In particular, the "unitarity" and CP conditions used in the analysis of the $K_L \to \mu^+\mu^-$ puzzle are contained in such an S matrix. Thus, any results relating K_L decays to K_S decays are valid in a pure S-matrix formulation.

V. ACKNOWLEDGMENT

The author wishes to express his deep gratitude to Professor William D. McGlinn for suggesting the problem and various helpful discussions.

---

[1]W. D. McGlinn and D. F. Polis, Phys. Rev. Letters 22, 908 (1969); K. W. McVoy, *ibid.* 23, 56 (1969); L. Durand III and K. W. McVoy, *ibid.* 23, 59 (1969); G. C. Wick, Phys. Letters 30B, 126 (1969); Y. Dothan and D. Horn, Phys. Rev. D 1, 916 (1970); L. Wolfenstein, Phys. Rev. 188, 2536 (1969).

[2]L. Stodolsky, Phys. Rev. D 1, 2683 (1970).

[3]T. T. Gien, Progr. Theoret. Phys. (Kyoto) 45, 1203 (1971). See also W. Kerler, Phys. Rev. D 3, 2909 (1971).

[4]Sudhir Kumar, Phys. Rev. D 5, 2128 (1972).

[5]J. S. Bell and J. Steinberger, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High Energy Laboratory, Chilton, Berkshire, England, 1966), pp. 195–222.

[6]See, for example, M. K. Gaillard, Phys. Letters 36B, 114 (1971), and references cited therein.

[7]Equation (15) is precisely the condition necessary to show the validity of the unitarity condition used by Gaillard (Ref. 6), $-i(T_{if} - T^*_{\hat{i}\hat{f}}) = \sum_n T_{in} T_{\hat{n}\hat{f}} = \sum_n T^*_{\hat{i}\hat{n}} T_{nf}$.

[8]It may be worthwhile to note that McVoy's choice (Ref. 1) of normalization, i.e., $g_i^\dagger g_i = h_i^\dagger h_i = 1$ implying $|\lambda_i| = 1$ is not consistent with the exact unitarity conditions (2). This inconsistency, however, seems to disappear in the later choice of Durand and McVoy (Ref. 1) which, by choosing $g_i^\dagger g_i = h_i^\dagger h_i$, implies $g_i^\dagger g_i = (1 + |\alpha|^2)^{1/2}$ and $|\lambda_i| = 1$. The unitarity conditions, Eqs. (12) and (13) of Durand and McVoy (Ref. 1), are therefore exact and, contrary to their remarks, they are independent of any CPT assumption.

[9]See, for example, R. G. Sachs, Ann. Phys. (N.Y.) 22, 239 (1963), and P. K. Kabir, *The CP Puzzle* (Academic, London and New York, 1968).

PHYSICAL REVIEW D VOLUME 6, NUMBER 7 1 OCTOBER 1972

Why Do Resonances Scale?

M. Elitzur*†
The Rockefeller University, New York, New York 10021

and

L. Susskind
Belfer Graduate School, Yeshiva University, New York 10033

(Received 31 March 1972)

Scaling of hadronic resonance excitation is discussed within the framework of a simple potential-theoretic parton model.

I. INTRODUCTION

The experimental scaling behavior of deep-inelastic electroproduction cross sections[1,2] and diffractive resonance excitations[3] raises a number of questions which we will consider in this paper:

(1) What is the scale of energies and momentum transfers which determines the onset of scaling?

(2) What is the role of resonances in the scaling phenomena?

(3) In terms of which kinematic variables does the scaling work best?

The experimental answer to question (1) is that scaling sets in for values of ν and Q^2 in excess of 1 GeV^2. This suggests that the important scale in the problem is the hadronic level spacing[4] (inverse Regge slope).

Questions (2) and (3) were answered by Bloom and Gilman,[5] who observed that in the scaling variable

$$\omega' = \frac{2M\nu}{Q^2} + \frac{M^2}{Q^2}$$

$$= 1 + \frac{s}{Q^2} \tag{1}$$

scaling occurs over a particularly large range of phase space. In fact the use of ω' even allows the prominent resonance contributions to oscillate about the universal curve. This is to be contrasted with the earlier use of

$$\omega = \frac{2M\nu}{Q^2} \tag{2}$$

as a scaling variable.[2] Although scaling is quite pronounced in terms of ω, once the resonance region is reached scaling fails for ω and the resonances lie off the asymptotic curve. ω' is also the relevant variable in the analysis of diffractive resonance production,[3] where the hadronic form factors were found to scale in terms of the equivalent variable $t/M_R{}^2$.

