Entropy of the Dirac field in a Kerr-Newman black hole

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Thinking of the Dirac equations in Newman-Penrose formalism, there are four components of the wave function $\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}$. We consider that the total free energy is summed up from the free energies of the four components. Then the entropy of the Dirac field in a Kerr-Newman black hole is calculated via the brick-wall method. We find that the calculation of the Dirac field is similar to that of a scalar field. It also requires a radial and an angular cutoff ϵ and δ. Moreover, if the cutoff values $\overline{\epsilon}$ and δ satisfy the same proper relation in the scalar field between them, the resulting entropy is 7/2 times the Klein-Gordon field.

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I. INTRODUCTION

Since the pioneering works of Bekenstein and Hawking [1,2], which proved that the entropy of a black hole is proportional to its surface area, there have been many methods used to study the statistical origin of black hole entropy [3–5]. One possible method is the brick-wall model [5] by 't Hooft in which the black hole entropy is identified with the entropy of a thermal gas of quantum field excitations outside the event horizon. This method has been applied to various black hole models [6–9]. However, all these calculations are for scalar fields outside the black hole.

In this paper, we give the calculation of the entropy of the Dirac field in the Kerr-Newman black hole. On the basis of [10], we discuss the more general case—the Kerr-Newman black hole. Because the field on the rotating black hole has two kinds of modes, the superradiant (SR) and nonsuperradiant (NSR) mode [11], the free energy of Dirac field is composed of the SR part F_{SR} and the NSR part F_{NSR}. This makes the calculation more complex. Furthermore, there are not many symmetries in the Kerr-Newman black hole as in the Schwarzschild black hole. This makes the calculation more difficult. From the Dirac equations in Newman-Penrose formalism, we conclude that the entropy of the Dirac field is also proportional to the surface area of black hole. On the same condition of scalar field, it is 7/2 times that of scalar field. This result is inaccordance with [12] which is calculated by another method [13].

Although the massless Klein-Gordon equation and Dirac equation in the $(1+1)$-dimensional charged black hole have been solved in [14], it is necessary to calculate a real model such as the Kerr-Newman black hole. The entropy calculated by the brick-wall method is a result of statistical physics. If particles are fermionic, there is also a kind of entropy called the fermonic one. Our results also show that the fermionic entropy completely has the form as that of bosonic entropy [6], except that the coefficient is different. However, the difference of the coefficient is not the same as that of [14].

II. DIRAC EQUATION IN KERR-NEWMAN SPACETIME

In a curve spacetime, Dirac equations of spinor dynamics in Newman-Penrose formalism are as follows [15]:

$$(D+\varepsilon-\rho)\Psi_{11}+(\overline{\delta}+\pi-\alpha)\psi_{12}=\frac{i}{\sqrt{2}}\mu_0\Psi_{21},$$

$$(\Delta'+\mu-\gamma)\Psi_{12}+(\delta+\beta-\tau)\Psi_{11}=\frac{i}{\sqrt{2}}\mu_0\Psi_{22},$$

$$(D+\varepsilon^*-\rho^*)\Psi_{22}-(\delta+\pi^*-\alpha^*)\Psi_{21}=\frac{i}{\sqrt{2}}\mu_0\Psi_{12},$$

$$(\Delta'+\mu^*-\gamma^*)\Psi_{21}-(\overline{\delta}+\beta^*-\tau^*)\Psi_{22}=\frac{i}{\sqrt{2}}\mu_0\Psi_{11}, \tag{1}$$

where μ_0 is the mass of spinor, Ψ_{11}, Ψ_{12}, Ψ_{21}, and Ψ_{22} are the four components of the wave function, D, Δ', δ, and $\overline{\delta}$ are usual differential operators, and α, β, γ, ε, ρ, π, μ, and τ are spin coefficients. All the operators and spin coefficients can be expressed in a null tetrad frame $(l^\mu, n^\mu, m^\mu, \overline{m}^\mu)$ as follows:

$$\alpha=\frac{1}{2}(l_{\mu;\nu}n^\mu\overline{m}^\nu-m_{\mu;\nu}\overline{m}^\mu\overline{m}^\nu), \rho=l_{\mu;\nu}m^\mu\overline{m}^\nu,$$

