[8]R. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).

[9]As for the relations between short-distance and light-cone singularities, see Refs. 7 and 8.

[10]J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[11]H. Sugawara, Phys. Rev. 170, 1659 (1968).

[12]C. G. Callan, R. F. Dashen, and D. H. Sharp, Phys. Rev. 165, 1883 (1968).

[13]H. Georgi, Phys. Rev. D 2, 2908 (1970); J. H. Lowenstein and B. Schroer, *ibid.* 3, 1981 (1971).

[14]These equations were obtained by C. M. Sommerfield, Phys. Rev. 176, 2019 (1968), Eq. (1).

[15]This is the only place in the article where normal ordering refers to the positive- and negative-frequency components of the free-spinor field rather than to the frequency components of the currents j.

[16]Use of the vector field (4.3) avoids the technical infrared difficulties of the scalar field, introduced in Ref. 5, whose derivative would be (4.3).

[17]To be consistent with this viewpoint the normal ordering in Eq. (2.14) should be replaced by, e.g.,

$$:j_+(u)\psi_2(u,v):=\lim_{l\to 0}\tfrac{1}{2}[j_+(u+l)\psi_2(u,v)+\psi_2(u,v)j_+(u-l)]$$

as may be verified using the commutator of $j^{(\pm)}$ with ψ.

[18]In an interesting recent work, A. H. Mueller and T. L. Trueman, Phys. Rev. D 4, 1635 (1971), argue that anomalous dimensions in the massive Thirring model are the same as in the massless model, but that the anomalous dimension of ψ changes with the gauge transformation of the massive Thirring model $\psi \to \exp(-ig\chi)\psi$. They conclude that the anomalous dimension is without physical significance. However, the local fields $R[\bar{\psi}\psi] = R[\psi_2^\dagger\psi_1] + R[\psi_1^\dagger\psi_2]$ and $R[\bar{\psi}\gamma_5\psi] = R[\psi_2^\dagger\psi_1] - R[\psi_1^\dagger\psi_2]$ have anomalous dimension $\bar{a}/a$ and are invariant under the gauge transformation. So if the limit is smooth as the mass goes to zero, the massive Thirring model has an observable anomalous dimension because $\partial_\mu\epsilon^{\mu\nu}j_\nu$ is proportional to $R[\bar{\psi}\gamma_5\psi]$.

[19]This section and the related Appendixes (A and B) are primarily the work of one of us (G.F.D.A.).

[20]We shall not distinguish between $\mathcal{H}_{\underline{q}}$ and its canonical image as a subspace of $\mathcal{H}$.

[21]We shall always by this wording refer to the Weyl algebra.

[22]See, e.g., H. Reeh, Forstschr. Physik 16, 357 (1968).

[23]A coherent $\|g))_{\underline{n},\pm}$ is a state which satisfies

$$A^{\underline{n}}_{\underline{J},\pm}(h)\|g))_{\underline{n},\pm} = (g,h)\,\|g))_{\underline{n},\pm}\,.$$

[24]$\Omega^{\underline{n}}_{\underline{J}}$ is the (unique) vector in $\mathcal{H}_n$ such that $A^{\underline{n}}_{\underline{J},\pm}(f)\Omega^{\underline{n}}_{\underline{J}} = 0, \forall f$.

[25]Diximier, *C*-Algebras* (Gouthier-Villars, 1969), Proposition 5.1.2.

[26]It should be noted that, for $\mathcal{H}_{\underline{q}} = \mathcal{H}_{\alpha + n_1\underline{e}^1 + n_2\underline{e}^2}$, $\alpha \in R\times R$, one can in the same way construct a field $\psi(x;\underline{\alpha})$ solution of the Thirring equations. The "solution" $\psi(x;\underline{\alpha})$ is not equivalent to $\psi(x)\equiv\psi(x;\underline{0})$ if $\underline{\alpha} + n_1\underline{e}^1 + n_2\underline{e}^2 \neq 0$ for any choice of n_1, $n_2 \in Z$. In fact, both fields will be irreducible, but in the Hilbert space of $\psi(x;\underline{\alpha})$ there is no vector invariant under space-time translations.

[27]We do not know whether strong duality, i.e., $[a^{\underline{n}}(J)]' = [a^{\underline{n}}(J)]''$ holds.