In this paper we will show that the observed answers to questions (1)–(3) can be understood in terms of a parton picture in the infinite-momentum frame. Using the correspondence between two-dimensional Galilean physics[6] and infinite-momentum mechanics we formulate a nonrelativistic (NR) model in terms of Lorentz-covariant quantities.

Our model is not meant as a detailed picture of the hadron, and we do not attempt to fit any experimental data with it. Our purpose is to understand the intuitive content of scaling of resonance excitation.

The rules of correspondence between infinite-momentum physics and two-dimensional nonrelativistic physics are as follows[6]:

(1) Nonrelativistic mass μ is identified with longitudinal fraction η $[=(P_0+P_3)/\sqrt{2}]$:

$$\mu \leftrightarrow \eta . \tag{3}$$

(2) Two-dimensional nonrelativistic momentum q is identified with transverse momentum Q. Similarly, transverse position X and nonrelativistic 2-dimensional position x are identified:

$$Q \leftrightarrow q, \quad X \leftrightarrow x . \tag{4}$$

(3) The nonrelativistic energy of a bound system is

$$P^2/2\mu + E , \tag{5}$$

where μ is the sum of the constituent masses and E is the internal energy of the system. This is identified with the infinite-momentum energy

$$\begin{aligned} H &= (P_0 - P_3)/\sqrt{2} \\ &= (P^2 + M^2)/2\eta \\ &= P^2/2\eta + H_{\text{internal}} , \end{aligned}$$

where M is the rest mass of the bound state. Using Eqs. (3) and (5), we identify the internal energy:

$$E \leftrightarrow \frac{M^2}{2\eta} \quad \text{or} \quad 2\mu E \leftrightarrow M^2 . \tag{6}$$

Consider next the absorption of a purely transverse momentum Q by a hadron of transverse momentum P and longitudinal fraction η. Define

$$\nu = P\cdot Q \xleftrightarrow[\text{NR analog}]{} p\cdot q , \tag{7}$$

$$s = M^2 - 2PQ - Q^2 \leftrightarrow 2\mu E - 2p\cdot q - q^2 , \tag{8}$$

$$\omega = \frac{2\nu M}{Q^2} \leftrightarrow \frac{2\nu}{q^2}(2\mu E)^{1/2} , \tag{9}$$

$$\omega' = 1 + \frac{s}{Q^2} \leftrightarrow 1 + \frac{s}{q^2} . \tag{10}$$

We are interested in the transition probability to excite a resonant state.

II. THE MODEL

We will apply the rules of correspondence (3), (4), and (6) to a process in which a nonrelativistic bound state absorbs a momentum q. For simplicity the bound system consists of a pair of point particles of masses μ_1 and μ_2 bound by a potential $V(|x_1 - x_2|)$ which is deep enough to bind many states. According to the parton model[7] the invariant cross section $F(\omega', Q^2)$ in the corresponding relativistic problem tends to a function $F(\omega')$ when Q^2 and $Q\cdot P$ become large. Furthermore, $F(\omega')$ (Ref. 8) is proportional to the mean number of partons with longitudinal fraction satisfying

$$\frac{1}{\omega'} = \frac{\eta_P}{\eta_H} \equiv \beta_P , \tag{11}$$

where η_P (η_H) is the longitudinal fraction of the parton (hadron). Thus we may expect in the corresponding nonrelativistic problem that the asymptotic scaling curve is nonzero only when

$$\frac{1}{\omega'} = \frac{\mu_1}{\mu_1+\mu_2} \quad \text{or} \quad \frac{1}{\omega'} = \frac{\mu_2}{\mu_1+\mu_2} . \tag{12}$$

The nonrelativistic scaling curve will then be of the form

$$F(\omega') \xrightarrow[\text{Bj}]{} a_1 \delta\left(\omega' - \frac{\mu_1+\mu_2}{\mu_1}\right) + a_2\delta\left(\omega' - \frac{\mu_1+\mu_2}{\mu_2}\right). \tag{13}$$

A smooth scaling curve can be obtained by imagining that the system is a continuous superposition of states in which the total mass $(\mu_1+\mu_2)$ is proportioned differently into μ_1 and μ_2. We shall be content to study the approach to the limit (13).

