

Vector non-Abelian Chern-Simons duality

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Abelian Chern-Simons gauge theory is known to possess an “S-self-dual” action where its coupling constant k is inverted, i.e., $k \leftrightarrow 1/k$. Here a vector non-Abelian duality is found in the pure non-Abelian Chern-Simons action at the classical level. The dimensional reduction of the dual Chern-Simons action to two dimensions constitutes a dual Wess-Zumino-Witten action already given in the literature.

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I. INTRODUCTION

Duality is a very important tool in the study of nonperturbative physics in quantum field and string theories (for a review, see for instance [1]). In this context, duality helps to describe the strong coupling limit of some supersymmetric field and string theories. Thus, it is important to determine if a theory does admit dual versions. In order to do that, the Roček-Verlinde procedure is very useful [2]. One general signature to know whether a system can be described through “dual” variables is the presence of a *global* symmetry. This symmetry can be made local to construct a more general Lagrangian with additional variables (Lagrange multiplier fields) and a bigger symmetry. From this *parent* Lagrangian, the original Lagrangian and its associated dual Lagrangian can be obtained. This global symmetry can be *Abelian* or *non-Abelian* and, according to this, the above mentioned dualization procedure is called *Abelian* or *non-Abelian duality*. Abelian duality is nowadays well understood (for a review see for instance [3,4]). However non-Abelian duality has a more complicated structure. Non-Abelian duality was originally proposed by de la Ossa and Quevedo in [5]. Its global structure was investigated in [6] and further worked out in [7,8]. In particular, in Ref. [8], the structure of non-Abelian duality of Wess-Zumino-Witten (WZW) models was studied in detail at the quantum level. A nontrivial generalization of the non-Abelian T duality is the Poisson-Lie T duality, which was considered by Klimčík and Severa in a series of papers [9].

On the other hand, Chern-Simons gauge theory has been used to describe a wide range of phenomena in three dimensions. These range from condensed matter systems in low dimensions and particularly in the fractional quantum Hall effect and superconductivity (see for instance [10]) to $(2+1)$ -dimensional gravity [11]. On the mathematical side,

Chern-Simons theory has been very useful for constructing knot and link invariants [12]. The study of duality in the Abelian Chern-Simons action was first introduced in Ref. [13] and further studied in Refs. [14,15]. In particular, in Ref. [15], the effects of T duality in the fractional quantum Hall effect were computed. This duality works by interchanging the level k of the Chern-Simons theory to $1/k$. The generalization to the non-Abelian Chern-Simons and supersymmetric Chern-Simons cases has been worked out in Refs. [16,17]. In these papers, it was found that at the classical level, the non-Abelian dual theories are also non-Abelian Chern-Simons theories with inverted level, just as in the Abelian case. Recently, some new non-Abelian dualities in two-dimensional models were discovered by dimensionally reducing certain three-dimensional non-Abelian dual systems [18]. These results seem to be relevant to massive type-IIA supergravity. Another recent application of non-Abelian duality concerns the dual descriptions of Belavin-Polyakov instantons [19].

A very different duality of the Chern-Simons action has also been discussed by Kapustin and Strassler in the context of the mirror symmetry of the Abelian gauge theory in three dimensions [20]. In this latter paper it is found that $\mathcal{N}=3$ Chern-Simons QED and $\mathcal{N}=4$ QED are in fact S-dual with the mapping $k \leftrightarrow 1/k$. In this case the non-Abelian vector duality generalization still remains an open problem.

It is well known from Ref. [12] that for compact groups, the quantization of Chern-Simons gauge theory consists of the finite-dimensional Hilbert state constructed from the conformal blocks of the associated two-dimensional rational conformal field theory (RCFT). Other features of CFT, such as the conformal anomaly and the duality of conformal blocks, can also be reinterpreted by means of the Chern-Simons three-dimensional theory [21]. It is natural to ask whether the Roček-Verlinde procedure for the WZW models can be carried over from the Chern-Simons theory perspective. In this paper we find a positive answer to this question.