$$\beta=\frac{1}{2}(l_{\mu;\nu}n^\mu m^\nu-m_{\mu;\nu}\overline{m}^\mu m^\nu), \pi=-n_{\mu;\nu}\overline{m}^\mu l^\nu,$$

$$\gamma=\frac{1}{2}(l_{\mu;\nu}n^\mu n^\nu-m_{\mu;\nu}\overline{m}^\mu n^\nu), \mu=-n_{\mu;\nu}\overline{m}^\mu m^\nu,$$

$$\varepsilon=\frac{1}{2}(l_{\mu;\nu}n^\mu l^\nu-m_{\mu;\nu}\overline{m}^\mu l^\nu), \tau=l_{\mu;\nu}m^\mu n^\nu,$$

$$D=l^\mu\partial_\mu, \Delta'=n^\mu\partial_\mu, \delta=m^\mu\partial_\mu, \overline{\delta}=\overline{m}^\mu\partial_\mu. \tag{2}$$

In the Boyer-Lindquist coordinate system, the line element of the Kerr-Newman spacetime is given by

$$
\begin{aligned}
ds^2 &= g_{tt}dt^2 + 2g_{t\varphi}dt\,d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \\
&= -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dt\,d\varphi \\
&\quad + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2,
\end{aligned} \tag{3}
$$

where $\Sigma(r,\theta) = r^2 + a^2\cos^2\theta$, $\Delta(r) = r^2 - 2Mr + a^2 + Q^2$, and M,a,q are the mass, the angular momentum per unit mass, and the charge of the black hole, respectively.

For a description of Kerr-Newman spacetime in a Newman-Penrose formalism, we first need to choose a null tetrad frame. As in Ref. [15], we choose the null tetrad frame as follows:

$$
\begin{aligned}
l^\mu &= \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a), \\
n^\mu &= \frac{1}{2\Sigma}(r^2+a^2, -\Delta, 0, a), \\
m^\mu &= \frac{1}{\sqrt{2}\sigma}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right), \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}\sigma^*}\left(-ia\sin\theta, 0, 1, -\frac{i}{\sin\theta}\right),
\end{aligned} \tag{4}
$$

where $\sigma = r + ia\cos\theta$.

The covariant form of the basis vector is

$$
\begin{aligned}
l_\mu &= \frac{1}{\Delta}(\Delta, -\Sigma, 0, -a\Delta\sin^2\theta), \\
n_\mu &= \frac{1}{2\Sigma}(\Delta, \Sigma, 0, -a\Delta\sin^2\theta), \\
m_\mu &= \frac{1}{\sqrt{2}\sigma}(ia\sin\theta, 0, -\Sigma, -i(r^2+a^2)\sin\theta), \\
\bar{m}_\mu &= \frac{1}{\sqrt{2}\sigma^*}(-ia\sin\theta, 0, -\Sigma, i(r^2+a^2)\sin\theta).
\end{aligned} \tag{5}
$$

The null tetrad frame can be used to write out the Dirac equation in Newman-Penrose formalism. On the other hand, because of the symmetry of Kerr-Newman spacetime, we can put the four components of wave function in the form [15]

$$
\begin{aligned}
\Psi_{11} &= e^{-i(Et-m\varphi)}(r - ia\cos\theta)^{-1}\psi_{11}(r,\theta), \\
\Psi_{12} &= e^{-i(Et-m\varphi)}\psi_{12}(r,\theta), \\
\Psi_{21} &= e^{-i(Et-m\varphi)}\psi_{21}(r,\theta), \\
\Psi_{22} &= e^{-i(Et-m\varphi)}(r + ia\cos\theta)^{-1}\psi_{22}(r,\theta).
\end{aligned} \tag{6}
$$

