Constraints on the gravitational force on photons from microlensing

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(Received 11 February 2000; revised manuscript received 20 August 2001; published 27 February 2002)

Gravitational microlensing observations are used to study the radial dependence of the gravitational force acting on photons. The Newton inverse square law is shown to be valid at a few % level at a distance scale of a few AU (where 1 AU$=1.496\times10^8$ km is the astronomical unit). Constraints on possible non-Newtonian terms are derived.

DOI: 10.1103/PhysRevD.65.062002 PACS number(s): 04.80.Cc

I. INTRODUCTION

In the past decade, interest in experimental studies of deviations from Newton's law of gravitation, the so-called inverse square law, has been renewed by claims of detection of new macroscopic forces (see for instance [1] and references therein). The inverse square law has been tested over scales ranging from 1 cm to 10^8 km. At astronomical scales, the most stringent limits on possible deviations come from planetary and rocket motions [2].

The status of Newtonian gravity is more controversial at larger scales. Some authors have suggested to modify the inverse square law in an attempt to explain the galactic rotation curves [3–5] (the so-called "dark matter" problem). It has been noted that the gravitational deflection of light is an important test of these models [6,7].

The gravitational force on photons is known from studies of the deflection of light by the Sun at the scale of a fraction of an AU (where 1 AU$=1.496\times10^8$ km is the astronomical unit) and by gravitational lensing observations on larger scales. The deflection of light by the Sun has been shown to be consistent with the inverse square law [8].

Gravitational lensing (see Ref. [9] for a review) is a consequence of the gravitational deflection of light. It has evolved from a theoretical speculation to being an important tool for observational astrophysics. Lensing by structures on scales differing as much as clusters of galaxies and stars ("microlensing") has been observed. Lensing by galaxies and clusters of galaxies has been used by Dar [10] to investigate the validity of the inverse square law at the kpc (3×10^{16} km) scale and above. This author compared the mass inferred from the size of the Einstein ring to the velocity dispersion inside the lensing system. An agreement was found at the 30% level. However, as mentioned previously, the validity of the inverse square law at the galactic scale has been questioned by several authors [3].

Microlensing is sensitive to gravitational interactions at a scale of a few to a few tens of AU. Hence it would fill a small part of the gap between the solar system scale (where Newton's law is known to be accurate at the 10^{-9} level [1]) and the galactic scale. The purpose of this paper is to address the following question: can we learn something on the gravitational interaction of photons from microlensing experiments?

Microlensing observations have an obvious drawback: the lensing mass cannot be estimated in two independent ways. Hence, microlensing observations are not sensitive to the absolute value of the gravitational constant. On the other hand, the impact parameter of the lensed photons changes with time when the lens moves in front of the source. This allows tests of the radial dependence of the gravitational force acting on photons.

II. THE GRAVITATIONAL LENS EQUATION

A. Angular position of images

The simplest lensing configuration is shown in Fig. 1. A source star located at S is lensed by a massive lens star (mass M) at L and observed at O. The Schwarzschild radius r_S and the Einstein radius r_E are defined by

$$r_S=\frac{2GM}{c^2} \tag{1}$$

$$r_E=\sqrt{\frac{2r_S D_{LS} D_{OL}}{D_{OS}}} \tag{2}$$

$$D_{OS}=D_{OL}+D_{LS}. \tag{3}$$

Since we want to study non-Newtonian gravity, G in the above equation is not necessarily the same as Newton's constant measured in the laboratory. The image positions r are given by the normalized lens equation:

$$r-r_o=\alpha(r) \tag{4}$$

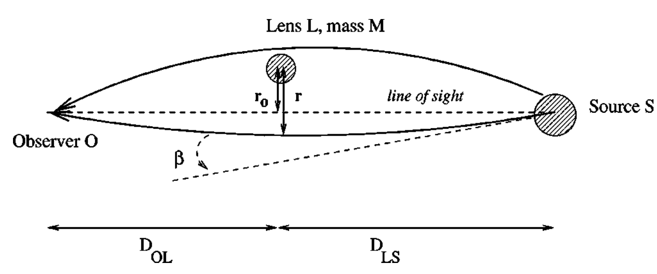

FIG. 1. An observer located at O receives photons from a source star at S. The source star is lensed by a lens star at L. The distance between the observer and the lens is D_{OL}, the distance between the lens and the source is D_{LS}. The lens plane is normal to the line of sight OS.