We address the problem of non-Abelian duality in the Chern-Simons gauge theory in three dimensions, with a compact and simple gauge group. In the process we derive the non-Abelian duality of WZW models found in Ref. [8], from

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the Chern-Simons perspective. To be specific, we exploit the fact that the Chern-Simons action is invariant under global transformations of the connection in the adjoint representation. This symmetry is gauged out, and the dual action is then obtained. In order to verify the proposed non-Abelian Chern-Simons duality and its possible consequences, we reduce the parent action to its 2D counterpart. It turns out that it coincides with the duality found in Ref. [8], where its consequences were computed. The reduction to 2D RCFT may help us to understand properly the structure of non-Abelian duality in three dimensions. In [22], a mirror symmetry of 3D Chern-Simons theories with a geometric interpretation, realized as a brane configuration of D-branes and NS-branes, was found. Our results would be relevant to find a relation of [22] to the geometrical WZW models realized as brane configurations [23].

This paper is organized as follows. In Sec. II we briefly review the necessary tools of non-Abelian duality (we follow Ref. [4]) which will be useful in the subsequent sections. Sections III and IV are the main contribution of this paper. In Sec. III basically we find the dual non-Abelian Chern-Simons action. In Sec. IV we reduce the dual action to two dimensions. Finally in Sec. V we give our concluding remarks.

II. NON-ABELIAN DUALITY

In this section we will briefly recall the basics of non-Abelian duality. Non-Abelian duality was first proposed in Ref. [5], in the context of the target space duality in string theory, and further developed in [6–9]. The starting point is a given nonlinear sigma model described by a Lagrangian L consisting of M world-sheet scalar fields X^M and with non-constant target space metric $G_{MN}(X)$. This metric is assumed to possess a group of non-Abelian isometries $\mathbf{G}$. Let n be the index denoting the isometric directions. Then scalar fields transform under the global group $\mathbf{G}$ as $X^m \rightarrow g_n^m X^n$ with $g_n^m \in \mathbf{G}$. Following the Roček-Verlinde procedure, one can gauge out a non-Abelian subgroup $\mathbf{H}$ of $\mathbf{G}$, with $\partial X^m \rightarrow \mathcal{D}X^m = \partial X^m + A^\alpha (T_\alpha)_n^m X^n$. The procedure also incorporates into the action a term $\int \text{tr}(\Lambda F)$ where $F = \partial \bar{A} - \bar{\partial} A + [A, \bar{A}]$ and Λ is a two-index Lagrange multiplier field. The gauge field is a Lie algebra $[\text{Lie}(G)]$ valued field in the adjoint representation of $\mathbf{H}$.

The partition function is given by

$$Z = \int \frac{\mathcal{D}X}{V_{\mathbf{G}}} \int \mathcal{D}\Lambda \mathcal{D}A \mathcal{D}\bar{A} \exp \left\{ -i \left(S_{\text{gauged}}[X, A, \bar{A}] + \int \text{tr}(\Lambda F) \right) \right\}. \quad (1)$$

The original action can, as usual, be found by integration of the Lagrange multiplier Λ . The dual theory can be obtained by integrating over the gauge fields A and $\bar{A}$. It yields

$$Z = \int \mathcal{D}X \mathcal{D}\Lambda \delta[\mathcal{F}] \det \frac{\delta \mathcal{F}}{\delta \omega} \exp(-i S'[X, \Lambda]) \det(f^{-1}), \quad (2)$$

where $\mathcal{F}$ is the gauge fixing function, ω represents the parameters of the group of isometries, f is a matrix-valued coefficient of the quadratic term in the gauge fields and $S'[X, \Lambda]$ is given by

$$S'[X, \Lambda] = S[X] - \frac{1}{4\pi\alpha'} \int \bar{J}_\alpha (f^{-1})^{\alpha\beta} J_\beta, \quad (3)$$

where J and $\bar{J}$ are currents coupled to $\bar{A}$ and A respectively.

In the next section we will show that the non-Abelian Chern-Simons theory possesses *exactly* this non-Abelian duality structure. Thus, it will constitute a new example of this sort of duality.

III. NON-ABELIAN CHERN-SIMONS DUAL ACTION

Consider the pure Chern-Simons action

$$L = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left(AdA + \frac{2}{3} A^3 \right), \quad (4)$$

where A is a connection on the G bundle E over $\mathcal{M}$, G is a compact and simple Lie group and $\mathcal{M}$ is an oriented arbitrary three-manifold with nonempty boundary $\partial\mathcal{M} \neq \emptyset$. “Tr” is an invariant quadratic form on the Lie algebra $\mathcal{G} = \text{Lie}(G)$ of G . The wedge product is omitted from this action.

