aligned} \tag{11}$$

We determine the counterterms by requiring that

$$\Delta_\pi^{-1}(m_\pi^2) = 0 \tag{12a}$$

and

$$\frac{d}{dp^2}\Delta_\pi^{-1}(m_\pi^2) = 1 . \tag{12b}$$

From Eq. (8a), $\delta\mu_\pi^2$ and Z_M are

$$\delta\mu_\pi^2 = \lambda^2[B_\sigma(0)+5B_\pi(0)] + 4\lambda^2(\lambda^2v^2)B_{\pi\sigma}(m_\pi^2) + 2g^2B_{NN}^{\pi}(m_\pi^2), \tag{13}$$

$$Z_M = 1 - 4\lambda^2(\lambda^2v^2)B'_{\pi\sigma}(m_\pi^2) - 2g^2B_{NN}^{\pi\prime}(m_\pi^2), \tag{14}$$

and it is obvious that

$$\Delta_\pi^{-1}(p^2) = p^2 - m_\pi^2 + 2\lambda^2M^2B_{\pi\sigma}^{R}(p^2) + 2g^2B_{NN}^{\pi R}(p^2), \tag{15}$$

where

$$\begin{aligned} B_{\pi\sigma}^{R}(p^2) &= B_{\pi\sigma}(p^2) - B_{\pi\sigma}(m_\pi^2) \\ &\quad - (p^2-m_\pi^2)B'_{\pi\sigma}(m_\pi^2) \\ &= O((p^2-m_\pi^2)^2). \end{aligned}$$

We subtract $\Gamma_\sigma(k_1, k_2, k_3)$ at a point where all external momenta are zero and assume that

$$\Gamma_\sigma(000) = 1 + 3\lambda^2[B_{\sigma\sigma}(0) - B_{\pi\pi}(0)] . \tag{16}$$

This implies

$$Z_\lambda = 1 + 12\lambda^2B_{\sigma\sigma}(0) - \frac{g^3}{\lambda^2 v}(8m)B_{NN}(0) . \tag{17}$$

With Eq. (17) we have

$$\begin{aligned} \Gamma_\sigma(k_1k_2k_3) &= 1 - 5\lambda^2\overline{B}_{\pi\pi}(k_3^2) - 3\lambda^2\overline{B}_{\sigma\sigma}(k_3^2) - 2\lambda^2\overline{B}_{\pi\sigma}(k_1^2) - 2\lambda^2\overline{B}_{\pi\sigma}(k_2^2) \\ &\quad - 4\lambda^2(\lambda^2v^2)[C_{\pi\pi\sigma}(k_1k_2) + 3C_{\sigma\sigma\pi}(k_1k_2)] - 2\frac{g^3}{\lambda^2 v}\overline{E}_{NNN}(k_1k_2k_3), \end{aligned} \tag{18}$$

where

$$\overline{B}_{xy}(p^2) = B_{xy}(p^2) - B_{\sigma\sigma}(0)$$

and

$$\overline{E}_{NNN}(k_1k_2k_3) = E_{NNN}(k_1k_2k_3) + 4mB_{NN}(0) .$$

Using the values of $\delta\mu_\pi^2$, Z_λ, and Z_M, one has

$$\begin{aligned} \Delta_\sigma^{-1}(p^2) &= p^2 - (m_\pi^2+M^2) + 9\lambda^2M^2\overline{B}_{\sigma\sigma}(p^2) + 3\lambda^2M^2\overline{B}_{\pi\pi}(p^2) + 2\lambda^2M^2[B_{\pi\sigma}(0) - B_{\pi\sigma}(m_\pi^2)] + 2g^2B_{NN}^{\pi R}(p^2) \\ &\quad - 16g^2m^2\overline{B}_{NN}(p^2) - 2\lambda^2M^2(p^2-m_\pi^2)B'_{\pi\sigma}(m_\pi^2) . \end{aligned} \tag{19}$$

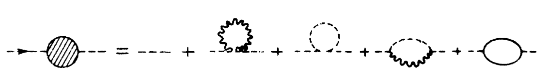
FIG. 2. One-loop pion propagator.

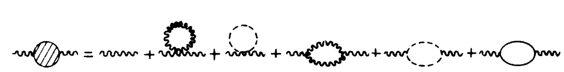
FIG. 3. One-loop σ propagator.

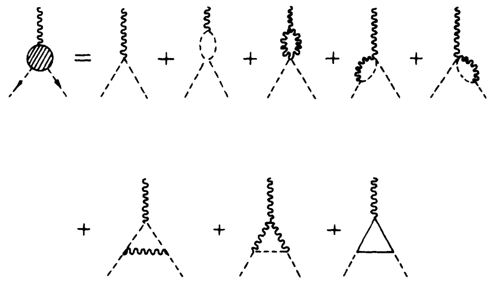

FIG. 4. One-loop $\sigma\pi\pi$ vertex.

It is easy to see that Δ_π^{-1}, Γ_σ, and Δ_σ^{-1} in Eqs. (15), (18), and (19), respectively, are all finite.

The inverse nucleon propagator is (see Fig. 5)

$$S_F^{-1}(\not{p}) = \not{p} - m - 3g^2 B_{\pi N}(\not{p}) - g^2 B_{\sigma N}(\not{p}) - \delta m + (Z_F - 1)(\not{p} - m), \tag{20}$$

where

$$B_{\pi N}(\not{p}) = \frac{1}{i}\int_l \frac{-\not{l} + m}{(l^2 - m^2)[(p-l)^2 - m_\pi^2]},$$
$$B_{\sigma N}(\not{p}) = \frac{1}{i}\int_l \frac{-\not{l} - m}{(l^2 - m^2)[(p-l)^2 - m_\sigma^2]}. \tag{21}$$

Let us define

$$\Sigma_F(\not{p}) = 3g^2 B_{\pi N}(\not{p}) + g^2 B_{\sigma N}(\not{p}). \tag{22}$$

Due to Lorentz covariance, we will have

$$\Sigma_F(\not{p}) = \not{p}\Sigma_V(p^2) + \Sigma_S(p^2). \tag{23}$$

We determine the counterterms by requiring that

$$S_F^{-1}(\not{p})|_{\not{p}=m} = 0, \quad \frac{\partial}{\partial\not{p}} S_F^{-1}\Big|_{\not{p}=m} = 1. \tag{24}$$

From Eq. (20),

$$\delta m = -m\Sigma_V(m^2) - \Sigma_S(m^2),$$
$$Z_F = 1 + \Sigma_V(m^2) + 2m^2\Sigma_V'(m^2) + 2m\Sigma_S'(m^2), \tag{25}$$

therefore,

$$S_F^{-1}(\not{p}) = \not{p} - m - \Sigma_F^R(\not{p})$$
$$= \not{p} - m - [\Sigma_S(p^2) - \Sigma_S(m^2) - 2m(\not{p} - m)\Sigma_S'(m^2)] - \{\not{p}[\Sigma_V(p^2) - \Sigma_V(m^2)] - 2m^2(\not{p} - m)\Sigma_V'(m^2)\}, \tag{26}$$

and it is finite.

The πNN vertex is (Fig. 6)

$$T^\alpha_{\pi NN}(kp_1p_2) = \bar{u}(p_2)\Gamma^\alpha_{\pi NN}u(p_1), \tag{27}$$

$$\Gamma^\alpha_{\pi NN}(kp_1p_2) = \gamma_5\tau_\alpha[g + g^3 C_{NN\pi}(p_1p_2) + g^3 C_{NN\sigma}(p_1p_2) + 2\lambda^2 v g^2 C_{\sigma\pi N}(p_1p_2) + 2\lambda^2 v g^2 C_{\pi\sigma N}(p_1p_2)] + \gamma_5\tau_\alpha g(Z_g - 1), \tag{28}$$

where the auxiliary functions $C_{NN\pi}(p_1, p_2)$ etc. are given in Appendix A.

Z_g is determined by the condition that the Ward identity

$$v\Gamma^\alpha_{\pi NN}(op_1p_2) = \gamma_5\tau_\alpha[m + \Sigma_S(p_1^2) + \delta m + m(Z_F - 1)] \tag{29}$$

is still satisfied after renormalization. By using $vg = m$, which is the Ward identity in lowest order, and by identifying the constants, one has

$$vg(Z_g - 1) = \delta m + m(Z_F - 1)$$

or

$$Z_g = 1 + \frac{\delta m}{m} + (Z_F - 1). \tag{30}$$

Z_g thus determined also cancels the logarithmic divergence in the square brackets of Eq. (28). Therefore $\Gamma^\alpha_{\pi NN}$ is finite.

If one wants to identify g with the physical quantity $g_{\pi NN}$, one can define $\Gamma^\alpha_{\pi NN}(m_\pi^2, m, m) = g_{\pi NN}$ and get the relation of g and $g_{\pi NN}$.

The σNN vertex is given as (Fig. 7)

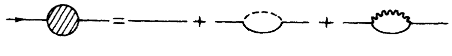

FIG. 5. One-loop nucleon propagator.

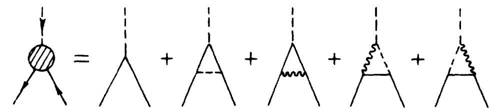

FIG. 6. One-loop πNN vertex.