where r_o is the projected position of the source in the plane perpendicular to the line of sight containing the lens, and α is related to the deflection angle β by

$$\alpha = \frac{D_{LS} D_{OL}}{D_{OS}} \beta. \tag{5}$$

B. The deflection angle

In the weak field approximation, the deflection angle is related to the gravitational potential Φ by

$$\beta = -2 \int_S^O \partial_\perp \Phi dz \tag{6}$$

where z is a coordinate along the source-observer axis. For a Schwarzschild lens, the potential is

$$\Phi \propto \frac{1}{r} \tag{7}$$

which gives the well known result [18]

$$\alpha(r) = \frac{r_E^2}{r} \left[1 + O\left(\frac{r_S}{r} \right) \right]. \tag{8}$$

Since deviations from the inverse square law are not expected to be large, the class of lenses for which

$$\alpha(r) = \frac{r_E^2}{r} [1 + f(r)], \quad f(r) \ll 1 \tag{9}$$

[hereafter called quasi Schwarzschild (QS) lenses], is especially important.

C. Models of deviations from the inverse square law

Three models for a deviation from the inverse square law are considered in this paper. The first model (model I) is a power law model:

$$\Phi \propto r^{-\delta}. \tag{10}$$

This model is convenient for lensing analysis, since the deflection angle is also found to follow a power law with index $-\delta$. When $\delta \sim 1$, this model is equivalent to a QS lens with

$$f(r) = (1-\delta) \ln \frac{|r|}{r_E}. \tag{11}$$

The second model (model II) is a Yukawa type model

$$\Phi \propto \frac{1}{r} (1 + \epsilon e^{-r/\lambda}) \tag{12}$$

where ϵ and λ are phenomenological parameters. Potentials of this type have been proposed to describe short range [12,13,11] or long range [4] modifications of gravity. The deflection angle induced by the model II potential can be expressed with the modified Bessel function K_1:

$$\beta \propto \frac{1}{r} \left[1 + \epsilon \frac{r}{\lambda} K_1 \left(\frac{r}{\lambda} \right) \right]. \tag{13}$$

The last model (model III) simply adds an extra $1/r^2$ term to the ordinary $1/r$ deflection:

$$\beta \propto \frac{1}{r} \left(1 + \epsilon' \frac{r_E}{r} \right). \tag{14}$$

An extra $1/r^2$ term exists also in general relativity [19], but it is predicted to be very small. On the other hand, this term might be non-negligible for some exotic lenses such as lenses with scalar charges [14].

III. SCHWARZSCHILD LENSES

For a Schwarzschild lens, Eq. (4) is readily solved. Two images r_+ and r_- are found. However, these images are not detected individually in microlensing observations. Instead, the experiments take advantage of the apparent magnification of the flux of the lensed source.

The magnification factor is given by

$$A(r_o) = \left| \frac{r_+}{r_o} \frac{dr_+}{dr_o} \right| + \left| \frac{r_-}{r_o} \frac{dr_-}{dr_o} \right|. \tag{15}$$

The lensing is detected when the lens moves in front of the source with a transverse velocity v_t. For Schwarzschild lenses, the shape of the "light curve" (magnification A versus time t) is given by the well known formula [15]

$$A = \frac{u^2+2}{u\sqrt{u^2+4}} \tag{16}$$

$$u^2 = u_o^2 + \frac{v_t^2 (t-t_o)^2}{r_E^2} \tag{17}$$

where v_t is the transverse velocity of the lens. The Einstein radius crossing time defined by $t_E = r_E / v_t$ is the only physical observable.

The microlensing events found by astronomical surveys such as EROS, MACHO or OGLE [16] fit well the expected shape. Since our goal is to constrain the deviations from Newtonian gravity, the expected magnification for QS lenses has to be computed and compared to the shape (16). The first step is to solve the lens equation (4). This equation has in general more than two solutions, even in the case of QS lenses (see e.g. [14]). It is easy to show that models I and II (with $\epsilon \ll 1$) have only 2 solutions for Eq. (4). Model III has 2 solutions for $\epsilon' > 0$, but can have either 2 or 4 solutions for $\epsilon' < 0$. The lens equation for model III with $\epsilon' < 0$ has 4 solutions if

$$|r_0| < \left| \frac{r_E}{4\epsilon'} - 2\epsilon' r_E \right|. \tag{18}$$

The right-hand side of this inequality can be arbitrarily large for small ϵ', so that this inequality holds for practical values of r_0 and ϵ'. Discarding this case (model III with $\epsilon'<0$), the lens equation (4) has only 2 solutions for the models considered in this paper. These solutions can be expressed as infinite series in two special cases: large source distances and QS lenses. In the next 2 sections, these special cases are investigated in turn.