[28]This last property follows from Lemma 1, Remark 3. See also the paragraphs immediatly preceding Eq. (5.3), and Ref. 25.

[29]G. F. Dell'Antonio and S. Doplicher, J. Math. Phys. 8, 663 (1967).

[30]We take $\tilde{J}_\pm(p)$ to vanish for $p \geq \Lambda$.

PHYSICAL REVIEW D VOLUME 6, NUMBER 4 15 AUGUST 1972

A Basic Discontinuity Equation*

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(Received 31 March 1972)

We derive from field theory a formula that expresses the discontinuity of any multiparticle scattering function across any normal threshold cut in terms of specified limits of scattering functions for other processes. The special case that expresses any inclusive cross section as a discontinuity has been used extensively in recent work on high-energy processes. Other cases of the general formula also appear to have important implications, which are briefly discussed.

I. INTRODUCTION

Recent studies of high-energy processes based on the work of Mueller have exploited a formula that expresses any inclusive cross section as a discontinuity in an appropraite multiparticle scattering function.[1] This formula is a special case of the general discontinuity equation displayed in Fig. 1. We describe this equation in detail in the next section. It has been discussed earlier

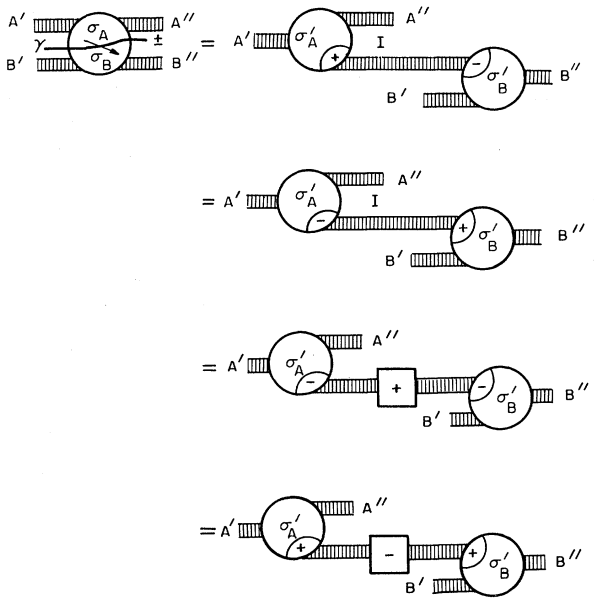

FIG. 1. Diagrammatic representation of the four forms of basic discontinuity equation.

from the S-matrix viewpoint in several unpublished works,[2] but has never actually been derived. The aim of the present work is to describe it in detail, to point out its relevance to the theory of high-energy processes, and to derive it from field theory. Our field-theoretic derivation yields the off-mass-shell version of the on-mass-shell formula discussed earlier.

The basic discontinuity equation can be briefly described in nondiagrammatic terms as follows: Let A', B', A'', and B'' represent arbitrary sets of particles. Let $S_C(p_{A'}, p_{B'}; \sigma; p_{A''}, p_{B''})$ represent a particular boundary value of the analytically continued (off-mass-shell) connected component of the S matrix for the process $A'+B' \to A''+B''$. This boundary value is the one obtained by approaching the point $p \equiv (p_{A'}, p_{B'}; p_{A''}, p_{B''})$ from within the cone Γ_σ, which lies in the physical sheet. The location of the cone Γ_σ is specified by the set of signs $\sigma = (\sigma_1, \ldots, \sigma_{\gamma'}, \ldots, \sigma_N)$: Γ_σ lies in the intersection of the upper half planes of the signed channel-energy variables $E_{\gamma'} \times \sigma_{\gamma'}$ $\operatorname{sgn} \mathrm{Re} E_{\gamma'}$. Let us denote by γ the channel defined by $A = A' + A''$ (or equivalently by $B = B' + B''$). Let us divide σ into the four parts $\sigma \equiv (\sigma_\gamma, \sigma_A, \sigma_B, \sigma_C)$, where σ_A is the set of signs for the channels defined by nonempty proper subsets of A, σ_B is the set of signs for the channels defined by nonempty proper subsets of B, and σ_C is the set of remaining signs $\sigma_{\gamma'}$. The functions $S_C(p_{A'}; \sigma'_{A+I}; p_{A''}, p_I)$ and $S_C(p_{B'}, p_I; \sigma'_{B+I}; p_{B''})$ are defined, analogously, as boundary values of the connected components of the S matrices for the processes $A' \to A'' + I$ and $B' + I \to B''$, respectively. In terms of these definitions the first basic discontinuity equation (top line of Fig. 1) says