The initial wave function is a product of c.m. and internal wave functions:

$$\Psi_i(x_1, x_2) = \psi_0(x_1 - x_2)\exp\left[ip\cdot\left(\frac{\mu_1 x_1 + \mu_2 x_2}{\mu_1+\mu_2}\right)\right] . \tag{14}$$

Using Eqs. (3) and (11) we identify the nonrelativistic mass ratios as

$$\frac{\mu_i}{\mu_1+\mu_2} \equiv \beta_i \quad (i=1,2)\,. \tag{15}$$

Assume the momentum q is absorbed by particle 1. The wave function becomes

$$e^{iq\cdot x_1}\Psi_i(x_1, x_2)\,. \tag{16}$$

To compute the transition matrix element into the nth level we project Eq. (16) onto

$$\Psi_f(x_1,x_2) = \psi_n(x_1-x_2)\exp[i(p+q)\cdot(\beta_1 x_1+\beta_2 x_2)]\,. \tag{17}$$

Since in the relativistic problem the 4-momentum Q absorbed by the hadron has no longitudinal or time component[9] the matrix element must contain an energy-conserving δ function which reads

$$\delta(p^2+2(\mu_1+\mu_2)E_0-(p+q)^2-2(\mu_1+\mu_2)E_n)\,, \tag{18}$$

which, according to Eq. (8), may be written

$$\delta(s-M_n^{\,2})\,. \tag{19}$$

The result is

$$\delta(s-M_n^{\,2})F_1(n,q^2) = \delta(s-M_n^{\,2})\int d^2x\, e^{iq\cdot x(1-\beta_1)}\psi_n^*(x)\psi_0(x)\,, \tag{20}$$

where $x=x_1-x_2$. The transition probability is $|F_1(n,q^2)|^2$. We will use the notation

$$q_i=(1-\beta_i)q \quad (i=1,2)\,. \tag{21}$$

The Schrödinger equation for $\psi_n(x)$ is

$$[-(1/2\mu^*)\nabla^2+V(x)]\,\psi_n(x)=E_n\psi_n(x)\,, \tag{22}$$

where the reduced mass μ^* is defined by

$$\mu^*=\frac{\mu_1\mu_2}{\mu_1+\mu_2} = \beta_1\beta_2(\mu_1+\mu_2)\,. \tag{23}$$

III. THE MATRIX ELEMENT

We consider the transition matrix element $F_1(n,q^2)$ in Eq. (20):

$$F_1(n,q^2)=\int d^2x\, e^{iq_1\cdot x}\psi_n^*(x)\psi_0(x)\,. \tag{24}$$

To simplify the discussion we first consider the one-dimensional problem.

A. One-Dimensional Matrix Element

Since the wave function of a bound state drops exponentially to zero outside the potential,[10] we will approximate Eq. (24) by restricting the integral to be over the region where the potential is nonzero (see Fig. 1)[11]:

$$F_1(n,q^2)=\int_{x_a}^{x_b} e^{iq_1\cdot x}\psi_n^*(x)\psi_0(x)\,dx\,. \tag{25}$$

The wave function ψ_n is known to have $n-1$ nodes inside the potential.[10] Therefore it is clear that

$$F_1(n,q^2)\underset{n\to\infty}{\longrightarrow}0\,,\quad q^2=\text{const} \tag{26a}$$

$$F_1(n,q^2)\underset{q^2\to\infty}{\longrightarrow}0\,,\quad n=\text{const}\,. \tag{26b}$$