IV. MAGNIFICATION OF QS LENSES AT LARGE AND SMALL DISTANCES

A. Solutions of the lens equation at large distances

As written above, the lens equation (4) is assumed to have only 2 solutions. One of the solutions, denoted by r_+, goes continuously to r_o when r_0 is large compared to r_E. r_+ is easily found by application of the Lagrange inversion formula [17]:

$$r_+=r_o+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^{n-1}}{dr_o^{n-1}}[\alpha(r_o)]^n. \tag{19}$$

The second solution of the lens equation is denoted by r_- and goes continuously to 0 when r_0 is large compared to r_E. To obtain this solution, the lens equation is rewritten as

$$y=\alpha(r) \tag{20}$$

$$y=-r_o+\alpha^{-1}(y). \tag{21}$$

The Lagrange inversion formula now gives:

$$r_-=-\alpha^{-1}(r_o)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{d^{n-1}}{dr_o^{n-1}}\times\left(\frac{d\alpha^{-1}(r_o)}{dr_o}[\alpha^{-1}(r_o)]^n\right) \tag{22}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^{n-1}}{dr_o^{n-1}}[\alpha^{-1}(r_o)]^n. \tag{23}$$

To simplify the calculation, we now specialize to model I lenses to obtain the magnification in the limits of large and small source distances. However, the conclusion obtained in the next section can be shown to hold qualitatively for model II and model III QS lenses.

B. Application to model I lenses

The explicit values of $r_\pm$ for power law model I are:

$$r_+=r_o+\frac{r_E^{1+\delta}}{r_o^{\delta}}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}[(n\delta)\ldots(n\delta+n-2)]}{n!}\frac{r_E^{n(\delta+1)}}{r_o^{n(\delta+1)-1}} \tag{24}$$

$$r_-=-r_E\left(\left(\frac{r_E}{r_o}\right)^{1/\delta}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\left[\left(\frac{n}{\delta}\right)\ldots\left(\frac{n}{\delta}+n-2\right)\right]}{n!}\left(\frac{r_E}{r_o}\right)^{[(n/\delta)+n-1]}\right). \tag{25}$$

It is clear that these expressions have a restricted validity range in r_o. Using the methods of Ref. [17], the series for r_+ are found to converge for

$$r_o\geqslant\frac{\delta+1}{\delta^{\delta/(\delta+1)}}r_E \tag{26}$$

while the series for r_- converge if the same inequality is true with δ changed into $1/\delta$. In the worst case (that of Schwarzschild lenses), the expansions converge for $r_o\geqslant 2r_E$.

For a Schwarzschild lens, the expressions for $r_\pm$ simplify to

$$r_+=r_o+\sum_{n=1}^{\infty}\frac{[2(n-1)!(-1)^{n-1}]}{n!(n-1)!}\frac{r_E^{2n}}{r_o^{2n-1}} \tag{27}$$

$$r_-=-\left(\sum_{n=1}^{\infty}\frac{[2(n-1)!(-1)^{n-1}]}{n!(n-1)!}\frac{r_E^{2n}}{r_o^{2n-1}}\right) \tag{28}$$

$$=r_o-r_+ \tag{29}$$

a well known result.

The magnification factor for model I is

$$A(r_o)=\left|\frac{r_+}{r_o}\frac{dr_+}{dr_o}\right|+(r_+\leftrightarrow r_-) \tag{30}$$

$$=\left|\frac{r_+}{r_o\left[1+\delta\left(\frac{r_E}{r_+}\right)^{\delta+1}\right]}\right|+(r_+\leftrightarrow r_-) \tag{31}$$

$$=\left|\frac{r_+^2}{r_o[(1+\delta)r_+-\delta r_o]}\right|+(r_+\leftrightarrow r_-). \tag{32}$$

In the high magnification limit ($r_o\ll r_E$), the magnification can be written as

$$A_{hm}=\frac{r_E'}{r_o} \tag{33}$$

where r'_E is defined by $r'_E = 2r_E/(1+\delta)$. Thus, in the high magnification limit, a lens with a power law interaction cannot be distinguished from a Schwarzschild lens with Einstein radius r'_E.