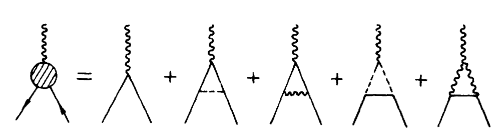

FIG. 7. One-loop σNN vertex.

$$T_{\sigma NN}(kp_1p_2)=\bar{u}(p_2)\Gamma_{\sigma NN}(p_1p_2)u(p_1)\,, \tag{31}$$

$$\Gamma_{\sigma NN}(p_1p_2)=-i\left[g+3g^3D_{NN\pi}(p_1p_2)+g^3D_{NN\sigma}(p_1p_2)+6g^2\lambda^2vD_{\pi\pi N}(p_1p_2)+6g^2\lambda^2vD_{\sigma\sigma N}(p_1p_2)\right]-ig(Z_g-1)\,. \tag{32}$$

$D_{NN\pi}(p_1,p_2)$ etc. is given in Appendix A. By using Z_g obtained before we can show that $\Gamma_{\sigma NN}(p_1p_2)$ is also finite.

The proper π^4 vertex is (see Fig. 8)

$$V_{\alpha\beta\gamma\delta}(k_1k_2k_3k_4)=-2i\lambda^2[\delta_{\alpha\beta}\delta_{\gamma\delta}V(k_1k_2;k_3k_4)+\delta_{\alpha\gamma}\delta_{\beta\delta}V(k_1k_3;k_2k_4)+\delta_{\alpha\delta}\delta_{\beta\gamma}V(k_1k_4;k_2k_3)]\,, \tag{33}$$

where

$$\begin{aligned}V(k_1k_2;k_3k_4)=1&-\lambda^2[7\bar{B}_{\pi\pi}(s)+2\bar{B}_{\pi\pi}(t)+2\bar{B}_{\pi\pi}(u)+\bar{B}_{\sigma\sigma}(s)]\\&-2\lambda^2M^2[C_{\sigma\sigma\pi}(k_1k_2)+C_{\sigma\sigma\pi}(k_3k_4)+C_{\pi\pi\sigma}(k_1k_2)+C_{\pi\pi\sigma}(k_3k_4)\\&\qquad+C_{\pi\pi\sigma}(k_2k_4)+C_{\pi\pi\sigma}(k_1k_3)+C_{\pi\pi\sigma}(k_2k_3)+C_{\pi\pi\sigma}(k_1k_4)]\\&-2\lambda^2(M^2)^2[D_{\pi\sigma\pi\sigma}(k_1k_2;k_3k_4)+D_{\pi\sigma\pi\sigma}(k_1k_2;k_4k_3)]\\&+\frac{g^4}{\lambda^2}[D_{NNNN}(k_1k_2;k_3k_4)+D_{NNNN}(k_1k_2;k_4k_3)-8B_{NN}(0)]\,.\end{aligned} \tag{34}$$

In Eq. (34), we have used the value of Z_λ from Eq. (17). We see that $V(k_1k_2;k_3k_4)$ is finite.

Finally, the proper πN scattering amplitude is (Fig. 9)

$$\begin{aligned}V_{\pi N\pi N}(k_1p_1;k_2p_2)=\;&ig^4(2\tau_\alpha\tau_\beta+\tau_\beta\tau_\alpha)E_{NNN\pi}(k_1p_1;k_2p_2)+ig^4(2\tau_\beta\tau_\alpha+\tau_\alpha\tau_\beta)E_{NNN\pi}(-k_2p_1;-k_1p_2)\\&+ig^4\tau_\beta\tau_\alpha E_{NNN\sigma}(k_1p_1;k_2p_2)+ig^4\tau_\alpha\tau_\beta E_{NNN\sigma}(-k_2p_1;-k_1p_2)\\&+i\,10g^2\lambda^2\delta_{\alpha\beta}(-)D_{\pi\pi N}(p_1p_2)+i\,2\lambda^2g^2\delta_{\alpha\beta}(-)D_{\sigma\sigma N}(p_1p_2)\\&+i\,(2\lambda^2v)^2g^2\tau_\beta\tau_\alpha E_{\pi\sigma\pi N}(k_1p_1;k_2p_2)+i\,(2\lambda^2v)^2g^2\tau_\alpha\tau_\beta E_{\pi\sigma\pi N}(-k_2p_1;-k_1p_2)\\&+i\,(2\lambda^2v)^2g^2\delta_{\alpha\beta}E_{\sigma\pi\sigma N}(k_1p_1;k_2p_2)+i\,(2\lambda^2v)^2g^2\delta_{\alpha\beta}E_{\sigma\pi\sigma N}(-k_2p_1;-k_1p_2)\\&+i\,(2\lambda^2v)g^3\tau_\beta\tau_\alpha E_{\pi NN\sigma}(k_1p_1;k_2p_2)+i\,(2\lambda^2v)g^3\tau_\alpha\tau_\beta E_{\sigma NN\pi}(k_1p_1;k_2p_2)\\&+i\,(2\lambda^2v)g^3\tau_\beta\tau_\alpha E_{NN\pi\sigma}(k_1p_1;k_2p_2)+i\,(2\lambda^2v)g^3\tau_\alpha\tau_\beta E_{NN\sigma\pi}(k_1p_1;k_2p_2)\,.\end{aligned} \tag{35}$$

The right-hand side of Eq. (35) is finite.

In conclusion, we have carried out the renormalization up to one loop explicitly. The finite expressions of Δ_π^{-1}, Δ_σ^{-1}, S_F^{-1}, Γ_σ, $\Gamma_{\pi NN}$, $\Gamma_{\sigma NN}$, $V_{4\pi}$, and $V_{\pi N\pi N}$, in terms of many auxiliary quantities, are given in Eqs. (15), (19), (26), (18), (28), (32), (34), and (35), respectively. We study the low-energy limit of these auxiliary quantities in the Appendix B. In the next section, we shall determine the low-energy limit of the two-point functions and $\pi\pi$ and πN scattering amplitudes.

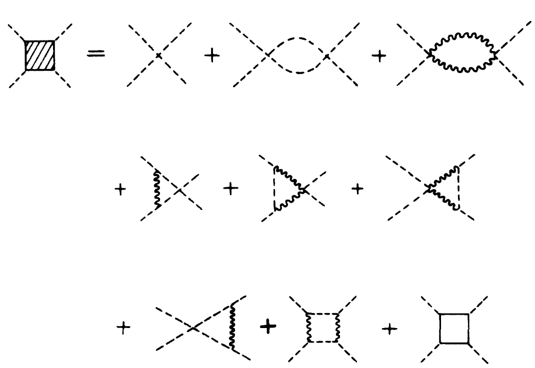

FIG. 8. One-loop $\pi\pi$ amplitude (one-particle irreducible).

IV. LOW-ENERGY LIMIT

In this section, we give the low-energy-limit results of π and N propagators, $\pi\pi$ scattering amplitude, and πN scattering amplitude. We refer to Appendix B for details of individual diagrams calculations.

The σ model is $SU(2)\otimes SU(2)$-symmetric. This symmetry is broken by a term which is linear in σ and is characterized by a parameter c. In the limit $c=0$, pions are Goldstone bosons. When $c\neq 0$, the symmetry-breaking term gives the pion a mass squared that is proportional to c. If k is the external pion momentum and p is the external nucleon momentum, we characterize the softness of the pion and the departure of the nucleon from its

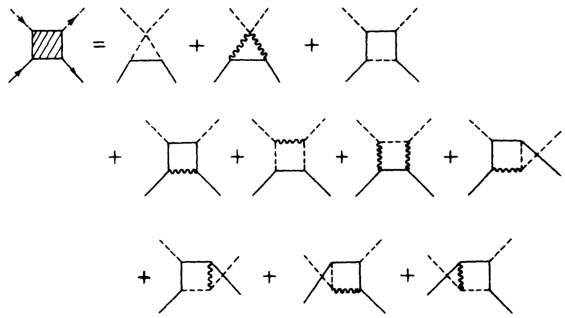

FIG. 9. One-loop πN amplitude (one-particle irreducible).

mass shell by a single parameter ζ, i.e.,

$$k = \zeta\phi$$

and

$$(\not p - m)u(p) = \zeta\not\phi u(p)$$
$$\bar u(p)(\not p - m) = \zeta\bar u(p)\not\phi\,,$$

where ϕ is finite.

The low-energy limit is defined as the limit where $\zeta^2 \to 0$ and $c \to 0$ simultaneously. We shall calculate the amplitudes to the nonvanishing order in this limit. We want to show that no contributions of the order $c\ln c$, $\zeta\ln c$, and $\zeta^2\ln c$ exist in these amplitudes. As a result of this, the nonanalytic corrections vanish like $\zeta^3\ln c$, $\zeta^2 c\ln c$, or $c^2\ln c$.