This is due to the increasing number of oscillations in ψ_n in case (26a) and of $e^{iq_1\cdot x}$ in case (26b). There is however a direction in the n-q^2 plane where the oscillations of ψ_n and $e^{iq_1\cdot x}$ can be matched. Along this direction (which we refer to as the scaling direction) constructive interference will keep $F_1(n,q^2)$ from falling as quickly as in other directions. To find this direction we note that the number of nodes between x_a and x_b of the factor e^{iq_1x} is

$$N=\frac{1}{\pi}(x_b-x_a)q_1\,. \tag{27}$$

Matching this against the number of nodes of ψ_n we obtain

$$\frac{n}{q_1}\simeq\frac{1}{\pi}(x_b-x_a)\,. \tag{28}$$

n may be related to the energy E_n of the level n by the Bohr-Sommerfeld quantization rule or the WKB[12] approximation:

$$\int_{x_a}^{x_b}[2\mu^*(E_n-V(x))]^{1/2}dx=(n+\tfrac{1}{2})\pi\,. \tag{29}$$

The conditions for the validity of Eq. (29) are that n be large and that the slope of the potential vary slowly. For large n we can expand the integrand of (29) and use the mean-value theorem to write

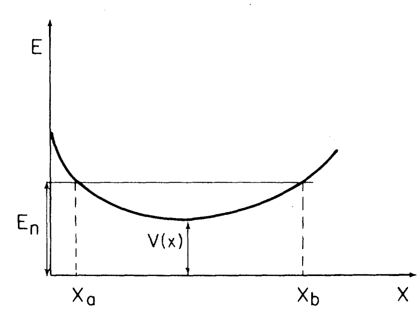

FIG. 1. Significant range of integration for the excitation matrix element.

$$(x_b - x_a)(2\mu^* E_n)^{1/2}(1 - \tfrac{1}{2}\bar{V}/E_n + \cdots) = (n + \tfrac{1}{2})\pi\,, \tag{30}$$

where $\bar{V}$ is some mean value for the potential between x_a and x_b. Equation (30) gives the number of nodes for a highly excited level:

$$n \simeq \frac{x_b - x_a}{\pi}\,(2\mu^* E)^{1/2}\,. \tag{31}$$

Comparing Eqs. (31) and (28) we derive a scaling direction in the E-q^2 plane which reads

$$\begin{aligned} 1 &= \frac{q_1^{\,2}}{2\mu^* E_n} \\ &= \frac{q^2(1-\beta_1)^2}{2\beta_1\beta_2(\mu_1+\mu_2)E_n}\,. \end{aligned} \tag{32}$$

Using Eqs. (3), (4), (6), and (15) we may write the relativistic analog of Eq. (32) as

$$\frac{Q^2(1-\beta_1)}{M_n^{\,2}\beta_1} = 1\,. \tag{33}$$

Recalling that the energy-conserving δ function in Eq. (19) restricts $M_n^{\,2} = s$, we may finally write

$$\frac{1}{\beta_1} = 1 + \frac{s}{q^2} \equiv \omega'\,. \tag{34}$$

Hence we see that the condition for constructive interference between ψ_n and e^{iq_1x} defines a scaling direction in the (s, q^2) plane. [Compare with Eq. (11).] The emergence of the Bloom-Gilman variable ω' appears natural in our discussion.

One further constraint on $F_1(n, q^2)$ is derived from the unitarity of e^{iq_1x}:

$$\sum_n |F_1(n, q^2)|^2 = 1\,. \tag{35}$$

We will now replace the discrete variable n by the continuous variable ω'. We define

$$T_1(\omega', q^2) = \frac{dn}{d\omega'}\,|F_1(n, q^2)|^2\,, \tag{36}$$

where $dn/d\omega'$ is the density of states.

Then

$$\int_1^\infty T_1(\omega', q^2)\,d\omega' = 1 \tag{37}$$

for any q^2. This is the analog of the Adler sum rule.

Since there is only one value of ω' for which F will be significant when $q^2 \to \infty$ we must have

$$\lim_{q^2\to\infty} T_1(\omega', q^2) = \delta(\omega' - 1/\beta_1)\,. \tag{38}$$

The factor $dn/d\omega'$ is given by

$$\begin{aligned} \frac{dn}{d\omega'} &= \frac{dn}{dM_n^{\,2}}\,\frac{dM_n^{\,2}}{d\omega'} \\ &= \frac{dn}{dM_n^{\,2}}\,q^2 \\ &= M_n^{\,2}\,\frac{dn}{dM_n^{\,2}}\,\frac{1}{\omega'-1}\,. \end{aligned} \tag{39}$$