In the low magnification limit ($r_o \gg 2r_E$), the $r_\pm$ solutions can be expanded in powers of $(r_E/r_o)^\delta$ and $(r_E/r_o)^{1/\delta}$. Keeping only the leading terms of the expansion, the magnification is given by

$$A_{lm} = 1 + (1-\delta)\frac{r_E}{r_o}^{\delta+1} + \delta(2\delta-1)\frac{r_E}{r_o}^{2(\delta+1)} + \frac{1}{\delta}\frac{r_E}{r_o}^{2[(1/\delta)+1]}. \tag{34}$$

For a QS lens ($\delta = 1+\epsilon, \epsilon \ll 1$), this can be rewritten as

$$A_{lm} = 1 - \epsilon\left(\frac{r_E}{r_o}\right)^2 + 2\left(\frac{r_E}{r_o}\right)^4(1+\epsilon) + O\left(\epsilon^2, \left(\frac{r_E}{r_o}\right)^6\right). \tag{35}$$

This has to be compared to the magnification of a Schwarzschild lens:

$$A_{lm}^{Schw} = 1 + 2\left(\frac{r_E}{r_o}\right)^4 + O\left(\left(\frac{r_E}{r_o}\right)^6\right). \tag{36}$$

Thus, in the low magnification limit, the magnification is the sum of the magnification of a Schwarzschild lens ($\epsilon = 0$) and of an additional term. A pure Schwarzschild lens has a magnification which is fourth order in r_E/r_o. The non-Newtonian term of the magnification [Eq. (35)] is both first order in ϵ and second order in r_E/r_o, and can become dominant at large enough r_o.

The result obtained in this section shows that Newtonian and non-Newtonian lenses are significantly different in the low magnification regime. However, the microlensing events are best observed in the high magnification regime. In the next section, we show that the shape (16) has a simple generalization for QS lenses.

V. FIRST ORDER SOLUTIONS FOR QS LENSES

It is natural to compare the two solutions $r_\pm$ of the lens equation (4) [with α given by Eq. (9)] to the solutions $r_\pm^o$ of the lens equation for Schwarzschild lenses. Changing the variable r to $y = r - (r_E/r)$, changes the lens equation into a set of 2 equations:

$$y - r_o = \frac{f[\frac{1}{2}(y+\sqrt{y^2+4r_E^2})]}{\frac{1}{2}(y+\sqrt{y^2+4r_E^2})} \quad (y>0) \tag{37}$$

$$y - r_o = \frac{f[\frac{1}{2}(y-\sqrt{y^2+4r_E^2})]}{\frac{1}{2}(y-\sqrt{y^2+4r_E^2})} \quad (y<0). \tag{38}$$

Since $f \ll 1$, this set of equations is formally solvable by the Lagrange inversion formula. Transforming back to the r variable, the solutions are

$$r_\pm^1 = r_\pm^o + \sum_{n=1}^{\infty}\frac{(\pm 1)^n}{n!}\frac{d^{n-1}}{dr_o^{n-1}}\left(\frac{f(r_\pm^o)r_E^2}{\sqrt{r_o^2+4r_E^2}}\right)^n. \tag{39}$$

A necessary condition for Eq. (39) to hold is that r_+^1 (r_-^1) and r_+^o (r_-^o) have the same sign.

The magnification of a QS lens can be expressed as

$$A_{QS} = \frac{1}{|r_o|}\left(\left|\frac{r_+^2}{2r_+ - r_o - r_E^2 f'(r_+)}\right|\right) + (r_+ \leftrightarrow r_-). \tag{40}$$

Replacing $r_\pm$ by their values from Eq. (39) and keeping only leading order terms in f gives:

$$A_{QS} = \frac{r_o^2+2r_E^2}{r_o\sqrt{r_o^2+4r_E^2}} + \frac{r_E^2[r_+^{o\,2}f'(r_+^o) - r_-^{o\,2}f'(r_-^o)]}{r_o(r_o^2+4r_E^2)} + \frac{2r_E^4[f(r_+^o)+f(r_-^o)]}{r_o(r_o^2+4r_E^2)^{3/2}}. \tag{41}$$

In the case of the power law model I, a straightforward calculation shows that

$$A_{QS}^I = \frac{r_o^2+(2-\epsilon)r_E^2}{r_o\sqrt{r_o^2+4r_E^2}}. \tag{42}$$

The magnification has also a simple expression for model III,

$$A_{QS}^{III} = \frac{r_o^2+2r_E^2}{r_o\sqrt{r_o^2+4r_E^2}} + 4\epsilon'\frac{r_E^3}{r_o(r_o^2+4r_E^2)}, \tag{43}$$

while the expression for model II is more involved:

$$A_{QS}^{II} = \frac{r_o^2+2r_E^2}{r_o\sqrt{r_o^2+4r_E^2}} + \left(\frac{\epsilon}{\lambda^2}\right)\frac{r_E^2\left[\sum_\pm |r_\pm^0|^3 K_0\left(\left|\frac{r_\pm^0}{\lambda}\right|\right)\right]}{r_o(r_o^2+4r_E^2)} + 2\left(\frac{\epsilon}{\lambda}\right)\frac{r_E^4\left[\sum_\pm |r_\pm^0| K_1\left(\left|\frac{r_\pm^0}{\lambda}\right|\right)\right]}{r_o(r_o^2+4r_E^2)^{3/2}}. \tag{44}$$