The following remark is in order: If a bare diagram does not involve any internal pion line, it gives no nonanalytic correction. The reason is, in this case, the pion mass appears only through the subtraction constants (we subtract at the point where $p_i^2 = m_\pi^2$). After the internal-momentum integration, the denominator of the integrand in the resultant multidimensional Feynman integral is of the form

$$[m_i^2 x + m_j^2(1-x) - m_\pi^2 x(1-x)]^n\,,$$

where $n \geqslant 0$ and $m_{i,j}$ are σ mass or nucleon mass. Because of the smallness of the physical pion mass as compared with $m_{i,j}$, the denominator does not vanish for all x; therefore, there is no infrared divergence as $m_\pi^2 \to 0$ for this diagram. As an example, we study the nucleon loop in the pion inverse propagator. Equations (11), (15), and (B4) together show that the above conclusion is correct.

The S-matrix elements are analytic in the external momenta. We can expand these matrix elements in a Taylor series about the appropriate momenta, e.g., we have

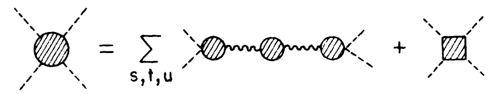

FIG. 10. One-loop $\pi\pi$ scattering amplitude.

$$V(z_i) = V(0) + \sum_i z_i \frac{\partial}{\partial z_i} V(z_i)\bigg|_{z_i=0} + \cdots\,,$$

where z_i's are all the possible invariant momenta squared formed by the external pion momenta. Since the nonanalytic behavior is no worse than $\ln m_\pi^2$, we shall calculate the coefficient of $\ln m_\pi^2$ or $m_\pi^2 \ln m_\pi^2$ only in the above expansion.

In studying the low-energy limit, we first consider the second-order pion inverse propagator. From Eqs. (15) and (B3), it is easy to see that

$$\begin{aligned}\Delta_\pi^{-1}(p^2) &\sim p^2 - m_\pi^2 + 2\lambda^2 M^2 B^R_{\pi\sigma}(p^2) + \cdots \\ &\sim p^2 - m_\pi^2 \\ &\quad + (p^2 - m_\pi^2)^2 \frac{\lambda^2}{16\pi^2 M^2}\, 2\frac{m_\pi^2}{M^2}\ln\frac{m_\pi^2}{M^2} + \cdots\,.\end{aligned} \tag{36}$$

The nonanalytic correction is of the order of

$$(p^2 - m_\pi^2)^2 m_\pi^2 \ln(m_\pi^2/M^2) \quad\text{or}\quad (\zeta^2 - c)^2 c\ln c\,.$$

Next we turn to the inverse nucleon propagator. From Eqs. (26) and (B8), it is equal to

$$S_F^{-1}(\not p) \sim \not p - m - 3g^2 \frac{(\not p - m)^2}{2} B''_{\pi N}(m) + \cdots \tag{37}$$

$$\sim \not p - m - (\not p - m)^2 \frac{6g^2}{16\pi^2 m}\left(\frac{m_\pi^2}{m^2}\right)\ln\frac{m_\pi^2}{m^2} + \cdots\,. \tag{38}$$

Again, the nonanalytic correction is of the order of $\zeta^2 c\ln c$.

Notice that in both the π and nucleon propagators, double subtractions at their corresponding physical masses give rise to extra momentum factors. We no longer have them in the vertices considered below.

We would like to study the scattering amplitudes of $\pi\pi$ and πN which are given diagrammatically in Figs. (10) and (11), respectively. To this end, we need the Taylor expansion of several renormalized vertices. The behavior of individual diagrams is

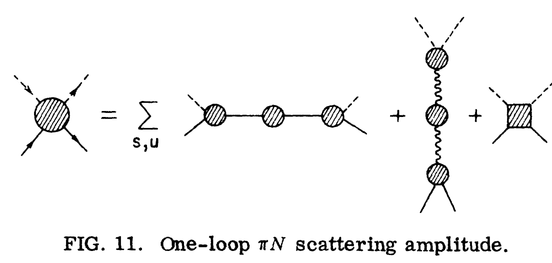

FIG. 11. One-loop πN scattering amplitude.

given in Appendix B and the expressions for each relevant vertex are summarized below.
From Eqs. (18), (19), (34), (B10)–(B16), we have

$$\Gamma_\sigma(k_1k_2k_3) \sim -5\lambda^2\overline{B}_{\pi\pi}(k_3^2) - 2\lambda^2\overline{B}_{\pi\sigma}(k_1^2) - 2\lambda^2\overline{B}_{\pi\sigma}(k_2^2) - 4\lambda^2(\lambda^2v^2)[C_{\pi\pi\sigma}(k_1k_2) + 3C_{\sigma\sigma\pi}(k_1k_2)] + \cdots$$
$$\sim \frac{\lambda^2}{16\pi^2}\left(3\ln\frac{m_\pi^2}{m_\sigma^2} + \frac{1}{m_\sigma^2}(k_3^2 - k_1^2 - k_2^2)\ln\frac{m_\pi^2}{m_\sigma^2}\right) + O(\zeta^4\ln c, \zeta^2 c\ln c, c^2\ln c), \tag{39}$$

$$\Delta_\sigma^{-1}(p^2) \sim 3\lambda^2M^2\overline{B}_{\pi\pi}(p^2) + 2\lambda^2M^2[B_{\pi\sigma}(0) - B_{\pi\sigma}(m_\pi^2)] - 2\lambda^2M^2(p^2 - m_\pi^2)B'_{\pi\sigma}(m_\pi^2) + \cdots$$
$$\sim -\frac{3\lambda^2}{16\pi^2}m_\sigma^2\left(\ln\frac{m_\pi^2}{m_\sigma^2} - \frac{m_\pi^2}{m_\sigma^2}\ln\frac{m_\pi^2}{m_\sigma^2}\right) + O(\zeta^4\ln c, \zeta^2 c\ln c, c^2\ln c), \tag{40}$$

$$V(k_1k_2; k_3k_4) \sim -\lambda^2[7\overline{B}_{\pi\pi}(s) + 2\overline{B}_{\pi\pi}(t) + 2\overline{B}_{\pi\pi}(u)] - 2\lambda^2M^2[C_{\sigma\sigma\pi}(k_1k_2) + C_{\sigma\sigma\pi}(k_3k_4) + C_{\pi\pi\sigma}(k_1k_2) + C_{\pi\pi\sigma}(k_3k_4)$$
$$+ C_{\pi\pi\sigma}(k_2k_4) + C_{\pi\pi\sigma}(k_1k_3) + C_{\pi\pi\sigma}(k_2k_3) + C_{\pi\pi\sigma}(k_1k_4)]$$
$$- 2\lambda^2(M^2)^2[D_{\pi\sigma\pi\sigma}(k_1k_2; k_3k_4) + D_{\pi\sigma\pi\sigma}(k_1k_2; k_4k_3)] + \cdots$$
$$\sim \frac{\lambda^2}{16\pi^2}\left(3\ln\frac{m_\pi^2}{m_\sigma^2} + \frac{1}{m_\sigma^2}(2k_1\cdot k_2 + 2k_3\cdot k_4)\ln\frac{m_\pi^2}{m_\sigma^2}\right) + O(\zeta^4\ln c, \zeta^2 c\ln c, c^2\ln c). \tag{41}$$

The T matrix for $\pi\pi$ scattering (σ reducible) is

$$T_{\alpha\beta\gamma\delta}(k_1k_2k_3k_4) = \delta_{\alpha\beta}\delta_{\gamma\delta}T(k_1k_2; k_3k_4) + \delta_{\alpha\gamma}\delta_{\beta\delta}T(k_1k_3; k_2k_4) + \delta_{\alpha\delta}\delta_{\beta\gamma}T(k_1k_4; k_2k_3), \tag{42}$$

where

$$T(k_1k_2; k_3k_4) = -2i\lambda^2V(k_1k_2; k_3k_4) + T^{\alpha\beta}(k_1k_2)i\Delta_\sigma(k_1 + k_2)T^{\alpha\beta}(k_3k_4)$$
$$\sim -2i\lambda^2\frac{\lambda^2}{16\pi^2}\ln\frac{m_\pi^2}{m_\sigma^2}\left[3 + \frac{1}{m_\sigma^2}(2k_1\cdot k_2 + 2k_3\cdot k_4) - \frac{m_\sigma^2 - m_\pi^2}{m_\sigma^2 - (k_1 + k_2)^2}\left(6 + \frac{1}{m_\sigma^2}(2k_1\cdot k_2 + 2k_3\cdot k_4)\right)\right.$$
$$\left. + \frac{m_\sigma^2(m_\sigma^2 - m_\pi^2)}{[m_\sigma^2 - (k_1 + k_2)^2]^2}\left(3 - 3\frac{m_\pi^2}{m_\sigma^2}\right)\right] + O(\zeta^4\ln c, \zeta^2 c\ln c, c^2\ln c)$$
$$= 0 + O(\zeta^4\ln c, \zeta^2 c\ln c, c^2\ln c). \tag{43}$$

From Eqs. (39)–(43), we observe that the renormalized vertices contain nonanalytic correction of the form $\ln c$, $\zeta^2\ln c$, and $c\ln c$, which cancel exactly in the observable $\pi\pi$ scattering amplitude. This exact cancellation is due to the symmetry of the Lagrangian and we conclude that the nonanalytic correction to the $\pi\pi$ scattering is at least of the order of $\zeta^4\ln c$, $\zeta^2 c\ln c$, or $c^2\ln c$.