Now we can calculate differential operators and spin coefficients in Eq. (2) by Eqs. (4) and (5). Then by putting Eqs. (2) and (6) into Eq. (1), we can obtain the following simple equations:

$$
\begin{aligned}
&\hat{P}_1\psi_{11} + \frac{1}{\sqrt{2}}\hat{Q}_1\psi_{12} = \frac{1}{\sqrt{2}}(i\mu_0 r + a\mu_0\cos\theta)\psi_{21}, \\
&\Delta\hat{P}_2\psi_{12} - \sqrt{2}\hat{Q}_2\psi_{11} = -\sqrt{2}(i\mu_0 r + a\mu_0\cos\theta)\psi_{22}, \\
&\hat{P}_1\psi_{22} - \frac{1}{\sqrt{2}}\hat{Q}_2\psi_{21} = \frac{1}{\sqrt{2}}(i\mu_0 r - a\mu_0\cos\theta)\psi_{12}, \\
&\Delta\hat{P}_2\psi_{21} + \sqrt{2}\hat{Q}_1\psi_{22} = -\sqrt{2}(i\mu_0 r - a\mu_0\cos\theta)\psi_{11},
\end{aligned} \tag{7}
$$

where $\hat{P}_1 = \partial_r - iK_1/\Delta$, $\hat{P}_2 = \partial_r + iK_1/\Delta + (r-M)/\Delta$, $\hat{Q}_1 = \partial_\theta - K_2 + (1/2)\cot\theta$, $\hat{Q}_2 = \partial_\theta + K_2 + (1/2)\cot\theta$, $K_1 = (r^2 + a^2)E - am$, and $K_2 = aE\sin\theta - m/\sin\theta$.

It is now apparent that the variables can be separated by writing [15]

$$
\begin{aligned}
\psi_{11}(r,\theta) &= R_{-1/2}(r)\Theta_{-1/2}(\theta), \\
\psi_{12}(r,\theta) &= R_{+1/2}(r)\Theta_{+1/2}(\theta), \\
\psi_{21}(r,\theta) &= R_{+1/2}(r)\Theta_{-1/2}(\theta), \\
\psi_{22}(r,\theta) &= R_{-1/2}(r)\Theta_{+1/2}(\theta),
\end{aligned} \tag{8}
$$

where $R_{\pm 1/2}(r)$ and $\Theta_{\pm 1/2}(\theta)$ are functions only of r and θ, respectively. For simplicity, we set $\mu_0 = 0$ because the calculation of entropy usually uses the small-mass limit. This approximation makes the equation more simple, thus

$$
\begin{aligned}
&\hat{P}_1\Delta\hat{P}_2R_{+1/2}(r) = \lambda R_{+1/2}(r), \\
&\Delta\hat{P}_2\hat{P}_1R_{-1/2}(r) = \lambda R_{-1/2}(r), \\
&\hat{Q}_2\hat{Q}_1\Theta_{+1/2}(\theta) + \lambda\Theta_{+1/2}(\theta) = 0, \\
&\hat{Q}_1\hat{Q}_2\Theta_{-1/2}(\theta) + \lambda\Theta_{-1/2}(\theta) = 0,
\end{aligned} \tag{9}
$$

where λ is a constant of separating variants. We have thus reduced the solution of the Dirac equation in Kerr-Newman geometry to the solution of four decoupled equations.

Expanding the decoupled equations, we obtain

$$
\frac{d^2\Theta_{-1/2}}{d\theta^2} + \cot\theta\frac{d\Theta_{-1/2}}{d\theta} + \left[aE\cos\theta + m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} + \lambda\right]\Theta_{-1/2} = 0,
$$

$$
\frac{d^2\,\Theta_{+1/2}}{d\theta^2} + \cot\theta\frac{d\Theta_{+1/2}}{d\theta} + \left[-aE\cos\theta - m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} + \lambda\right]\Theta_{+1/2} = 0,
$$

$$\Delta \frac{d^2R_{+1/2}}{dr^2}+3(r-M)\frac{dR_{+1/2}}{dr}+\left\{(2iEr+1)+\frac{[(r^2+a^2)E-am]^2}{\Delta}-i\frac{[(r^2+a^2)E-am](r-M)}{\Delta}\right\}R_{+1/2}=\lambda R_{+1/2},$$

$$\Delta \frac{d^2R_{-1/2}}{dr^2}+(r-M)\frac{dR_{-1/2}}{dr}+\left\{-2iEr+\frac{[(r^2+a^2)E-am]^2}{\Delta}+i\frac{[(r^2+a^2)E-am](r-M)}{\Delta}\right\}R_{-1/2}=\lambda R_{-1/2}. \tag{10}$$