VI. CONSTRAINTS ON THE GRAVITATIONAL FORCE FROM A MICROLENSING EVENT

The detailed study of all available microlensing data and the extraction of the best possible constraints is outside the scope of this paper and will be the subject of future work. A simple simulation shows that the light curve of a bright microlensed source (with photometric errors of ~0.5%), monitored with a time sampling of 3 points per night, should constrain the parameters of model I and II at the level of ~5%. The parameter ϵ of model III can be constrained at the ~10% level for $\lambda \sim r_E$. The limits can be improved by combining the results of several light curves.

It is interesting to compare these predictions with the lim-

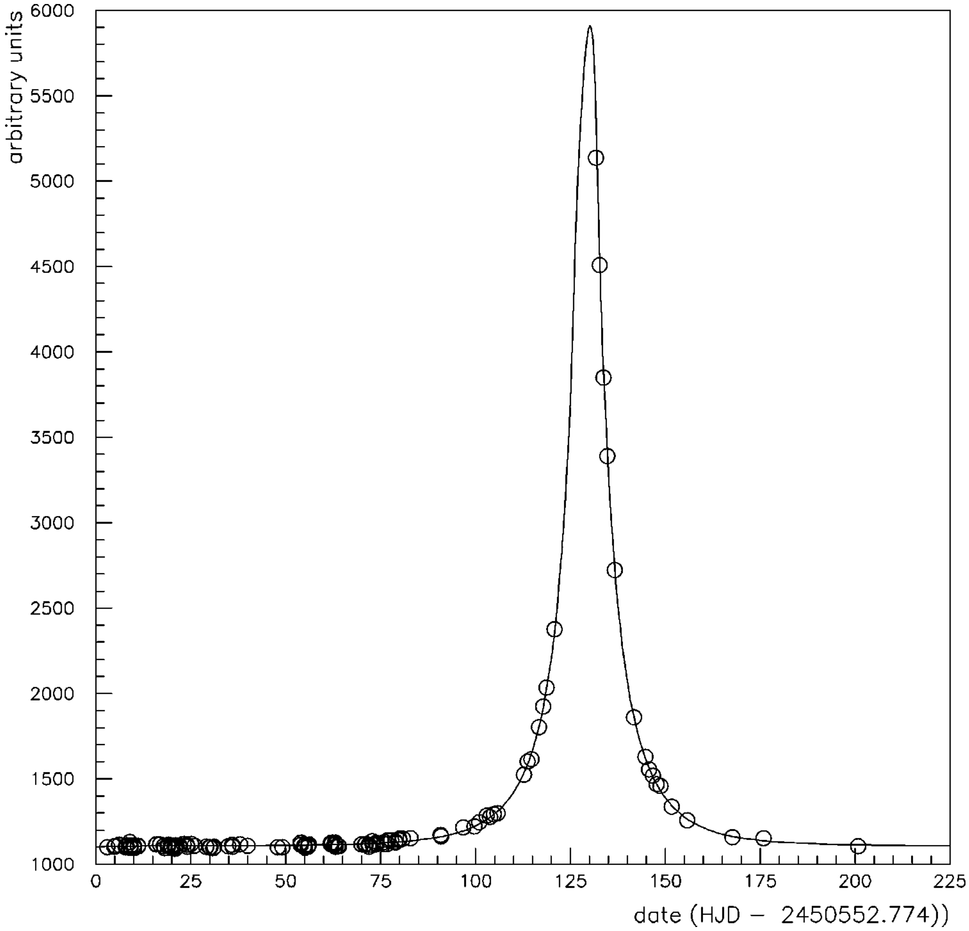

FIG. 2. Microlensing observation OGLE BUL-SC39 361372. The solid line shows the overall fit to a Schwarzschild lens. Only the first 225 days of data are shown for clarity.

its obtained from a real life event. For this, we use data from the Galactic bulge microlensing events catalog [20] released by the OGLE Collaboration. This catalog, which contains 214 events, is available on the web.[1] The event BUL-SC39 361372 has a very bright source star, with a I band magnitude $I_o = 13.392$, and accordingly very small photometric errors ($<1\%$). The light curve of BUL-SC39 361372 is shown in Fig. 2.