The πN scattering amplitude can be divided into three parts:

$$\begin{aligned}
T_{\alpha\beta}(k_1p_1; k_2p_2) &= T_{\alpha\beta}^{(t)} + T_{\alpha\beta}^{(s)} + T_{\alpha\beta}^{(u)},\\
T_{\alpha\beta}^{(t)} &= T_{\sigma NN}(t, p_1p_2)i\Delta_\sigma(t)T_{\pi\pi\sigma}^{\alpha\beta}(t, k_1^2k_2^2) + i\,10g^2\lambda^2\delta_{\alpha\beta}(-)D_{\pi\pi N}(p_1p_2)\\
&\quad + i(2\lambda^2v)^2g^2\tau_\beta\tau_\alpha E_{\pi\sigma\pi N}(k_1p_1; k_2p_2) + i(2\lambda^2v)^2g^2\tau_\alpha\tau_\beta E_{\pi\sigma\pi N}(-k_2p_1; -k_1p_2),\\
T_{\alpha\beta}^{(s)} &= T_{\pi NN}^{\alpha}(k_1^2, \not{p}_1, \not{k}_1 + \not{p}_1)iS_F(\not{p}_1 + \not{k}_1)T_{\pi NN}^{\beta}(k_2^2, \not{k}_1 + \not{p}_1, \not{p}_2)\\
&\quad + ig^4(2\tau_\alpha\tau_\beta + \tau_\beta\tau_\alpha)E_{NNN\pi}(k_1p_1; k_2p_2) + i(2\lambda^2v)^2g^2\delta_{\alpha\beta}E_{\sigma\pi\sigma N}(k_1p_1; k_2p_2)\\
&\quad + i(2\lambda^2v)g^3\tau_\alpha\tau_\beta E_{\sigma NN\pi}(k_1p_1; k_2p_2) + i(2\lambda^2v)g^3\tau_\alpha\tau_\beta E_{NN\sigma\pi}(k_1p_1; k_2p_2),
\end{aligned} \tag{44}$$

$$T_{\alpha\beta}^{(u)}(k_1, k_2) = T_{\beta\alpha}^{(s)}(-k_2, -k_1). \tag{45}$$

From Eq. (40),

$$\Delta_\sigma^{-1}(p^2) \sim -3\frac{\lambda^2}{16\pi^2}M^2\ln m_\pi^2. \tag{46}$$

From Eq. (39),

$$T^{\alpha\beta}_{\pi\pi\sigma}=-2i\lambda^2 v\,\Gamma_\sigma\delta_{\alpha\beta},$$

$$\Gamma_\sigma(tk_1^2k_2^2)\sim 1+\frac{\lambda^2}{16\pi^2}\left(3\ln m_\pi^2+\frac{1}{M^2}(t-k_1^2-k_2^2)\ln m_\pi^2\right). \tag{47}$$

From Eqs. (32), (B19), (B23), and (B26),

$$\begin{aligned}\Gamma_{\sigma NN}(kp_1p_2)\sim -i\Big[\,g+\Big(3g^3D_{NN\pi}(p_1p_2)-\frac{g}{m}\Sigma_S^{(\pi)}(m^2)\Big)+6g^2\lambda^2vD_{\pi\pi N}(p_1p_2)+2m^2g\Sigma_V'(m^2)+2mg\Sigma_S'(m^2)\Big]\\
\sim -i\Big[\,g+\frac{3g^3}{16\pi^2}\Big(-\frac{m_\pi^2}{m^2}-\frac{1}{2m^2}(\not{Q}_1^2+\not{Q}_2^2+\not{Q}_2\not{Q}_1)\Big)\ln m_\pi^2\\
-\frac{6g\lambda^2}{16\pi^2}\ln m_\pi^2\Big(-\frac{1}{2}+\frac{1}{2}\frac{m_\pi^2}{m^2}+\tfrac{1}{2}(\not{Q}_1+\not{Q}_2)-\frac{1}{4}\frac{k^2}{m^2}-\frac{1}{2m^2}(\not{Q}_1^2+\not{Q}_2^2)-\frac{\not{Q}_2\not{Q}_1}{m^2}\Big)\Big],\end{aligned} \tag{48}$$

where $\not{Q}_1$ etc. are defined in Eq. (B27). Together with Eqs. (B26) and (B31),

$$T^{(t)}_{\alpha\beta}\sim i\frac{\delta_{\alpha\beta}}{16\pi^2}\frac{g^4}{m}\ln\frac{m_\pi^2}{m^2}\left(-2\frac{m_\pi^2}{m^2}-\frac{3}{2m^2}(\not{Q}_1^2+\not{Q}_2^2+\not{Q}_2\not{Q}_1)\right)+O(c^2\ln c,\ \zeta c\ln c,\ \zeta^3\ln c). \tag{49}$$

From Eqs. (28),

$$\begin{aligned}\Gamma^\alpha_{\pi NN}(kp_1p_2)\sim \tau_\alpha\gamma_5\Big[g+\Big(g^3C_{NN\pi}(p_1p_2)+g^3C_{NN\sigma}(p_1p_2)-\frac{g}{m}\Sigma_S(m^2)\Big)\\
+2\lambda^2vg^2C_{\sigma\pi N}(p_1p_2)+2\lambda^2vg^2C_{\pi\sigma N}(p_1p_2)+2m^2g\Sigma_V'(m^2)+2mg\Sigma_S'(m^2)\Big],\end{aligned} \tag{50}$$

and from Eqs. (B19), (B33)–(B35),

$$\Gamma^\alpha_{\pi NN}(kp_1p_2)\sim\tau_\alpha\gamma_5\left[g-6\frac{g^3}{16\pi^2}\ln\frac{m_\pi^2}{m^2}\left(\frac{m_\pi^2}{m^2}\right)+\frac{g^3}{16\pi^2}\ln\frac{m_\pi^2}{m^2}\left(-3\frac{\not{Q}_2}{m}+4\frac{\not{Q}_2^2}{m^2}+\frac{1}{2}\frac{\not{Q}_2\not{Q}_1}{m^2}\right)\right]. \tag{51}$$

From Eqs. (45) and (51), the vertex corrections to the s-channel amplitude is

$$ig^4\frac{\tau_\beta\tau_\alpha}{16\pi^2m}\ln\frac{m_\pi^2}{m^2}\left(6\frac{m_\pi^2}{m^2}+3\frac{\not{Q}_3}{m}-\frac{22}{4}\frac{\not{Q}_3^2}{m^2}-\frac{1}{4m^2}(\not{Q}_3\not{Q}_1+\not{Q}_2\not{Q}_3)\right)+O(c^2\ln c,\ \zeta c\ln c,\ \zeta^3\ln c). \tag{52}$$

The nucleon self-energy insertion gives

$$ig^2\gamma_5\frac{1}{\not{p}_3-m-\Sigma_F^R(\not{p}_3)}\gamma_5\sim ig^2\frac{1}{\not{p}_3+m}[\gamma_5\Sigma_F^R(\not{p}_3)\gamma_5]\frac{1}{\not{p}_3+m}.$$

From Eq. (B22),

$$\sim\frac{ig^4}{m}\frac{1}{16\pi^2}\ln\frac{m_\pi^2}{m^2}\left(-3\frac{m_\pi^2}{m^2}-\frac{3}{2}\frac{\not{Q}_3}{m}+\frac{15}{4}\frac{\not{Q}_3^2}{m^2}\right). \tag{53}$$

Equations (52) and (53) together with Eqs. (B36), (B37), and (B38) give

$$T^{(s)}_{\alpha\beta}\sim i\frac{g^4}{16\pi^2m}\ln\frac{m_\pi^2}{m^2}\left((-\tfrac{3}{2}\delta_{\alpha\beta}+\tfrac{5}{2}\tau_\beta\tau_\alpha)\frac{m_\pi^2}{m^2}+(2\tau_\alpha\tau_\beta+\tau_\beta\tau_\alpha)\tfrac{1}{4}(\not{Q}_1^2+\not{Q}_2^2+\not{Q}_2\not{Q}_1)\right)+O(c^2\ln c,\ \zeta c\ln c,\ \zeta^3\ln c). \tag{54}$$

It is easy to see from Eqs. (49) and (54) that

$$T_{\alpha\beta}\sim 0+O(c^2\ln c,\ \zeta c\ln c,\ \zeta^3\ln c). \tag{55}$$

Again we conclude that the nonanalytic correction to πN scattering amplitude is of higher order.