The level density for the usual nonrelativistic potentials generally behaves as some power of the energy so that $M_n^{\,2}dn/dM_n^{\,2}$ will be proportional to n. Thus

$$T_1(\omega' q^2) \sim \frac{n}{\omega'-1}\,|F_1(n, q^2)|^2\,. \tag{40}$$

Thus in order that Eq. (38) be satisfied, $|F_1(n, q^2)|$ must fall no faster than $n^{-1/2}$ along the scaling direction. It is also evident that the rate at which the scaling limit (38) occurs is governed by the behavior of $F_1(n, q^2)$ along the scaling direction. Suppose $F_1(n, q^2)$ behaves like

$$F_1(n, q^2) \to n^C \tag{41}$$

along the scaling direction. In order that T_1 tend to a δ function, the width of T_1 on the ω axis must satisfy

$$\Delta\omega'\,\frac{n}{\omega'-1}\,n^{2C} \to 1 \tag{42}$$

or

$$\Delta\omega' \to \frac{1}{n^{2C+1}}\,. \tag{43}$$

Thus, the larger C is, the faster T_1 will converge to a narrow δ function.

If we take into account the contribution of the second particle the full amplitude is

$$F(n, q^2) = F_1(n, q^2) + F_2(n, q^2)\,. \tag{44}$$

This leads to the following expression for the full transition probability T:

$$T(\omega', q^2) = T_1(\omega', q^2) + T_2(\omega', q^2) + T_{12}(\omega', q^2)\,, \tag{45}$$

where the interference term is

$$T_{12}(\omega', q^2) = [F_1(n, q^2)F_2^*(n, q^2) + F_1^*(n, q^2)F_2(n, q^2)]\frac{dn}{d\omega'}\,. \tag{46}$$

In the deep-inelastic limit T_{12} will vanish relative to T_1 and T_2. For $\beta_1 \neq \beta_2$ this occurs because it is the product of two factors which survive only in non-overlapping regions. When $\beta_1 = \beta_2$ the parity of successive levels alternate, and the amplitude is

$$T(\omega', q^2) = 2T_1(\omega', q^2)[1 + (-1)^n]\,. \tag{47}$$

In both cases, we get the expected relation, Eq. (13), for the transition probability.

B. Harmonic Oscillator

The simplest case to discuss, apart from the almost trivial square well, is the harmonic oscillator. The Hamiltonian is conveniently written in the form

$$H = \frac{P_1^{\,2}}{2\mu_1} + \frac{P_2^{\,2}}{2\mu_2} + \frac{g^2}{2\mu^*}(x_1 - x_2)^2 . \tag{48}$$

The matrix element $F_1(n, q^2)$ is

$$F_1(n, q^2) = \langle n | e^{iq_1 x} | 0 \rangle . \tag{49}$$

Expressing the state $|n\rangle$ and the coordinate x in terms of creation and annihilation operators, we get

$$F_1(n, q^2) = i^n \exp\left(\frac{-q_1^{\,2}}{4g}\right)\left[\left(\frac{q_1^{\,2}}{2g}\right)^n \frac{1}{n!}\right]^{1/2} . \tag{50}$$

From Eq. (50) we see that if $q_1^{\,2}$ or n tends to ∞ with the other remaining fixed the matrix element vanishes, at least exponentially. To find the scaling direction from Eq. (50) we apply Stirling's formula to get

$$F_1(n, q^2) \to i^n \exp\left[\tfrac{1}{2}\left(1 - \frac{q_1^{\,2}}{2gn}\right)\right]\left(\frac{q_1^{\,2}}{2gn}\right)^{n/2}(2\pi n)^{-1/4} . \tag{51}$$

Evidently the scaling direction is

$$\frac{q_1^{\,2}}{2gn} = 1 , \tag{52}$$

and in this direction F falls like $n^{-1/4}$. This is well above the unitarity limit $n^{-1/2}$. The levels for the Hamiltonian in Eq. (48) are

$$E_n = \frac{g}{\mu^*}(n + \tfrac{1}{2}) . \tag{53}$$

The scaling direction then satisfies $q^2(1-\beta_1)^2/2E_n\mu^* = 1$, as in Eq. (32). Equivalently, $\omega' = 1/\beta_1$.