All the fits presented here are performed with the MINUIT program [21]. The light curve has first been fitted to the shape (16), which corresponds to a Newtonian gravitational potential. A magnification at maximum A_{max} and an Einstein crossing time t_E of

$$A_{max} = 5.48 \pm 0.06 \tag{45}$$

$$t_E = 18.18 \pm 0.05 \text{ days} \tag{46}$$

were found, in very good agreement with the values published by the OGLE Collaboration [20]. The χ^2 has a high value of 358 for 246 degrees of freedom. The residuals of the fit, displayed in Fig. 3, show no obvious trend as a function of time. From this, it is clear that the Schwarzschild lens model provides a reasonable fit to the data. The distribution of residuals is well fitted by a Gaussian with a zero mean and a standard deviation of 1.43. This may be due to an overestimate of the errors or to other systematic effects. The microlensing fit may be improved by using a more sophisticated lens model. The light curve may deviate from the shape (16) if the source star [22] or the lens [23] is in a binary system. Other possible effects include the orbital motion of the Earth (“microlensing parallax”) [24], or the finite size of the source star [25]. No rescaling of the errors was attempted for the fits of the non-Newtonian effects.

The fit of the power law model I gives

$$\delta - 1 = 0.014 \pm 0.018. \tag{47}$$

The gravitational potential is thus found to be Newtonian at the 2% level. The fit of model III parameter ϵ' gives

$$\epsilon' = 0.034 \pm 0.03. \tag{48}$$

The maximum contribution of an extra $\epsilon' r_E^2/r^2$ in the gravitational potential is thus

$$4 \times 10^{-3} \leqslant \epsilon' \leqslant 6.4 \times 10^{-2} (68\% \text{ C.L.}). \tag{49}$$

The fit of the Yukawa type model II is illustrated in Fig. 4. The parameter λ was varied between 0.2 r_E and 10 r_E. The other parameters were fitted. Parameter ϵ is less than 2.5 σ from zero everywhere, except near $\lambda = 1.25$ r_E. At this value of λ, ϵ is 4.5 standard deviations from zero. Param-