Let $\{T_a\}$, $a=1, \dots, \dim(G)$, be a basis of $\mathcal{G} = \text{Lie}(G)$ with $[T^a, T^b] = f_c^{ab} T^c$ and “Tr” the diagonal quadratic form $\text{Tr}(T_a T_b) = -2\delta_{ab}$. In local coordinates of $\mathcal{M}$, the action (4) is thus written as

$$L = \frac{k}{4\pi} \int_{\mathcal{M}} \varepsilon^{ijk} \left(A_i^a \partial_j A_{ka} + \frac{1}{3} f_{abc} A_i^a A_j^b A_k^c \right). \quad (5)$$

The partition function of this theory is given by

$$Z = \int \mathcal{D}A \exp \left(\frac{ik}{4\pi} \int_{\mathcal{M}} \text{Tr} \left(AdA + \frac{2}{3} A^3 \right) \right). \quad (6)$$

We intend to find a “dual” action to Eq. (5) using the non-Abelian generalization of the Roček-Verlinde procedure originally proposed in [5] and further developed in [6,8,24,25] (for some reviews see [3,4]). We begin by noting that the action (5) is invariant under the global transformations

$$A_i \rightarrow g^{-1} A_i g, \quad (7)$$

where g is a constant element of G . As in the standard procedure, we gauge a subgroup H of the above symmetry in the action (5), with algebra $\mathcal{H}$, and introduce the $\mathcal{H}$ -valued gauge field B_i to get the parent action:

$$L_D = \int_{\mathcal{M}} \varepsilon^{ijk} \left[\frac{k}{4\pi} \left(A_i^a \hat{D}_j A_{ka} + \frac{1}{3} f_{abc} A_i^a A_j^b A_k^c \right) + \frac{1}{2\pi} \chi_i^a F_{jka}(B) \right], \quad (8)$$

where the Lagrange multipliers χ_i^a and the field strength $F_{jk}^a(B) = \partial_j B_k^a - \partial_k B_j^a + f_{bc}^a B_j^b B_k^c$ are $\mathcal{H}$ -valued forms on $\mathcal{M}$, and $\hat{D}_i = \partial_i + [B_i, \cdot]$ are the corresponding covariant derivatives with respect to the B fields. This action is invariant under the symmetry transformations

$$B_i \rightarrow h^{-1} B_i h + h^{-1} \partial_i h, \quad A_i \rightarrow h^{-1} A_i h, \quad \chi \rightarrow h \chi h^{-1}, \quad (9)$$

that is,

$$\hat{D}_i A = h^{-1} \hat{D}_i A h, \quad (10)$$

for any element h of H .

The partition function for this system is given by

$$Z = \int \mathcal{D}A \mathcal{D}\chi \mathcal{D}B \exp(iL_D). \quad (11)$$

Integrating with respect to the Lagrange multipliers χ_i^a , we get the constraints

$$F_{jk}^a(B) = 0, \quad (12)$$

which lead to considering only flat gauge connections B_i of the form

$$B_i = h^{-1} \partial_i h. \quad (13)$$

Then locally the gauge fields are pure gauge, the gauge fixing $B_i^a = 0$ can be chosen, and we recover the original action (5).

On the other hand, the “dual” action can be obtained by integrating over the $\mathcal{H}$ -valued gauge fields B_i^a and then fixing the gauge. The relevant part of the action is

$$L_D = \int_{\mathcal{M}} \varepsilon^{ijk} \left(\dots + \frac{k}{8\pi} f_{abc} A_i^a A_j^b B_k^c + \frac{1}{2\pi} \chi_i^a \partial_j B_{ka} \right. \\ \left. + \frac{1}{4\pi} f_{abc} \chi_i^a B_j^b B_k^c + \dots \right). \quad (14)$$

This is a Gaussian integral and the integration defines the “dual” action L_D^* with partition function

$$Z = \int \mathcal{D}A \mathcal{D}\chi \det M^{-1/2} \exp(iL_D^*), \quad (15)$$

with the H -invariant dual action,

$$L_D^*[A, \chi] = L[A] + \frac{1}{4\pi} \int_{\mathcal{M}} J_i^a(\chi, A) M_{ab}^{-1 ij} J_j^b(\chi, A) \\ + \frac{1}{4\pi} \int_{\partial\mathcal{M}} N_a^i(\chi) M_{ij}^{-1 ab} (N_b^j(\chi) - 2J_b^j(\chi, A)), \quad (16)$$

where $L[A]$ is the standard Chern-Simons action (5),

$$J_a^k(\chi, A) = \varepsilon^{ijk} \left(\partial_i \chi_{ja} + \frac{k}{8\pi} f_{abc} A_i^b A_j^c \right), \quad (17)$$