V. DISCUSSION

We have shown explicitly that the total sum of nonanalytic corrections of the lowest orders ($\ln c$, $c\ln c$, $\zeta\ln c$, $\zeta^2\ln c$) to the renormalized pion propagator, nucleon propagator, $\pi\pi$ and πN scattering amplitudes in each case is zero. Since there should be no infrared divergence if we renormalized our model properly, the cancellation of $\ln m_\pi^2$ serves as a check of our renormalization procedure. Recently, Li and Pagels[5] indicated that there might be nonanalytic behavior as bad as $m_\pi^2\ln m_\pi^2$ in the matrix elements. In our model calculation, we observe the nonanalytic corrections in individual diagrams, but they sum up to zero in the physically interesting amplitudes. We thus conclude that the nonanalytic corrections are of higher order and that the low-energy limit of those amplitudes (of order ζ^2 and c) should be derived from the usual PhL. We interpret our result as follows: Although it is probably possible to construct a model in which the low-order nonanalytic correction (e.g., $c\ln c$) persists, and thus invalidates the PhL prediction, our result is a solid example (possibly the only workable model that has all the symmetry properties) demonstrating that the joint effect of symmetry and renormalization results in the consistent cancellation of this low-order correction. It is very interesting to study the explicit low-energy limit of the amplitudes in the renormalized σ model to check whether they are given correctly by the PhL and to study the nonanalytic correction to the σ term in the πN scattering. These and other related topics are presented elsewhere.[9]

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix, we collect some complicated expressions used in Sec. III.

$$\begin{aligned}
C_{NN\pi}(p_1p_2)&=\frac{1}{i}\int_l\frac{(\not p_2-\not l+m)(\not p_1-\not l-m)}{[(p_2-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\pi^2]},\\
C_{NN\sigma}(p_1p_2)&=\frac{1}{i}\int_l\frac{(\not p_2-\not l-m)(\not p_1-\not l+m)}{[(p_2-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\sigma^2]},\\
C_{\sigma\pi N}(p_1p_2)&=\frac{1}{i}\int_l\frac{\not l-m}{[(p_1-l)^2-m_\pi^2][(p_2-l)^2-m_\sigma^2][l^2-m^2]},\\
C_{\pi\sigma N}(p_1p_2)&=-\frac{1}{i}\int_l\frac{\not l+m}{[(p_1-l)^2-m_\sigma^2][(p_2-l)^2-m_\pi^2][l^2-m^2]}.
\end{aligned}\tag{A1}$$

$$\begin{aligned}
D_{NN\pi}(p_1p_2)&=\frac{1}{i}\int_l\frac{(\not p_2-\not l-m)(\not p_1-\not l-m)}{[(p_2-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\pi^2]},\\
D_{NN\sigma}(p_1p_2)&=-\frac{1}{i}\int_l\frac{(\not p_2-\not l+m)(\not p_1-\not l+m)}{[(p_2-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\sigma^2]},\\
D_{\pi\pi N}(p_1p_2)&=\frac{1}{i}\int_l\frac{-\not l+m}{[(p_1-l)^2-m_\pi^2][(p_2-l)^2-m_\pi^2][l^2-m^2]},\\
D_{\sigma\sigma N}(p_1p_2)&=\frac{1}{i}\int_l\frac{-\not l-m}{[(p_1-l)^2-m_\sigma^2][(p_2-l)^2-m_\sigma^2][l^2-m^2]}.
\end{aligned}\tag{A2}$$

$$\begin{aligned}
D_{\pi\sigma\pi\sigma}(k_1k_2;k_3k_4)&=\frac{1}{i}\int_l\frac{1}{[l^2-m_\pi^2][(l-\hat p_2)^2-m_\sigma^2][(l+\hat p_4)^2-m_\pi^2][(l+\hat p_3)^2-m_\sigma^2]},\\
D_{NNNN}(k_1k_2;k_3k_4)&=\frac{1}{i}\int_l\operatorname{Tr}\left\{\gamma_5\frac{1}{\not l-m}\gamma_5\frac{1}{\not l-\hat{\not p}_2-m}\gamma_5\frac{1}{\not l+\hat{\not p}_4-m}\gamma_5\frac{1}{\not l+\hat{\not p}_3-m}\right\},
\end{aligned}\tag{A3}$$

where

$$\hat p_2=k_2,\quad \hat p_3=-(k_2+k_3+k_4),\quad \hat p_4=-(k_2+k_3)$$

in the above two equations.

$$
\begin{aligned}
E_{NNN\pi}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{(\not p_2-\not l-m)(\not p_1+\not k_1-\not l+m)(\not p_1-\not l-m)}{[(p_2-l)^2-m^2][(p_1+k_1-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\pi{}^2]},\\
E_{NNN\sigma}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{(\not p_2-\not l+m)(\not p_1+\not k_1-\not l-m)(\not p_1-\not l+m)}{[(p_2-l)^2-m^2][(p_1+k_1-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\sigma{}^2]},\\
E_{\pi\sigma\pi N}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{\not l-m}{[l^2-m^2][(p_1-l)^2-m_\pi{}^2][(p_2-l)^2-m_\pi{}^2][(p_1+k_1-l)^2-m_\sigma{}^2]},\\
E_{\sigma\pi\sigma N}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{\not l+m}{[l^2-m^2][(p_1-l)^2-m_\sigma{}^2][(p_2-l)^2-m_\sigma{}^2][(p_1+k_1-l)^2-m_\pi{}^2]},\\
E_{\pi NN\sigma}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{(\not p_1+\not k_1-\not l-m)(\not p_1-\not l+m)}{[(p_1+k_1-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\sigma{}^2][(k_2-l)^2-m_\pi{}^2]},\\
E_{\sigma NN\pi}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{(\not p_1+\not k_1-\not l+m)(\not p_1-\not l-m)}{[(p_1+k_1-l)^2-m^2][(p_1-l)^2-m^2][l^2-m_\pi{}^2][(k_2-l)^2-m_\sigma{}^2]},\\
E_{NN\pi\sigma}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{(\not p_2-\not l+m)(\not p_1+\not k_1-\not l-m)}{[(p_2-l)^2-m^2][(p_1+k_1-l)^2-m^2][(k_1-l)^2-m_\pi{}^2][l^2-m_\sigma{}^2]},\\
E_{NN\sigma\pi}(k_1p_1;k_2p_2)&=\frac{1}{i}\int_l\frac{(\not p_2-\not l-m)(\not p_1+\not k_1-\not l+m)}{[(p_2-l)^2-m^2][(p_1+k_1-l)^2-m^2][(k_1-l)^2-m_\sigma{}^2][l^2-m_\pi{}^2]}.
\end{aligned}
\tag{A4}
$$

APPENDIX B

The standard Feynman parametrization gives

$$B_{\pi\sigma}(p^2)=\frac{1}{i}\int dx\int_l\frac{1}{[l^2+p^2x(1-x)-m_\pi{}^2x-m_\sigma{}^2(1-x)]^2},\tag{B1}$$

$$B''_{\pi\sigma}(p^2)=\frac{1}{16\pi^2}\int dx\frac{x^2(1-x)^2}{[m_\pi{}^2x+m_\sigma{}^2(1-x)-p^2x(1-x)]^2}.\tag{B2}$$

We assume that $m_\sigma>2m_\pi$. $B''_{\pi\sigma}(m_\pi{}^2)$ can be integrated exactly, the result is of the form

$$(1/m_\pi{}^4)[A(m_\pi{}^2/m_\sigma{}^2)+B(m_\pi{}^2/m_\sigma{}^2)\ln(m_\pi{}^2/m_\sigma{}^2)],$$

where A and B are analytic at $\beta\equiv m_\pi{}^2/m_\sigma{}^2=0$. We expand A to the order of β^2 and B to the order of β^3 and find that

$$B''_{\pi\sigma}(m_\pi{}^2)=\frac{1}{16\pi^2m_\sigma{}^4}\left(\frac{1}{3}+2\frac{m_\pi{}^2}{m_\sigma{}^2}\ln\frac{m_\pi{}^2}{m_\sigma{}^2}\right)+O(\beta,\beta^2\ln\beta).\tag{B3}$$

We also find that

$$\int_l\frac{1}{(l-p)^2-m^2}=\int_l\frac{1}{l^2-m^2}+\frac{1}{32\pi^2i}p^2-\frac{1}{16\pi^2i}(p^2)^2\int_l dx\left(\frac{3x(1-x)(1-2x)}{m^2-l^2x(1-x)}+\frac{2x^2(1-x)^2(1-2x)p^2}{[m^2-l^2x(1-x)]^2}\right).\tag{B4}$$

This is obtained by using the following identity sufficient times:

$$\frac{1}{(k-p)^2-m^2}=\frac{1}{k^2-m^2}+\frac{2k\cdot p-p^2}{[(k-p)^2-m^2][k^2-m^2]}.\tag{B5}$$

From Eq. (21) a translation in l gives

$$B_{\pi N}(\not p)=\not p\left(\frac{1}{64\pi^2}-\frac{1}{i}\int_l dx\frac{x}{[l^2-m^2(1-x)-m_\pi{}^2x+p^2x(1-x)]^2}\right)+\frac{m}{i}\int_l dx\frac{1}{[l^2-m^2(1-x)-m_\pi{}^2x+p^2x(1-x)]^2}.\tag{B6}$$

From this,

$$B''_{\pi N}(\not p;m_\pi{}^2m^2)=\frac{4}{32\pi^2}\int dx\,x(1-x)\left(\frac{m-3x\not p}{m_\pi{}^2x+m^2(1-x)-p^2x(1-x)}+\frac{2x(1-x)p^2(-x\not p+m)}{[m_\pi{}^2x+m^2(1-x)-p^2x(1-x)]^2}\right),\tag{B7}$$

$$B''_{\pi N}(m;m_\pi{}^2m^2)\sim\frac{4}{16\pi^2m}\left(\frac{m_\pi{}^2}{m^2}\right)\ln\frac{m_\pi{}^2}{m^2}+O(c^2\ln c).\tag{B8}$$