Now we can construct the wave function of four components in the Dirac field as

$$\begin{aligned}\Psi&=[\Psi_{11}\Psi_{12}\Psi_{21}\Psi_{22}]^T\\&=[(r-ia\cos\theta)^{-1}R_{-1/2}\Theta_{-1/2}\quad R_{+1/2}\Theta_{+1/2}\quad R_{+1/2}\Theta_{-1/2}\quad (r+ia\cos\theta)^{-1}R_{-1/2}\Theta_{+1/2}]^T\\&\quad\times\exp[-i(Et-m\varphi)].\end{aligned}$$

III. FREE ENERGY OF THE DIRAC FIELD IN THE KERR-NEWMAN BLACK HOLE

Here we have assumed that a free Dirac particle with mass μ_0 propagates in the Kerr-Newman black hole background. This means that the spinor is in the Hartle-Hawking vacuum whose temperature should be Hawking temperature T_H. According to the theory of canonical ensemble, the free energy of the system can be written as

$$\beta F=-\sum_{E,m}\ln[1+e^{-\beta(E-m\Omega_H)}],$$

where Ω_H is the angular velocity of the black hole horizon defined by $\Omega_H=-g_{t\varphi}/g_{\varphi\varphi}|_{r=r_H}=a/(r_H^2+a^2)$. Using the semiclassical theory and assuming that the energy E and the azimuthal quantum number m are continuous, we replace the sum by integration

$$\begin{aligned}\beta F&=-\int_0^{+\infty}g(E)dE\int dm\ln[1+e^{-\beta(E-m\Omega_H)}]\\&=-\int d\Gamma(E)\int dm\ln[1+e^{-\beta(E-m\Omega_H)}]\\&=-\Gamma(E)\int dm\ln[1+e^{-\beta(E-m\Omega_H)}]\Big|_0^{+\infty}\\&\quad+\int dm\int_0^{+\infty}dE\frac{-\beta\Gamma(E)e^{-\beta(E-m\Omega_H)}}{1+e^{-\beta(E-m\Omega_H)}}\\&=-\beta\int dm\int_0^{+\infty}dE\frac{\Gamma(E)}{e^{\beta(E-m\Omega_H)}+1},\end{aligned}\tag{11}$$

where $\Gamma(E)$ is the total number of microscopic states with energy less than E and $g(E)=d\Gamma(E)/dE$ is the density of state. We have integrated by parts in the third line of the above equation.

According to the semiclassical quantization rule, we have

$$\Gamma(E)=\int d\theta d\varphi\int dk_\theta\cdot n_r=\frac{1}{\pi}\int d\theta d\varphi\int dk_\theta\int k_r dr.$$

Then the expression of free energy can be written as

$$F=-\frac{1}{\pi}\int dm\int_0^{\infty}dE\int d\theta d\varphi\int dk_\theta\int dr\frac{k_r}{e^{\beta(E-m\Omega_H)}+1},\tag{12}$$

where k_r, k_θ are the radial and angular wave number.

Considering that the wave function of the Dirac field is composed of four components, we will calculate the free energy of the components respectively at first. The total free energy should be the sum of the four components.

From Eq. (10), we can obtain the following equations by WKB approximation $[R_{\pm1/2}(r)\Theta_{\pm1/2}(\theta)=e^{iS(r,\theta)}]$:

$$R_{-1/2}:-\Delta k_r^2+\frac{[(r^2+a^2)E-am]^2}{\Delta}=\lambda,$$

$$R_{+1/2}:-\Delta k_r^2+1+\frac{[(r^2+a^2)E-am]^2}{\Delta}=\lambda,$$

$$\begin{aligned}\Theta_{-1/2}:-k_\theta^2+\Big[&aE\cos\theta+m\frac{\cos\theta}{\sin^2\theta}-\frac{1}{2}\csc^2\theta+\frac{1}{4}\cot^2\theta\\&+2mEa-a^2E^2\sin^2\theta-\frac{m^2}{\sin^2\theta}+\lambda\Big]=0,\end{aligned}$$

$$\begin{aligned}\Theta_{+1/2}:-k_\theta^2+\Big[&-aE\cos\theta-m\frac{\cos\theta}{\sin^2\theta}-\frac{1}{2}\csc^2\theta+\frac{1}{4}\cot^2\theta\\&+2mEa-a^2E^2\sin^2\theta-\frac{m^2}{\sin^2\theta}+\lambda\Big]=0,\end{aligned}\tag{13}$$

where $k_r=\partial_r S(r,\theta)$ and $k_\theta=\partial_\theta S(r,\theta)$.