[1]Web address: ftp://astro.princeton.edu/ogle/ogle2/microlensing/gb/

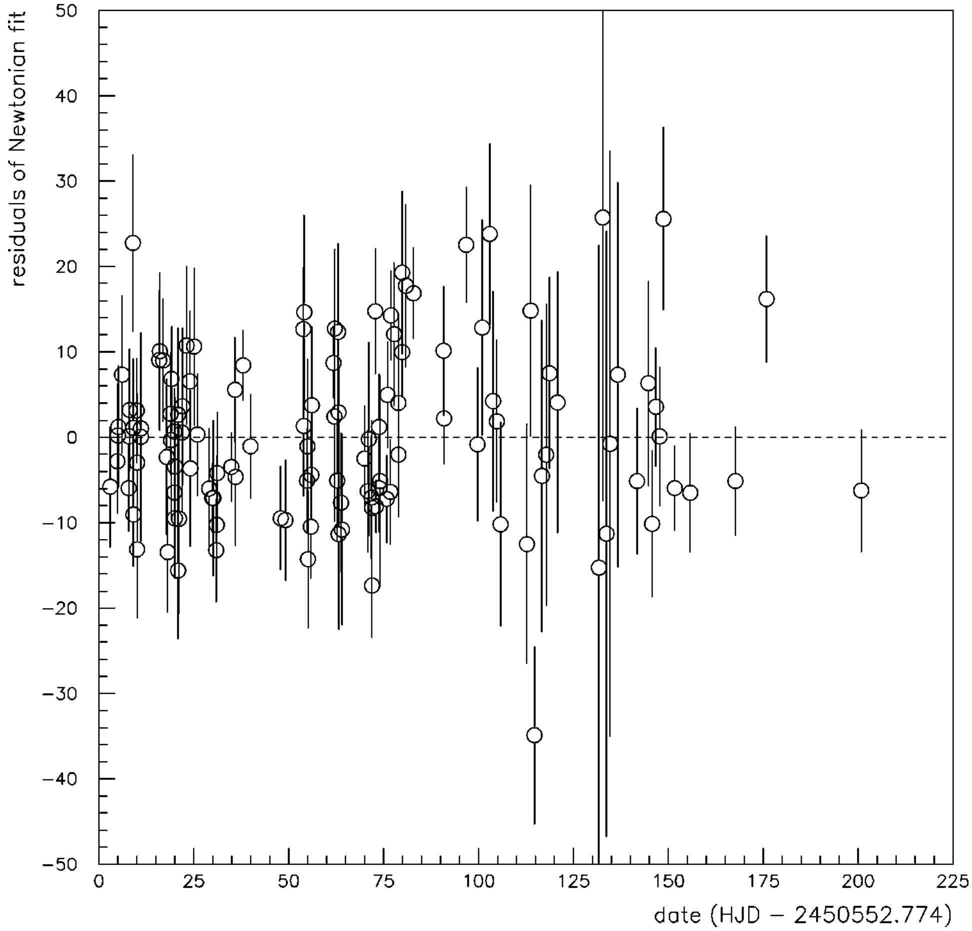

FIG. 3. Residuals of the Schwarzschild lens fit to microlens OGLE BUL-SC39 361372. These residuals do not show any obvious trend as a function of time. Only the first 225 days of data are shown for clarity.

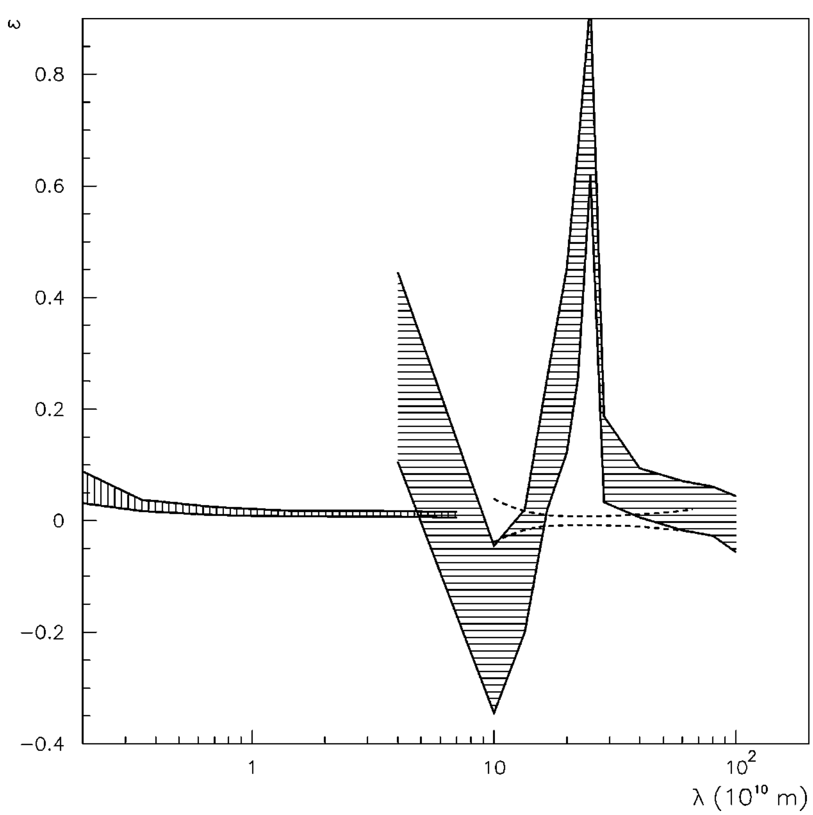

FIG. 4. 68% C.L. upper limits on the model II parameter ϵ as a function of λ. A typical value of $r_E = 1.4$ AU was assumed. The hashed zone with horizontal lines is the 68% C.L. accepted domain obtained from the analysis of the microlensing event OGLE BUL-SC39 361372 (this work). The 68% C.L. accepted domain of Ref. [8] is shown by vertical lines. The dashed lines show the boundaries of the 68% C.L. accepted domain for a simulated observation with 25 images per night on an 8 meter telescope.

eters ϵ and t_E are strongly correlated [see e.g. Eq. (33)]; because of this, the fitted parameters may sometimes drift away from the Schwarzschild lens solution. A 68% C.L. excluded domain can be derived from the size of the error bars, assuming that there is no non-Newtonian signal.

The limits on the parameters of model II are expressed in units of the Einstein radius r_E. They must be converted into physical units. While this is not possible on an event by event basis, an average value for r_E can be obtained. Furthermore, it is sufficient for our purpose to give an order of magnitude estimate. With typical values of

$$D_{OS}=8 \text{ kpc} \tag{50}$$

$$D_{LS}=3 \text{ kpc} \tag{51}$$

$$M=0.3 \ M_\odot \tag{52}$$

one finds $r_E \sim 2 \ 10^8$ km$\sim$1.4 AU. As shown in Fig. 4, bulge microlensing events such as OGLE BUL-SC39 361372 give significant constraints on ϵ for λ in the range $0.5\times 10^{10}-10^{12}$ m. The deflection of light by the Sun [8] gives constraints for λ in the range $10^8-0.5\times 10^{11}$ m. The upper limits on $|\epsilon|$ from Ref. [8], also shown in Fig. 4, are an order of magnitude smaller than the present work.