From Eq. (B1), it is easy to find that

$$B_{xy}(p^2) - B_{zw}(p_0^2) = -\frac{1}{16\pi^2}\int_0^1 d\alpha \ln\left|\frac{p^2\alpha(1-\alpha) - m_x^2\alpha - m_y^2(1-\alpha)}{p_0^2\alpha(1-\alpha) - m_z^2\alpha - m_w^2(1-\alpha)}\right|. \tag{B9}$$

Therefore,

$$\begin{aligned}
\bar{B}_{\pi\pi}(p^2) &= -\frac{1}{16\pi^2}\int dx \ln\left|\frac{p^2x(1-x) - m_\pi^2}{m_\sigma^2}\right|,\\
\bar{B}_{\pi\pi}(0) &= -\frac{1}{16\pi^2}\ln\frac{m_\pi^2}{m_\sigma^2},\\
\bar{B}'_{\pi\pi}(0) &= \frac{1}{16\pi^2}\,\frac{1}{6}\,\frac{1}{m_\pi^2},\\
\bar{B}_{\pi\sigma}(0) &= \frac{1}{16\pi^2}\left(\frac{m_\pi^2}{m_\sigma^2 - m_\pi^2}\ln\frac{m_\pi^2}{m_\sigma^2} + 1\right),\\
\bar{B}'_{\pi\sigma}(0) &= \frac{1}{16\pi^2}\left(\frac{m_\sigma^2 + m_\pi^2}{2(m_\sigma^2 - m_\pi^2)^2} + \frac{m_\sigma^2 m_\pi^2}{(m_\sigma^2 - m_\pi^2)^3}\ln\frac{m_\pi^2}{m_\sigma^2}\right).
\end{aligned} \tag{B10}$$

We have

$$C_{\pi\pi\sigma}(k_1^2k_2^2k_1\cdot k_2) = -\frac{1}{16\pi^2}\int dx\, ydy\, \frac{1}{m_\pi^2y + m_\sigma^2(1-y) - k_1^2xy(1-xy) - k_2^2(1-x)y[1-(1-x)y] - 2k_1\cdot k_2x(1-x)y^2}, \tag{B11}$$

$$\begin{aligned}
C_{\pi\pi\sigma}(000) &= \frac{1}{16\pi^2}\left(\frac{1}{m_\sigma^2 - m_\pi^2} + \frac{m_\sigma^2}{(m_\sigma^2 - m_\pi^2)^2}\ln\frac{m_\pi^2}{m_\sigma^2}\right);\\
\frac{\partial}{\partial k_1^2}C_{\pi\pi\sigma}(000) &= \frac{\partial}{\partial k_2^2}C_{\pi\pi\sigma}(000)\\
&= \frac{1}{16\pi^2}\left[-\frac{1}{2(m_\sigma^2 - m_\pi^2)^2}\left(1 + \frac{m_\sigma^2}{m_\pi^2}\right) + \frac{m_\sigma^2m_\pi^2}{(m_\sigma^2 - m_\pi^2)^4}\ln\frac{m_\pi^2}{m_\sigma^2}\right.\\
&\qquad\left. + \frac{1}{3(m_\sigma^2 - m_\pi^2)^3}\left(-\tfrac{1}{2}(m_\sigma^2 + m_\pi^2) + 3m_\sigma^2 + \frac{m_\sigma^4}{m_\pi^2}\right)\right],\\
\frac{\partial}{\partial(k_1\cdot k_2)}C_{\pi\pi\sigma}(000) &= -\frac{1}{16\pi^2}\left[\frac{m_\sigma^4}{(m_\sigma^2 - m_\pi^2)^4}\ln\frac{m_\pi^2}{m_\sigma^2} + \frac{1}{3}\,\frac{1}{(m_\sigma^2 - m_\pi^2)^3}\left(-\tfrac{1}{2}(m_\sigma^2 + m_\pi^2) + 3m_\sigma^2 + \frac{m_\sigma^4}{m_\pi^2}\right)\right].
\end{aligned} \tag{B12}$$

From Eq. (B9),

$$\begin{aligned}
B_{\pi\sigma}(0) - B_{\pi\sigma}(m_\pi^2) &= -\frac{1}{16\pi^2}\int dx \ln\left|\frac{m_\pi^2x + m_\sigma^2(1-x)}{m_\pi^2x^2 + m_\sigma^2(1-x)}\right|\\
&= -\frac{1}{16\pi^2}\left(-\frac{\beta}{1-\beta}\ln\beta - \ln\beta + 1 + \frac{1}{2\beta}\ln\beta - \frac{1}{2\beta}(1-4\beta)^{1/2}\ln\frac{2\beta}{|1 - 2\beta + (1-4\beta)^{1/2}|}\right), \qquad \beta = \frac{m_\pi^2}{m_\sigma^2},\\
&\sim 0 + O(\beta, \beta^2\ln\beta),
\end{aligned} \tag{B13}$$

$$\begin{aligned}
B'_{\pi\sigma}(m_\pi^2) &= \frac{1}{16\pi^2m_\sigma^2}\left(\frac{1}{2\beta}\ln\beta - \frac{1}{\beta} - \frac{1}{2\beta^2}\ln\beta - \frac{1}{2\beta^2}(1-3\beta)\frac{1}{(1-4\beta)^{1/2}}\ln\left|\frac{1 + (1-4\beta)^{1/2}}{1 - (1-4\beta)^{1/2}}\right|\right)\\
&\sim \frac{1}{16\pi^2m_\sigma^2}(\beta\ln\beta + \tfrac{1}{2}) + O(\beta, \beta^2\ln\beta).
\end{aligned} \tag{B14}$$

Standard calculation gives

$$\begin{aligned}
D_{\pi\sigma\pi\sigma}(k_1k_2; k_3k_4) &= \frac{1}{16\pi^2}\int dxdyz(1-z)dz\frac{1}{\Delta_1^2},\\
\Delta_1 &= m_\pi^2(1-z) + m_\sigma^2z - p_2^2yz(1-yz) - p_3^2(1-y)z[1-(1-y)z]\\
&\quad - p_4^2x(1-z)[1 - x(1-z)] - 2p_2\cdot p_3y(1-y)z^2 - 2p_2\cdot p_4xyz(1-z) + 2p_3\cdot p_4x(1-y)z(1-z),
\end{aligned} \tag{B15}$$

where

$$p_2=k_2\,,\quad p_3=-(k_2+k_3+k_4)\,,\quad p_4=-(k_2+k_3)\,,$$

$$D_{\pi\sigma\pi\sigma}(k_1k_2;\,k_3k_4)\sim-\frac{1}{16\pi^2 m_\sigma^{\;4}}\left[\ln\frac{m_\pi^{\;2}}{m_\sigma^{\;2}}+4\frac{m_\pi^{\;2}}{m_\sigma^{\;2}}\ln\frac{m_\pi^{\;2}}{m_\sigma^{\;2}}-\frac{1}{m_\sigma^{\;2}}(k_2\cdot k_3+k_1\cdot k_4)\ln\frac{m_\pi^{\;2}}{m_\sigma^{\;2}}\right]+O(\zeta^4\ln c,\,\zeta^2 c\ln c,\,c^2\ln c)\,. \tag{B16}$$

From Eqs. (21)–(23) and (B6),

$$\Sigma_V^{(\pi)}(p^2)=-\frac{3g^2}{i}\int dx\int_l\frac{x}{[l^2-m^2(1-x)-m_\pi^{\;2}x+p^2x(1-x)]^2}, \tag{B17}$$

$$\Sigma_S^{(\pi)}(p^2)=\frac{3g^2m}{i}\int dx\int_l\frac{1}{[l^2-m^2(1-x)-m_\pi^{\;2}x+p^2x(1-x)]^2}. \tag{B18}$$

Thus,

$$\begin{aligned}\Sigma_V^{(\pi)\prime}(m^2)&=\frac{3g^2}{32m^2\pi^2}\ln\frac{m_\pi^{\;2}}{m^2}\left(1-3\frac{m_\pi^{\;2}}{m^2}\right)+O(c^2\ln c)\,,\\ \Sigma_S^{(\pi)\prime}(m^2)&=-\frac{3g^2}{32m\pi^2}\ln\frac{m_\pi^{\;2}}{m^2}\left(1-\frac{m_\pi^{\;2}}{m^2}\right)+O(c^2\ln c)\,.\end{aligned} \tag{B19}$$

Also,

$$\begin{aligned}\Sigma_V^{(\pi)\prime\prime}(m^2)&=-\frac{3g^2}{32\pi^2m^4}3\ln\frac{m_\pi^{\;2}}{m^2}+O(c\ln c)\,,\\ \Sigma_S^{(\pi)\prime\prime}(m^2)&=\frac{3g^2}{32\pi^2m^3}2\ln\frac{m_\pi^{\;2}}{m^2}+O(c\ln c)\,.\end{aligned} \tag{B20}$$