So from Eqs. (8) and (13) we can obtain the following constraints between k_r and k_θ with respect to the four components of wave function:

$$\psi_{11}: k_r^2 = \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(aE\cos\theta + m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} - k_\theta^2 \right) \Big/ \Delta,$$

$$\psi_{12}: k_r^2 = \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(1 - aE\cos\theta - m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} - k_\theta^2 \right) \Big/ \Delta,$$

$$\psi_{21}: k_r^2 = \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(1 + aE\cos\theta + m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} - k_\theta^2 \right) \Big/ \Delta,$$

$$\psi_{22}: k_r^2 = \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(-aE\cos\theta - m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} - k_\theta^2 \right) \Big/ \Delta. \tag{14}$$

In Eq. (12), the integration with respect to k_θ must be carried out over the phase volume satisfying the following condition for every component of wave function:

$$\psi_{11}: \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(aE\cos\theta + m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Big/ \Delta \geqslant 0,$$

$$\psi_{12}: \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(1 - aE\cos\theta - m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Big/ \Delta \geqslant 0,$$

$$\psi_{21}: \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(1 + aE\cos\theta + m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Big/ \Delta \geqslant 0,$$

$$\psi_{22}: \frac{[(r^2+a^2)E-am]^2}{\Delta^2} + \left(-aE\cos\theta - m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Big/ \Delta \geqslant 0. \tag{15}$$

Due to $k_r^2 \geqslant 0$, the free energy of the four components can be rewritten as

$$F_{11} = -\frac{1}{2}\int dm \int d\theta d\varphi \int_0^\infty dE \int dr \frac{1/\Delta^{1/2}}{e^{\beta(E-m\Omega_H)}+1} \Bigg\{ [(r^2+a^2)E-am]^2/\Delta + \left(aE\cos\theta + m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Bigg\},$$

$$F_{12} = -\frac{1}{2}\int dm \int d\theta d\varphi \int_0^\infty dE \int dr \frac{1/\Delta^{1/2}}{e^{\beta(E-m\Omega_H)}+1} \Bigg\{ [(r^2+a^2)E-am]^2/\Delta + \left(1 - aE\cos\theta - m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Bigg\},$$

$$F_{21} = -\frac{1}{2}\int dm \int d\theta d\varphi \int_0^\infty dE \int dr \frac{1/\Delta^{1/2}}{e^{\beta(E-m\Omega_H)}+1} \Bigg\{ [(r^2+a^2)E-am]^2/\Delta + \left(1 + aE\cos\theta + m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Bigg\},$$

$$F_{22} = -\frac{1}{2}\int dm \int d\theta d\varphi \int_0^\infty dE \int dr \frac{1/\Delta^{1/2}}{e^{\beta(E-m\Omega_H)}+1} \Bigg\{ [(r^2+a^2)E-am]^2/\Delta + \left(-aE\cos\theta - m\frac{\cos\theta}{\sin^2\theta} - \frac{1}{2}\csc^2\theta + \frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right) \Bigg\}.$$

So the total free energy can be written as

$$F = F_{11} + F_{12} + F_{21} + F_{22}$$
$$\approx -2\int d\theta d\varphi \int dr \frac{\Sigma}{\Delta^{3/2}\sin^2\theta}\int dm \times \int dE \frac{g_{ab}T^aT^b}{e^{\beta(E-m\Omega_H)}+1}, \tag{16}$$

where $T^t = m, T^\varphi = E, g_{ab}T^aT^b = g_{\varphi\varphi}E^2 + 2g_{t\varphi}Em + g_{tt}m^2$, and indices a,b denote t and φ coordinates. The approximation in Eq. (16) depends upon $\frac{1}{4}\cot^2\theta \ll m^2/\sin^2\theta$.