The limits on model II parameters obtained from microlensing can be improved either by using all the available microlensing events or by taking data with better time sampling and better photometry. This can be achieved by monitoring a bright microlensed source with a dedicated 8-meter class telescope. A simple simulation (dashed lines in Fig. 4) shows that a 1% limit on ϵ can be reached by taking an average of 25 images per night with a photometric accuracy of 0.1%.

VII. CONCLUSION

In this paper, we have studied the radial dependence of the gravitational force acting on photons with observed microlensing events. Newton's inverse square law was found to be valid at the level of a few percent at a distance of about 1 AU. Some constraints on non-Newtonian gravity were derived. These constraints are much weaker than the limits from planetary and rocket data [2], but only slightly weaker than those derived from the gravitational deflection of light by the Sun [8]. Finally, the constraints might be improved by systematically using all available ''well measured'' microlensing observations or by taking data with good time-sampling on a dedicated 8 m class telescope.

ACKNOWLEDGMENTS

I would like to thank Marc Besancon for stimulating questions which started the work exposed in this paper. I am grateful to the OGLE Collaboration for kindly allowing me to use their data. I thank Alexis Amadon, Pierre Fayet, Dick Hubbard, Thierry Lasserre, Alain Milsztajn and James Rich for many useful comments.

[1] E.G. Adelberger, B.R. Heckel, C.W. Stubbs, and W.F. Rogers, Annu. Rev. Nucl. Part. Sci. **41**, 269 (1991).
[2] C. Talmadge, J.-P. Berthias, R.W. Hellings and E.M. Standish, Phys. Rev. Lett. **61**, 1159 (1988); D.R. Mikkelsen and M.J. Newman, Phys. Rev. D **16**, 919 (1977).
[3] J. Bekenstein and M. Milgrom, Astrophys. J. **286**, 7 (1984); see also A. Edery, Phys. Rev. Lett. **83**, 3990 (1990).
[4] R.H. Sanders, Astron. Astrophys. **136**, L21 (1984).
[5] D. Kazanas and P.D. Mannheim, Astrophys. J. **342**, 635 (1989); P.D. Mannheim, *ibid.* **479**, 659 (1997).
[6] J.D. Bekenstein and R.H. Sanders, Astrophys. J. **429**, 480 (1994).
[7] M.A. Walker, Astrophys. J. **430**, 463 (1994).
[8] C. Riveros and H. Vucetich, Phys. Rev. D **34**, 321 (1986).
[9] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer Verlag, Berlin, 1992).
[10] A. Dar, Nucl. Phys. B (Proc. Suppl.) **28A**, 321 (1992).
[11] J. Scherk, Phys. Lett. **88B**, 265 (1979).
[12] Y. Fujii, Nature (London), Phys. Sci. **234**, 5 (1971).
[13] J. O'Hanlon, Phys. Rev. Lett. **29**, 137 (1972).
[14] K.S. Virbhadra, D. Narasimha, and S.M. Chitre, Astron. Astrophys. **337**, 1 (1998).
[15] A. Einstein, Science **84**, 506 (1936); S. Liebes, Phys. Rev. **133B**, 835 (1964).
[16] See, for instance, B. Paczyński, Annu. Rev. Astron. Astrophys. **34**, 419 (1996).
[17] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1962).
[18] See, for instance, S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[19] R. Epstein and I.I. Shapiro, Phys. Rev. D **22**, 2947 (1980).
[20] A. Udalski, K. Zebrun, M. Szymański, M. Kubiak, G. Pietrzynski, I. Soszynski, and P. Wozniak, Acta Astron. **50**, 1 (2000).
[21] F. James, ''MINUIT, Function Minimization and Error Analysis,'' CERN Program Library Long Writeup D506, 1974.
[22] K. Griest and W. Hu, Astrophys. J. **397**, 362 (1992); **407**, 440 (1993).
[23] M. Dominik and A.C. Hirschfeld, Astron. Astrophys. **313**, 841 (1996).
[24] A. Gould, Astrophys. J. **392**, 442 (1992).
[25] H.J. Witt and S. Mao, Astrophys. J. **430**, 505 (1994).