Figure 2(a) illustrates $T_1(\omega', Q^2)$ as a function of ω' for different values of Q^2. The system is an oscillator with $\beta_1 = \frac{1}{3}$, $g = \frac{1}{9}$. This gives a realistic level spacing (inverse Regge slope) of 1 GeV^2 and a peak at $\omega' = 3$. The figure clearly demonstrates a rapid convergence to the scaling function $\delta(\omega' - 3)$. For purposes of comparison we have plotted T_1 against the older scaling variable ω [Fig. 2(b)]. It is apparent that scaling is much better in terms of the Bloom-Gilman variable ω'. Figure 3 demonstrates the other contributions to T: 3(a) shows T_2, which peaks at $\omega' = 1.5$ (corresponding to $\beta_2 = \frac{2}{3}$); 3(b) shows the interference term T_{12}, which oscillates with decreasing amplitude; 3(c) is the full amplitude T at 4 and 8 GeV^2. The structure of two δ functions is clearly emerging. It is apparent from the figures that the contribution at smaller β approaches the limit more slowly. This is a direct consequence of Eq. (40).

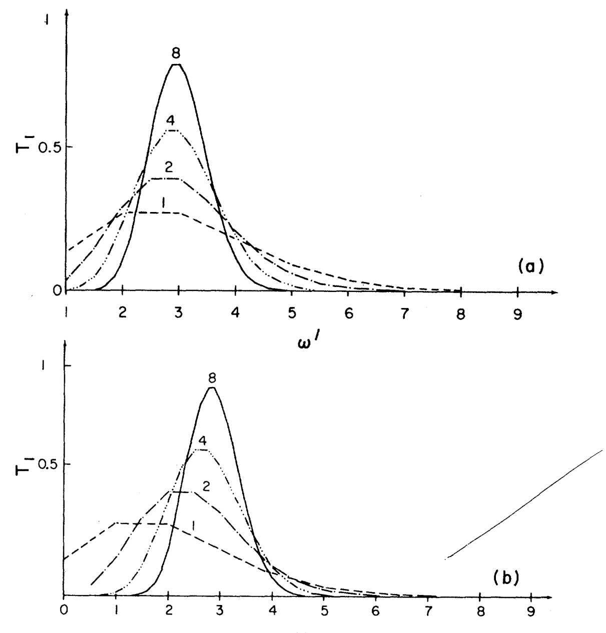

FIG. 2. The excitation probability T_1 for a harmonic oscillator with parameters $\beta_1 = \frac{1}{3}$, $g = \frac{1}{9}$. T_1 is plotted for different values of Q^2 as denoted on the curves. In (a) T_1 is shown as a function of ω', and in (b) as a function of ω.

III. TWO-DIMENSIONAL MATRIX ELEMENT

The two-dimensional problem is quite similar to the one-dimensional, to which it can be reduced due to the isotropy of the model. For instance, in the harmonic oscillator case we can choose q to lie along the x axis, and then there are no y excitations and the treatment we gave applies as it is. In the general case the wave function can be separated as

$$\psi_{l,n} = e^{il\theta} f_{l,n}(r) , \tag{54}$$

with $f_{l,n}(r)$ the radial function. The exponential factor can be expanded:

$$e^{iq_1 r\cos\theta} = \sum_{m=-\infty}^{+\infty} i^m J_m(q_1 r) e^{im\theta} , \tag{55}$$

and therefore

$$F_1(q^2, n) = 2\pi i^l \int_0^\infty r\,dr\,J_l(q_1 r) f_{ln}(r)\psi_0(r) ,$$

where $\psi_0(r)$ is the ground-state wave function. The substitution

$$f_{l,n}(r) = y_{l,n}(r)/\sqrt{r} \tag{56}$$

reduces the Schrödinger equation of the system to the one-dimensional case

$$y_{l,n}''(r) + [2\mu^*(E - V(r)) - (l^2 - \tfrac{1}{4})/r^2] y_{l,n}(r) = 0 , \tag{57}$$