On the other hand, we can easily sum up Eq. (15) to the following expression:

$$\frac{[(r^2+a^2)E - am]^2}{\Delta^2} + \left(-\frac{1}{4}\cot^2\theta + 2mEa - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta}\right) \Big/ \Delta \geqslant 0,$$

and considering $\frac{1}{4}\cot^2\theta \ll m^2/\sin^2\theta$, we can make it more simple as

$$g_{ab}T^aT^b \geqslant 0. \tag{17}$$

This expression is very useful in calculating the integrations of F_{NSR} and F_{SR} in the next part. The free energy can be composed from two parts as $F = F_{\mathrm{NSR}} + F_{\mathrm{SR}}$. The SR modes are characterized by the condition of $0 \leqslant E \leqslant m\Omega_H$ and $m > 0$, while the NSR modes satisfy $E > m\Omega_H$ for arbitrary m [16]. In these integration ranges, we can calculate the free energy of the two modes. Certainly we would like to adopt the brick-wall model that means $\Psi = 0$ if $r \leqslant r_H + \epsilon(\theta)$ or $r \geqslant L$, where $L \gg r_H$ and $\epsilon(\theta)$ is a small, positive parameter.

A. Free energies of NSR and SR modes

From Eq. (16), the NSR part of the free energy F_{NSR} is given by

$$F_{\mathrm{NSR}} = -2\int d\theta d\varphi \int_{r_H+\epsilon}^{L} dr \frac{\Sigma}{\Delta^{3/2}\sin^2\theta}\int dm \int dE \frac{g_{ab}T^aT^b}{e^{\beta(E-m\Omega_H)}+1}$$
$$\approx -\frac{7}{180}\frac{\pi^4}{\beta^4}\int d\theta d\varphi \frac{(r_H^2+a^2)^4 \sin\theta \cdot \epsilon^{-1}}{(r_H^2+a^2\cos^2\theta)(r_H-M)^2}. \tag{18}$$

Here it is necessary to point out that we have neglected the usual contribution from the distant vacuum surrounding the system including L^3 and have only kept the leading part including ϵ^{-1}.

Now let us calculate the remaining SR part of the free energy F_{SR} as

$$F_{\mathrm{SR}} = -2\int d\theta d\varphi \int_{r_H+\epsilon}^{L} dr \frac{\Sigma}{\Delta^{3/2}\sin^2\theta}\int_{>0} dm \int_0^{m\Omega_H} dE \frac{g_{ab}T^aT^b}{e^{\beta(E-m\Omega_H)}+1}$$
$$\approx \frac{7}{360}\frac{\pi^4}{\beta^4}\int d\theta d\varphi \frac{(r_H^2+a^2)^4 \sin\theta \cdot \epsilon^{-1}}{(r_H^2+a^2\cos^2\theta)(r_H-M)^2} + \frac{B}{\epsilon_m^4}, \tag{19}$$

where ϵ_m is a regulating infinitesimal parameter of factor $e^{-\epsilon_m m}$, which is introduced in the above integration to avoid the divergence of the free energy due to the m integration, B is an expression not including β, and we also kept the leading part including ϵ^{-1}.

The calculation of Eqs. (18) and (19) is very complex, but it is similar to that of [6]. The difference between them is only the factor $e^{\beta(E-m\Omega_H)} - 1$ and $e^{\beta(E-m\Omega_H)} + 1$ in the integrated function. However, it is important to point out that Eq. (17) is necessary to calculate these two integrations. To understand more, please refer to Eqs. (9)–(19) in [6].

B. Angular cutoff δ and entropy

As mentioned above, the total free energy is summed up from F_{NSR} and F_{SR}. So we can think about the integration of θ and φ. First of all, the brick-wall cutoff ϵ is introduced in order to eliminate divergences arising from the infinite growth of the density of states close to the horizon. But in the Kerr-Newman black hole case, the brick-wall cutoff depends on the angular coordinate θ as $\epsilon(\theta) = \bar{\epsilon} g(\theta)\sin^n\theta$ to avoid divergence, where $\bar{\epsilon}$ is a small constant, $n \geqslant 2$, and $g(\theta)$ is a regular function. As a result, we again encounter the unexpected divergence of $\epsilon(\theta)$ at $\theta = 0, \pi$. This difficulty can be avoided and the idea of the brick-wall method can be maintained by introducing another cutoff on the θ-integration range, for example, an angular cutoff δ. The expression of $\epsilon(\theta)$ and the idea of angular cutoff have been discussed very clearly in physical meaning in Ref. [6]. Then we can obtain the total free energy from Eqs. (18) and (19) as

$$F=F_{\mathrm{NSR}}+F_{\mathrm{SR}}$$
$$\approx-\frac{7}{360}\frac{\pi^4}{\beta^4}\int_\delta^{\pi-\delta}d\theta\int_0^{2\pi}d\varphi$$
$$\times\frac{(r_H^2+a^2)^4\sin\theta\cdot\epsilon(\theta)^{-1}}{(r_H^2+a^2\cos^2\theta)(r_H-M)^2}+\frac{B}{\epsilon_m^4}.$$