$N_a^i = \varepsilon^{ijk} n_j \chi_{ka}$, with n_i the normal to $\partial\mathcal{M}$, and M^{-1} is the inverse matrix of

$$M_{ab}^{ij} = \frac{1}{2} \varepsilon^{ijk} f_{abc} \chi_k^c. \quad (18)$$

Up to boundary terms, the form of the dual action (16) coincides with Eq. (3), with the difference that the factor determinant in Eq. (15) appears here with a square root due to the fact that the integral is Gaussian. The action (16) is still gauge invariant under H transformations and as usual, in order to obtain the true dual action, a gauge fixing has to be undertaken. However, as pointed out in [5] the dual action will not depend on the definite form of the gauge fixing.

The matrix (18) can be singular in general. Algebraically its inverse can be given, although not in a generally simple form. For example for $H = \text{SU}(2)$ it is given by

$$M_{ij}^{-1 ab} = \frac{1}{\det(M)} (\chi_i^a \chi_j^b - 2\chi_j^a \chi_i^b).$$

In the case that $\det(M)$ has singularities, counterterms have to be added in order to regularize the corresponding poles [24].

IV. DUAL WESS-ZUMINO-WITTEN ACTION

In order to understand the precise structure of the three-dimensional dual action (16), we compute the corresponding theory on the boundary $\partial\mathcal{M}$ of $\mathcal{M}$. To do this we follow Refs. [26,27]. If the manifold $\mathcal{M}$ has a boundary, we can consider the two dimensional theory corresponding to the action (8). In order to do that, we separate the time from the space components: $d = d_0 + \tilde{d}$, $A = A_0 + \tilde{A}$, $B = B_0 + \tilde{B}$ and $\chi = \chi_0 + \tilde{\chi}$; then the parent action (8) can be rewritten as

$$L_D = \int_{\mathcal{M}} \frac{1}{4\pi} \text{Tr} \left[-\frac{1}{2} \tilde{A} d_0 \tilde{A} + \tilde{\chi} d_0 \tilde{A} + \chi_0 \tilde{F}(\tilde{B}) - k A_0 (\tilde{d} \tilde{A} + \tilde{B} \tilde{A} \right. \\ \left. + \tilde{A} \tilde{B} + \tilde{A}^2) + B_0 (\tilde{d} \tilde{\chi} + \tilde{B} \tilde{\chi} + \tilde{\chi} \tilde{B} - k \tilde{A}^2) \right], \quad (19)$$

where we set the boundary conditions $A_0 = B_0 = 0$.

Thus, after integrating out the Lagrange multipliers, we have the following equations:

$$\tilde{F}(\tilde{B}) = \tilde{d} \tilde{B} + \tilde{B}^2 = 0, \quad (20)$$

$$G(\tilde{A}, \tilde{B}) \equiv \tilde{d} \tilde{A} + \tilde{B} \tilde{A} + \tilde{A} \tilde{B} + \tilde{A}^2 = 0, \quad (21)$$

$$H(\tilde{A}, \tilde{B}, \tilde{\chi}) \equiv \tilde{d} \tilde{\chi} + \tilde{B} \tilde{\chi} + \tilde{\chi} \tilde{B} - k \tilde{A}^2 = 0. \quad (22)$$

Further, we observe that if we define $\tilde{A} = \tilde{\phi} - \tilde{B}$ and $\tilde{\chi} = -k \tilde{A} + \tilde{\lambda}$, where $\tilde{\phi} \in \mathcal{G}$ and $\tilde{\lambda} \in \mathcal{H}$, then $G(\tilde{A}, \tilde{B}) = \tilde{F}(\tilde{\phi}) - \tilde{F}(\tilde{B})$ and $H(\tilde{A}, \tilde{B}, \tilde{\chi}) = -kG(\tilde{A}, \tilde{B}) + \tilde{d} \tilde{\lambda} + \tilde{B} \tilde{\lambda} + \tilde{\lambda} \tilde{B}$.