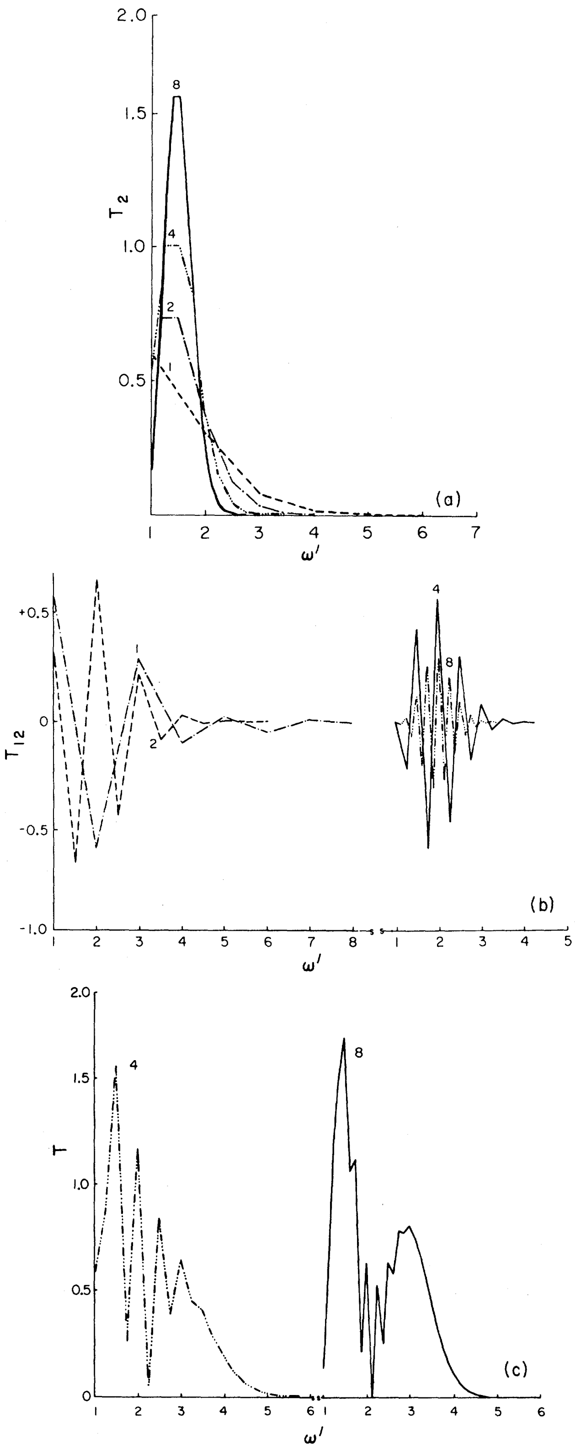

FIG. 3. In (a), (b), and (c) we show T_2, T_{12}, and T, respectively, for different values of Q^2, as denoted on the curves, as functions of ω'. The system is a harmonic oscillator with same parameters as in Fig. 2.

and all the properties of the one-dimensional equation can be used. In particular, $y_{l,n}$ vanishes exponentially outside the potential and has $n-1$ nodes inside it, for any value of l. The oscillations will be matched against those of the Bessel function, for which we use the large-argument approximation

$$J_l(q_1 r) \underset{\text{large } q_1 r}{\simeq} (2/\pi q_1 r)^{1/2} \cos[q_1 r - (l+\tfrac{1}{2})\pi] \qquad (58)$$

to get the equivalent of Eq. (27):

$$N = \frac{1}{\pi}(r_b - r_a) q_1 . \qquad (59)$$

The relation between n and the energy of the excited state can be achieved again using the WKB method applied to Eq. (59),[13]

$$\int_{r_a}^{r_b} dr [2\mu^*(E - V(r)) - l^2/r^2]^{1/2} = (n+\tfrac{1}{2})\pi , \qquad (60)$$

and this finishes the analogy with the one-dimensional problem. The scaling direction is therefore given again by $\omega' = 1/\beta_1$.

IV. CONCLUSIONS

We have discussed the scaling of hadronic resonance excitations in a simplified parton model. The resonances were identified with the excited levels of a nonrelativistic potential. The Galilean, infinite-momentum analog is the basis for the validity of this approach. The variable ω' emerged naturally in this approach and was found to be a better scaling variable than ω in a detailed numerical demonstration of the harmonic oscillator.