We can now obtain the entropy of the Dirac field on the Kerr-Newman black hole background from the standard formula

$$S=\beta^2\frac{\partial F}{\partial\beta}\bigg|_{\beta=\beta_H}.$$

So, we have the following expression:

$$S=\frac{7}{90}\frac{\pi^4}{\beta_H^3}\int_\delta^{\pi-\delta}d\theta\int_0^{2\pi}d\varphi\frac{(r_H^2+a^2)^4\sin\theta\cdot\epsilon(\theta)^{-1}}{(r_H^2+a^2\cos^2\theta)(r_H-M)^2}$$
$$=\frac{7\pi}{720}(r_H-M)(r_H^2+a^2)\int_\delta^{\pi-\delta}d\theta\int_0^{2\pi}d\varphi\frac{\sin\theta}{r_H^2+a^2\cos^2\theta}\frac{1}{\epsilon(\theta)}$$
$$=\frac{7}{2880}A_H\int_\delta^{\pi-\delta}d\theta\int_0^{2\pi}d\varphi\frac{\sin\theta(r_H-M)}{r_H^2+a^2\cos^2\theta}\frac{1}{\epsilon(\theta)},$$

where we have used $A_H=4\pi(r_H^2+a^2)$ and $\beta_H=2\pi(r_H^2+a^2)/(r_H-M)$.

We suppose the function $\epsilon(\theta)$ is

$$\epsilon(\theta)=\bar{\epsilon}\frac{r_H-M}{2(r_H^2+a^2\cos^2\theta)}\sin^2\theta,$$

and then we can obtain

$$S=\frac{7}{1440}A_H\int_\delta^{\pi-\delta}d\theta\int_0^{2\pi}d\varphi\frac{1}{\sin\theta}\frac{1}{\bar{\epsilon}},\tag{20}$$

where $\bar{\epsilon}$ is a small constant.

From Eq. (20), integrating θ, φ, we can obtain

$$S=\frac{7\pi}{360}A_H\frac{1}{\bar{\epsilon}}\ln\left(\cot\frac{\delta}{2}\right)\approx\frac{7\pi}{360}A_H\frac{1}{\bar{\epsilon}}\ln\left(\frac{2}{\delta}\right).\tag{21}$$

If we make the cutoff $\bar{\epsilon}$ and δ satisfy the same relation as [6]

$$\left(\frac{\delta}{2}\right)^{\bar{\epsilon}}=\exp\left(\frac{\pi}{45}\right),\tag{22}$$

the entropy of the Dirac field becomes 7/2 times the entropy of the scalar field, and is also proportional to the black hole area

$$S=\frac{7}{2}\frac{1}{4}A_H.\tag{23}$$

IV. CONCLUSIONS

We have studied the entropy of the Dirac field in the Kerr-Newman black hole. Thinking of the Dirac equations in Newman-Penrose formalism, there are four components of the wave function, which are ψ_{11}, ψ_{12}, ψ_{21}, and ψ_{22}. We considered that the total free energy is summed up from the free energies of the four components. Then the entropy of the Dirac field in Kerr-Newman black hole is calculated via the brick-wall method. If the cutoff values $\bar{\epsilon}$ and δ satisfy the same proper relation as the scalar field [6], the resulting entropy is 7/2 times that of the Klein-Gordon field and is also proportional to the black hole area. This result is in accordance with [12] which is calculated by another method [13]. Although the massless Klein-Gordon equation and the Dirac equation in the $(1+1)$-dimensional charged black hole have been solved in [14], it is necessary to calculate a real model such as the Kerr-Newman black hole. Moreover, our result is different from that of [14] in the relation between scalar and Dirac field. In the calculation, we also found that the F_{SR} is negative half of F_{NSR}. This is the same as Ref. [6].

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