Therefore the equations we have to set to zero are $\tilde{F}(\tilde{B}) = F(\tilde{\phi}) = 0$ and $\tilde{d}\tilde{\lambda} + \tilde{B}\tilde{\lambda} + \tilde{\lambda}\tilde{B} = 0$. The first two equations can be solved by $\tilde{B} = h^{-1}\tilde{d}h$ and $\tilde{\phi} = g^{-1}\tilde{d}g$, where $h \in H$ and $g \in G$. Now, if we insert these solutions into the last equation, we get

$$\tilde{d}\tilde{\lambda} + h^{-1}\tilde{d}h\tilde{\lambda} + \tilde{\lambda}h^{-1}\tilde{d}h = 0, \quad (23)$$

that is,

$$h\tilde{d}\tilde{\lambda}h^{-1} + \tilde{d}h\tilde{\lambda}h^{-1} + h\tilde{\lambda}h^{-1}\tilde{d}hh^{-1} = \tilde{d}(h\tilde{\lambda}h^{-1}) = 0. \quad (24)$$

Therefore

$$\tilde{\lambda} = h^{-1}\tilde{d}\alpha h, \quad \tilde{\chi} = -k\tilde{A} + h^{-1}\tilde{d}\alpha h \quad (25)$$

where $\alpha \in \mathcal{H}$. We get

$$L_D = \int_{\mathcal{M}} \frac{1}{4\pi} \text{Tr} \left[\frac{k}{2} g^{-1}\tilde{d}g d_0(g^{-1}\tilde{d}g) - \frac{k}{2} h^{-1}\tilde{d}h d_0(h^{-1}\tilde{d}h) + h^{-1}\tilde{d}\alpha h d_0(h^{-1}\tilde{d}h) \right], \quad (26)$$

which can be rewritten as

$$L_D = \int_{\mathcal{M}} \frac{1}{4\pi} \text{Tr} \left[\tilde{d} \left(\frac{k}{2} \tilde{d}g^{-1} d_0 g - \frac{k}{2} \tilde{d}h^{-1} d_0 h + \tilde{d}\alpha d_0 h h^{-1} \right) - \frac{k}{6} (h^{-1}dh)^3 + \frac{k}{6} (g^{-1}dg)^3 \right]. \quad (27)$$

Therefore, if for example our manifold is $\mathcal{M} = \mathbf{R} \times \mathbf{D}$, where $\mathbf{D}$ is a 2-disk, then if r and ϕ are the coordinates on the disk, we get the two dimensional parent action,

$$I_D = \int_{\mathbf{R} \times \partial \mathbf{D}} \frac{1}{4\pi} \text{Tr} \left(-\frac{k}{2} g^{-1} \partial_\phi g g^{-1} \partial_t g + \frac{k}{2} h^{-1} \partial_\phi h h^{-1} \partial_t h - \partial_\phi \alpha \partial_t h h^{-1} \right) d\phi dt + \frac{k}{24\pi} \int_{\mathbf{R} \times \mathbf{D}} \text{Tr}[(g^{-1}dg)^3 - (h^{-1}dh)^3]. \quad (28)$$

This parent action contains two WZW actions for g and h , as well as the α term. It coincides with the non-Abelian duality parent action for WZW given in [8] [see Eq. (4.16) of Ref. [8]].

From Eq. (25), we see that the field α corresponds to the field χ in Eq. (8) and thus its integration should give the WZW action as result. Indeed, the integration over α gives $\partial_\phi(h^{-1}\partial_t h) = 0$, whose solution is $h(t, \phi) = A(\phi)B(t)$. After substitution of this solution back into the parent action (28), the α term and the h WZW action vanish identically. Thus, as expected, the resulting action is the WZW action corresponding to the Chern-Simons action (4),

$$I_{\text{WZW}} = \frac{k}{2} \int \text{Tr} g^{-1} \partial_\phi g g^{-1} \partial_t g d\phi dt + \frac{k}{6} \text{Tr} \int (g^{-1}dg)^3. \quad (29)$$