From Eqs. (26), (B19), and (B20),

$$\begin{aligned}\gamma_5\Sigma_F^{(R)}(\not p)\gamma_5&=4m^2[\Sigma_S'(m^2)+m\Sigma_V'(m^2)]+2m\,(\not p-m)[2\Sigma_S'(m^2)]\\&\quad+(\not p-m)^2[\Sigma_S'(m^2)-3m\Sigma_V'(m^2)+2m\Sigma_S''(m^2)-2m^2\Sigma_V''(m^2)]+O(\zeta^3)\end{aligned} \tag{B21}$$

$$\sim\frac{3g^2}{16\pi^2}\ln\frac{m_\pi^{\;2}}{m^2}\left(-4\frac{m_\pi^{\;2}}{m^2}-2\frac{\not p-m}{m}+3\frac{(\not p-m)^2}{m^2}\right)+O(\zeta c\ln c,\,\zeta^3\ln c,\,c^2\ln c)\,. \tag{B22}$$

Equations (A2) and (B18) give

$$\begin{aligned}3g^3D_{NN\pi}(p_1p_2)-\frac{g}{m}\Sigma_S^{(\pi)}(m^2)=-\frac{3g^3}{16\pi^2}\int dxydy\{&[(\not p_2-m)(1-xy)-(\not p_1-m)(1-x)y-my]\\&\times[(\not p_1-m)(1-y+xy)-(\not p_2-m)xy-my]\,1/\Delta_2\\&+1+2\ln|\Delta_2/[m^2(1-x)^2+m_\pi^{\;2}x]|\}\,,\end{aligned} \tag{B23}$$

$$\Delta_2=m^2y^2+m_\pi^{\;2}(1-y)-p_1^{\;2}(1-x)y(1-y)-p_2^{\;2}xy(1-y)-k^2x(1-x)y^2\,.$$

This is of the form $\sum_{i=1}^7 A_i(k^2,p_1^{\;2},p_2^{\;2})P^{(i)}$, where $k=p_1-p_2$ and $P^{(1)}=1$, $P^{(2)}=\not p_1-m$, $P^{(3)}=\not p_2-m$, $P^{(4)}=(\not p_1-m)(\not p_2-m)$, $P^{(5)}=(\not p_2-m)(\not p_1-m)$, $P^{(6)}=(\not p_1-m)^2$, and $P^{(7)}=(\not p_2-m)^2$. We expand A_i in a Taylor series about $k^2=0$, $p_1^{\;2}=m^2$, and $p_2^{\;2}=m^2$ and use the relations $p_i^{\;2}-m^2=(\not p_i-m)^2+2m(\not p_i-m)$ to obtain

$$\begin{aligned}3g^3D_{NN\pi}(p_1p_2)-\frac{g}{m}\Sigma_S^{(\pi)}(m^2)&\sim 3g^3\frac{1}{32\pi^2}\ln\frac{m_\pi^{\;2}}{m^2}\bigg(2\frac{m_\pi^{\;2}}{m^2}+\frac{(\not p_2-m)(\not p_1-m)}{m^2}\\&\qquad-\frac{1}{2m^3}[(\not p_1-m)+(\not p_2-m)][(p_1^{\;2}-m^2)+(p_2^{\;2}-m^2)]\bigg)\\&\quad+O(c^2\ln c,\,\zeta c\ln c,\,\zeta^3\ln c)\\&=3g^3\frac{1}{16\pi^2}\ln\frac{m_\pi^{\;2}}{m^2}\left(\frac{m_\pi^{\;2}}{m^2}-\frac{1}{2m^2}(\not Q_1^{\;2}+\not Q_2^{\;2}+\not Q_2\not Q_1)\right)+\cdots\,,\end{aligned} \tag{B24}$$

where

$$\not Q_1=\not p_1-m\quad\text{and}\quad\not Q_2=\not p_2-m\,.$$

Similarly,

$$D_{\pi\pi N}(p_1 p_2)=\frac{1}{16\pi^2}\int dx(1-y)^2 dy\frac{1}{\Delta_3}[(\not{p}_1-m)x+(\not{p}_2-m)(1-x)-my],$$
$$\Delta_3=m^2y^2+m_\pi^2(1-y)-(p_1^2-m^2)xy(1-y)-(p_2^2-m^2)(1-x)y(1-y)-k^2x(1-x)(1-y)^2. \quad \text{(B25)}$$

Expanding the denominator, we have

$$D_{\pi\pi N}(p_1 p_2)\sim-\frac{1}{16\pi^2 m}\ln\frac{m_\pi^2}{m^2}\left(-\frac{1}{2}+\frac{1}{2}\frac{m_\pi^2}{m^2}+\tfrac{1}{2}(\not{Q}_1+\not{Q}_2)-\frac{1}{4}\frac{k^2}{m^2}-\frac{1}{2m^2}(\not{Q}_1^2+\not{Q}_2^2)-\frac{\not{Q}_2\not{Q}_1}{m^2}\right)$$
$$+O(c^2\ln c,\ \zeta c\ln c,\ \zeta^3\ln c). \quad \text{(B26)}$$

To study the nonanalytic behavior of the following seven diagrams, we first define some quantities:

$$\not{Q}_1=\not{p}_1-m,\qquad \not{Q}_2=\not{p}_2-m,\qquad \not{Q}_3=\not{p}_1+\not{k}_1-m,$$
$$\eta_1=p_1^2-m^2,\qquad \eta_2=p_2^2-m^2,\qquad \eta_3=(p_1+k_1)^2-m^2, \quad \text{(B27)}$$
$$\delta_1=k_1^2,\qquad \delta_2=k_2^2,\qquad \delta_3=(k_2-k_1)^2.$$

The first six quantities are of the order of ζ and the last three quantities are of the order of ζ^2.

The choice of Feynman parameters is a little tricky. We found that the following way is most convenient. We first combine the denominators with the same masses (say with parameter x) and we combine the one without dependence on external momentum last. After the momentum integration, we expand the integrand in a power series of the quantities defined in Eq. (B27) above, the coefficients of which are of the form

$$\int_0^1 dxdydz\frac{x^{m_1}y^{m_2}z^{m_3}}{Y^l}. \quad \text{(B28)}$$

The advantage of choosing these Feynman parameters is that Y is independent of x, linear in y, and quadratic in z. As a result, x and y can be integrated immediately and we have the following integrals:

$$I(m,n)\equiv\int dz\frac{z^m}{X^n},$$

where

$$X=az^2+bz+c\quad\text{with}\quad b^2-4ac<0. \quad \text{(B29)}$$

The coefficients of the $\ln|X|$ is given below

$$I(7,4)=\frac{1}{2a^4},\quad I(8,4)=-\frac{2b}{a^5},\quad I(9,4)=\left(-\frac{2c}{a^5}+\frac{5b^2}{a^6}\right),$$
$$I(5,3)=\frac{1}{2a^3},\quad I(6,3)=-\frac{3b}{2a^4},\quad I(7,3)=\left(-\frac{3c}{2a^4}+\frac{3b^2}{a^5}\right),$$
$$I(3,2)=\frac{1}{2a^2},\quad I(4,2)=-\frac{b}{a^3},\quad I(5,2)=\left(-\frac{c}{a^3}+\frac{3b^2}{2a^4}\right), \quad \text{(B30)}$$
$$I(1,1)=\frac{1}{2a},\quad I(2,1)=-\frac{b}{2a^2},\quad I(3,1)=\left(-\frac{c}{2a^2}+\frac{b^2}{2a^3}\right).$$

From these formulas, we can determine the coefficient of $\ln m_\pi^2$ directly.
We have

$$E_{\pi\sigma\pi N}(k_1p_1;k_2p_2)=\frac{1}{16\pi^2}\int dxydy(1-z)^2dz\frac{1}{\Delta_4^2}\{[\not{Q}_1x+\not{Q}_2(1-x)]y(1-z)+\not{Q}_3(1-y)(1-z)-mz\}, \quad \text{(B31)}$$

$$\Delta_4=[m_\pi^2 y+m_\sigma^2(1-y)](1-z)+m^2z^2 - \{[\eta_1 x+\eta_2(1-x)]y(1-z)z+\eta_3(1-y)(1-z)z+[\delta_1 x+\delta_2(1-x)]y(1-y)(1-z)^2+\delta_3 x(1-x)y^2(1-z)^2\},$$

$$E_{\pi\sigma\pi N}\sim-\frac{1}{32\pi^2}\ln\frac{m_\pi^2}{m^2}\left(\frac{1}{m_\sigma^4}\right)\Big[-m\left(\frac{m_\sigma^2}{m^2}+2\frac{m_\pi^2}{m^2}-\frac{m_\pi^2m_\sigma^2}{m^4}\right)+\tfrac{1}{2}(\not{Q}_1+\not{Q}_2)\left(-2\frac{m_\sigma^2}{m^2}+\frac{\eta_3}{m^2}\right)+m\frac{m_\sigma^2}{m^4}(\eta_1+\eta_2)$$
$$+\left(\frac{m_\sigma^2}{m^4}-\frac{1}{m^2}\right)[\not{Q}_1\eta_1+\not{Q}_2\eta_2+\tfrac{1}{2}(\not{Q}_1\eta_2+\not{Q}_2\eta_1)]+\frac{1}{2m^2}\not{Q}_3(\eta_1+\eta_2)-m\frac{1}{2m^2}(\delta_1+\delta_2)$$
$$-m\frac{1}{2m^2}\left(\frac{m_\sigma^2}{m^2}-1\right)\delta_3-m\left(\frac{m_\sigma^2}{m^6}-\frac{1}{2m^4}\right)(\eta_1^2+\eta_2^2+\eta_1\eta_2)-\frac{m}{2m^4}\eta_3(\eta_1+\eta_2)\Big]$$