The essence of scaling is that the wave function of a highly excited state looks very much like a free particle within the well. Although the detailed picture of a hadron as presented here is far from realistic, we believe that the essence of resonance scaling lies in the mechanisms described. They should be reflected in more elaborate many-particle models.

ACKNOWLEDGMENTS

One of us (M. E.) thanks L. Stodolsky for discussions which led to this study. We also thank J. D. Bjorken, J. Kogut, and Y. Aharonov for numerous helpful discussions.

*On leave from the Weizmann Institute, Rehovoth, Israel.

†Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-3505.

[1]E. D. Bloom *et al.*, MIT-SLAC Report No. SLAC-PUB-796, 1970 (unpublished), presented at the Fifteenth International Conference on High Energy Physics, Kiev, U.S.S.R., 1970.

[2]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[3]M. Elitzur, Phys. Rev. Letters 27, 895 (1971).

[4]The value of 1 GeV^2 suggests hadron mass, or equivalently hadron size, just as well as hadron level spacing. In the model presented the relevant scale is the level spacing.

[5]E. D. Bloom and F. J. Gilman, Phys. Rev. Letters 25, 1140 (1970).

[6]L. Susskind, Phys. Rev. 165, 1535 (1968); 165, 1547 (1968); J. Kogut and D. Soper, Phys. Rev. D 1, 2901 (1970).

[7]R. P. Feynman, Phys. Rev. Letters 23, 1415 (1969); J. Bjorken and E. Paschos, Phys. Rev. 185, 1975 (1969).

[8]In the original parton model (Ref. 7) it is $F(\omega)$ which is proportional to the mean number of partons with $\eta = 1/\omega$ in the deep-inelastic limit. In this limit, however, $\omega = \omega'$.

[9]In the scattering problems of interest the momentum transfer Q is spacelike. It is then always possible to find an infinite-momentum frame in which Q is purely transverse.

[10]A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961), Chap. 3.

[11]We choose the range of integration according to the support of ψ_n rather than ψ_0. This is because we are interested in the behavior of $F_1(n, q^2)$ when we vary n and q^2, and in this case $\psi_0(x)$ is just a common measure.

[12]Reference 10, Chap. VI.

[13]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 1101.

PHYSICAL REVIEW D VOLUME 6, NUMBER 7 1 OCTOBER 1972

A Class of Analytic Interference Models*

M. G. Donnelly† and R. E. Kreps
Department of Physics, University of Toronto, Toronto, Canada

(Received 5 June 1972)

We present a class of interference models with good analytic properties, arbitrary trajectory functions, no ancestors or ghosts, and a fair amount of freedom remaining. Asymptotics are good for bounded trajectories; for linearly rising trajectories there is trouble in one direction. As an example we do a small calculation on $X^0 \rightarrow \eta\pi\pi$.

I. INTRODUCTION

In the current state of the art of high-energy theory, many problems come down to constructing a model amplitude after one has extracted the known kinematic, spin and internal-symmetry factors. One tries to put in as many of the basics as possible – Lorentz invariance, analyticity, crossing symmetry, unitarity, Regge asymptotics – and one always has to violate one or more of these. This leads to various compromises, where one builds in some features and either ignores others (hopefully only temporarily) or tries to satisfy the others in some approximate sense.

One can also have a general scheme – e.g., Regge-cut machines,[1,2] eikonal generators,[2,3] multi-Regge equations[4,5] – in which one uses a basic model amplitude as an input to obtain a "better" output. Ultimately, of course, one wants to do a comparison with the experimental data, but this is usually done by starting from a model with good general properties and a small number of arbitrary parameters, and then juggling parameters to obtain the best possible fit to the data.

There are almost as many theoretical models as there are theorists, but in the last few years the bulk of the models has concentrated on crossing symmetry, duality, and narrow-resonance approximations.[6] One of the more critical diseases of these models is their difficulty in moving away from the narrow-resonance approximation. If one tries to use a complex trajectory function in order that resonances may have nonzero widths, one runs into the problem of "ancestors" – the resonance is predicted to occur in *all* partial waves at the same energy. It was in looking for ways around this difficulty that we came up with the model class to be described in detail in Sec. II. Other workers[7] have also found solutions to the ancestor problem, usually in the context of dual models. They have trouble with asymptotics, as we do also; they also have what are to our minds