In order to get the dual action to Eq. (29), it has to be integrated over the h field in Eq. (28) [8]. Let us denote by I_h the action containing h ; its variation with respect to h gives

$$\delta_h I_h = -\frac{1}{4\pi} \int \text{Tr} [h^{-1} \delta h (\partial_t \partial_\phi \alpha - k \partial_\phi (h^{-1} \partial_t h) + [h^{-1} \partial_t h, \partial_\phi \alpha])] dt d\phi = 0, \quad (30)$$

which is solved by

$$\partial_\phi \alpha = k h^{-1} \partial_\phi h, \quad (31)$$

whose substitution in the parent action gives

$$\tilde{I}_{\text{WZW}} = -\frac{1}{4\pi} \int_{\mathbf{R} \times \partial \mathbf{D}} \text{Tr} \left(\frac{k}{2} g^{-1} \partial_\phi g g^{-1} \partial_t g + \frac{k}{2} h^{-1} \partial_\phi h h^{-1} \partial_t h \right) d\phi dt + \frac{k}{24\pi} \int_{\mathbf{R} \times \mathbf{D}} \text{Tr}[(g^{-1}dg)^3 - (h^{-1}dh)^3]. \quad (32)$$

A detailed analysis done in [8], taking into account quantum corrections, in particular the ones due to the change of variable (31), shows that this dual non-Abelian WZW action corresponds exactly to a $(G/H)_k \times H_k$ WZW model.

V. CONCLUDING REMARKS

In this paper we have further investigated the structure of non-Abelian duality. We have found that one can associate a dual action with the non-Abelian Chern-Simons action, which constitutes a new example of this kind of duality. After solving the constraints, we find an explicit “dual” action for the original Chern-Simons (CS) action (5) in terms of the Lagrange multiplier variables χ_k^a , given by Eq. (16). This action contains a Chern-Simons term of the original fields A_i^a plus the χ action and a boundary term. It needs still to be gauge fixed in order to eliminate redundant degrees of freedom. In order to see which 2D CFT theory corresponds to this dual, we resort to the two dimensional theory on the boundary $\mathbf{R} \times \partial \mathbf{D}$ corresponding to the parent action. This procedure was performed in the Sec. IV and the non-Abelian action I_D obtained in Ref. [8] resulted.

We can summarize our results with the following diagram:

$$\begin{array}{ccc} \text{CS} & \xrightarrow{D} & \widetilde{\text{CS}} \\ \downarrow R & & \downarrow R \\ \text{WZW} & \xrightarrow{D} & \widetilde{\text{WZW}} \end{array}$$

where the mapping consisting in obtaining the dual action is denoted by D . R denotes the dimensional reduction of the dual action on the boundary $\mathbf{R} \times \partial\mathbf{D}$. The reduced parent action I_D Eq. (28) corresponds to two coupled WZW actions, just as was found in Ref. [8]. It is interesting to see that the above diagram commutes. The reason for this is that the dimensional reduction of $\widetilde{\text{CS}}$ theory, i.e., the $\widetilde{\text{WZW}}$ model, coincides with the dual action $\tilde{I}_{\text{WZW}}$ to the WZW model obtained in [8] which comes directly from dimensional reduction of the CS theory according to Refs. [26,27], and it is given by Eq. (32). This is so because both approaches have the same WZW parent action I_D [Eq. (28)].

It was the main aim of the paper to construct the non-Abelian dual theory of the non-Abelian Chern-Simons theory. Similarly to other examples, the dual action is a Chern-Simons action coupled to a Freedman-Townsend-like action Eq. (16). To find the utility of the non-Abelian Chern-Simons duality, it remains to apply it to some systems involving non-Abelian Chern-Simons theory. One of these examples would be to find a relation between some properties in the strong/weak coupling region of the Chern-Simons gauge theory and in the weak/strong region of the same theory. This could be of importance, for instance, in relating the topological invariants of knots and links defined in the strong coupling limit $1/k \rightarrow \infty$ (Jones polynomial) and those

defined in the weak coupling limit $1/k \rightarrow 0$ (Vassiliev invariants).

Finally, non-Abelian Chern-Simons actions with noncompact complex groups are relevant in the description of $(2+1)$ quantum gravity [11]. Dual actions for gravity and supergravity were found in [16,17]. We would like to apply the issues considered here to the Chern-Simons (super)gravity case and compare with the dual actions obtained in [16,17]. From Ref. [21] it is known that for the gravitational case, the associated Hilbert space is infinite dimensional. Even in this case CFT is of extreme importance to describe the gravity system [28]. Thus non-Abelian Chern-Simons duality and its two-dimensional reduction would be useful to address the gravitational case.

Also, it is well known that non-Abelian Chern-Simons gauge theory can be regarded as a topological string theory [29]. It is tantalizing to apply the dual Chern-Simons action in order to look for S -duality structure in the various topological sigma models involved. It would also be interesting to compare our results with that obtained by Mohammadi [30]. Some of these subjects are under current investigation.

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