$$=\frac{1}{16\pi^2M^2m}\ln\frac{m_\pi^2}{m^2}\Big[\frac{1}{2}+\frac{1}{2}\frac{m_\pi^2}{M^2}-\frac{1}{2}\frac{m_\pi^2}{m^2}-\frac{1}{2m}(\not{Q}_1+\not{Q}_2)+\frac{1}{2m^2}(\not{Q}_1^2+\not{Q}_2^2)$$
$$+\frac{1}{m^2}\not{Q}_2\not{Q}_1+\frac{t}{M^2}\left(-\frac{1}{4}+\frac{1}{4}\frac{M^2}{m^2}\right)+\frac{1}{4M^2}(k_1^2+k_2^2)\Big]+O(c^2\ln c,\ \zeta c\ln c,\ \zeta^3\ln c);\qquad\text{(B32)}$$

$$C_{\sigma\pi N}(p_1p_2)=-\frac{1}{16\pi^2}\int dx(1-y)dy\frac{1}{\Delta_5}\{[\not{Q}_1x+\not{Q}_2(1-x)](1-y)-my\}$$
$$\sim\frac{m}{16\pi^2M^2}\left(\ln\frac{m_\pi^2}{m^2}\right)\left(\frac{1}{2}\frac{m_\pi^2}{m^2}\right),\qquad\text{(B33)}$$

where

$$\Delta_5=[m_\pi^2x+m_\sigma^2(1-x)](1-y)+m^2y^2-\{[\eta_1x+\eta_2(1-x)]y(1-y)+tx(1-x)(1-y)^2\};$$

$$C_{\pi\sigma N}(p_1p_2)=\frac{1}{16\pi^2}\int dx(1-y)dy\frac{1}{\Delta_6}\{[\not{Q}_1(1-x)+\not{Q}_2x](1-y)+m(2-y)\}$$
$$\sim\frac{m}{16\pi^2M^2}\ln\frac{m_\pi^2}{m^2}\left(-\frac{3}{2}\frac{m_\pi^2}{m^2}-2\frac{\not{Q}_2}{m}+3\frac{\not{Q}_2^2}{m^2}\right),\qquad\text{(B34)}$$

where

$$\Delta_6=[m_\pi^2x+m_\sigma^2(1-x)](1-y)+m^2y^2-\{[\eta_1(1-x)+\eta_2x]y(1-y)+\delta_3x(1-x)(1-y)^2\};$$

$$g^3C_{NN\pi}(p_1p_2)-\frac{1}{v}\Sigma_S(m^2)+g^3C_{NN\sigma}(p_1p_2)$$
$$=-\frac{g^3}{8\pi^2}\int dx\ln\left|\frac{m^2(1-x)^2+m_\sigma^2x}{m^2(1-x)^2+m_\pi^2x}\right|-\frac{g^3}{8\pi^2}\int dxydy\ln\frac{|\Delta_7|}{|m^2(1-x)^2+m_\pi^2x|}$$
$$-\frac{g^3}{16\pi^2}\int dxydy\,\{1+[\not{Q}_2(1-(1-x)y)-\not{Q}_1xy+(2-y)m][\not{Q}_1(1-xy)-\not{Q}_2(1-x)y-my)]1/\Delta_7\}$$
$$\sim\frac{g^3}{16\pi^2}\ln\frac{m_\pi^2}{m^2}\left(\frac{m_\pi^2}{m^2}-\frac{\not{Q}_2}{m}+\frac{\not{Q}_2^2}{m^2}+\frac{1}{2}\frac{\not{Q}_2\not{Q}_1}{m^2}\right),\qquad\text{(B35)}$$

where

$$\Delta_7=m_\pi^2(1-y)+m^2y-\{[\eta_1x+\eta_2(1-x)]y(1-y)+\delta_3x(1-x)y^2\};$$

$$E_{\pi NN\sigma}(k_1p_1;k_2p_2)=-\frac{1}{16\pi^2}\int dxdyz(1-z)dz\Big\{\frac{2}{\Delta_8}-\frac{1}{\Delta_8^2}[(\not{Q}_3'+\not{k}_1+\not{k}_2)(1-xz)-\not{k}_2y(1-z)-\not{Q}_1(1-x)z-mz]$$
$$\times[\not{Q}_1(1-(1-x)z)-(\not{Q}_3'+\not{k}_1+\not{k}_2)xz-\not{k}_2y(1-z)+m(2-z)]\Big\}$$
$$\sim\frac{1}{16\pi^2}\ln\frac{m_\pi^2}{m^2}\left(\frac{1}{M^2}\right)\left(\frac{1}{2}\frac{m_\pi^2}{m^2}+\frac{\not{Q}_3'}{m}+\frac{\not{Q}_3'^2}{m^2}-\frac{1}{2}\frac{\not{Q}_2\not{Q}_3'}{m^2}\right),\qquad\text{(B36)}$$

where

$$\Delta_8=[m_\pi^2y+m_\sigma^2(1-y)](1-z)+m^2z^2-\{[\eta_1(1-y)+\eta_2x+\eta_3'(y-x)]z(1-z)+\delta_1xz[(1-x)z+(1-z)(1-y)] + \delta_2(1-z)[z(x-y)+y-y^2(1-z)-xyz]-\delta_3xz(1-z)(1-y)\},$$

$$\not{Q}_3'=\not{p}_1-\not{k}_1-m,\quad \eta_3'=(p_1-k_1)^2-m^2;$$

$$E_{\sigma\pi\sigma N}(k_1p_1;k_2p_2)=\frac{1}{16\pi^2}\int dx(1-y)dy(1-z)^2dz\frac{1}{\Delta_9^2}\{[\not{Q}_1x+\not{Q}_2(1-x)](1-y)(1-z)+\not{Q}_3y(1-z)+m(2-z)\}$$
$$\sim\frac{1}{16\pi^2M^4m}\ln\frac{m_\pi^2}{m^2}(-\tfrac{3}{2}m_\pi^2-2m\not{Q}_3+3\not{Q}_3^2),\qquad\text{(B37)}$$

where

$$\Delta_9=[m_\pi^2y+m_\sigma^2(1-y)](1-z)+m^2z^2 -\{[\eta_1x+\eta_2(1-x)](1-y)(1-z)z+\eta_3y(1-z)z+[\delta_1x+\delta_2(1-x)]y(1-y)(1-z)^2+\delta_3x(1-x)(1-y)^2(1-z)^2\};$$

$$E_{NNN\pi}(k_1p_1;k_2p_2)=\frac{1}{16\pi^2}\int dxydyz^2dz\bigg(\frac{1}{\Delta_{10}^2}\{\not{Q}_2-[\not{Q}_2x+\not{Q}_1(1-x)]yz-\not{Q}_3(1-y)z-mz\}$$
$$\times\{\not{Q}_3-[\not{Q}_2x+\not{Q}_1(1-x)]yz-\not{Q}_3(1-y)z+m(2-z)\}$$
$$\times\{\not{Q}_1-[\not{Q}_2x+\not{Q}_1(1-x)]yz-\not{Q}_3(1-y)z-mz\}$$
$$-\frac{2}{\Delta_{10}}\{\not{Q}_1+\not{Q}_2-\tfrac{1}{2}\not{Q}_3-\tfrac{3}{2}[\not{Q}_2x+\not{Q}_1(1-x)]yz-\tfrac{3}{2}\not{Q}_3(1-y)z-\tfrac{3}{2}mz+\tfrac{1}{2}m\}\bigg)$$
$$\sim\frac{1}{16\pi^2m}\ln\frac{m_\pi^2}{m^2}\left(-\frac{1}{2}\frac{m_\pi^2}{m^2}-\frac{1}{2}\frac{\not{Q}_3}{m}+\frac{1}{4m^2}(\not{Q}_1^2+\not{Q}_2^2+\not{Q}_3^2+\not{Q}_2\not{Q}_3+\not{Q}_3\not{Q}_1+\not{Q}_2\not{Q}_1)\right),\qquad\text{(B38)}$$

where

$$\Delta_{10}=m_\pi^2(1-z)+m^2z^2-\{[\eta_2x+\eta_1(1-x)]yz(1-z) +\eta_3(1-y)z(1-z)+[\delta_2x+\delta_1(1-x)]y(1-y)z^2+\delta_3x(1-x)y^2z^2\}.$$

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†Work supported in part by the National Science Foundation under Grant No. 32998X. Present address: Department of Physics, Brandeis University, Waltham, Mass. 02154.

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