$$S_C(p_{A'}, p_{B'}; (+1, \sigma_A, \sigma_B, \sigma_C); p_{A''}, p_{B''}) - S_C(p_{A'}, p_{B'}; (-1, \sigma_A, \sigma_B, \sigma_C); p_{A''}, p_{B''})$$

$$= \sum_I \int S_C(p_{A'}; (\sigma'_A, \sigma'_I = \{+\}); p_{A''}, p_I) S_C(p_I, p_{B'}; (\sigma'_B, \sigma'_I = \{-\}); p_{B''}), \tag{1.1}$$

where $\sum_I$ represents the sum over all possible sets of intermediate particles I, and the integral $\int$ is the same mass-shell integral that occurs in unitarity. (The precise weight factor in the unitarity integral depends on the conventions one adopts.) The other three forms are defined analogously, with the plus and minus boxes representing the full S matrix and its inverse, respectively, including both the connected and disconnected parts. The main content of this paper is, first, the formulation of the precise rules for evaluating all of the signs σ'_A and σ'_B occurring on the right-hand side of these discontinuity formulas in terms of the signs σ_A and σ_B occurring on the left-hand side, and, second, the proof that the resulting equations follow from field-theoretic principles. These equations constitute a closed set of equations on the physical-sheet boundary values of the analytically continued S matrix, in the sense that the quantities occurring on the right-hand side are again physical-sheet boundary values of the analytically continued S matrix. For each process there are a finite number of these boundary values, as contrasted to an infinite number of sheets. The discontinuity formula exploited by Mueller is the simplest one of these equations.

II. DESCRIPTION OF THE FORMULA

To provide a framework we recall some standard field-theoretic results. In field theory the scattering functions are related to the functions

$$\tau(x_1,\ldots,x_n)=\langle\Omega|\,T(A(x_1)\cdots A(x_n))\,|\Omega\rangle$$

$$=\sum_P\langle\Omega|A(x_{Pn})A(x_{P(n-1)})\cdots A(x_{P1})|\,\Omega\rangle\,\theta(x^0_{Pn}-x^0_{P(n-1)})\cdots\theta(x^0_{P2}-x^0_{P1})\,. \tag{2.1}$$

The variable x_i is a space-time four-vector, and x_i^0 is its time component. (This variable can be considered to be associated with a suppressed "type" variable, which specifies the type of particle associated with the index i.) The state $|\Omega\rangle$ is the vacuum, and T is the time-ordering operator, which is defined by the second part of (2.1). The function $\theta(x^0)$ is plus one for x^0 positive and zero for x^0 negative. The set $(P1,\ldots,Pn)$ is a permutation of the set of integers $(1,\ldots,n)$, and the sum is over all $n!$ such permutations.

The Fourier transform of $\tau(x_1,\ldots,x_n)\equiv\tau(x)$ is defined by

$$\tilde\tau(k_1,\ldots,k_n)\equiv\tilde\tau(k)\equiv\int d^4x_1\cdots d^4x_n\exp[-i(k_1x_1+\cdots+k_nx_n)]\tau(x_1,\ldots,x_n)\,, \tag{2.2}$$

where $k_ix_i=k_i^0x_i^0-\vec{\mathrm{k}}_i\cdot\vec{\mathrm{x}}_i$. By using the identity

$$k_1^0x_1^0+\cdots+k_n^0x_n^0=k_{P1}^0x_{P1}^0+\cdots+k_{Pn}^0x_{Pn}^0$$
$$=k_{P1}^0(x_{P1}^0-x_{P2}^0)+(k_{P1}^0+k_{P2}^0)(x_{P2}^0-x_{P3}^0)+(k_{P1}^0+k_{P2}^0+k_{P3}^0)(x_{P3}^0-x_{P4}^0)+\cdots+(k_{P1}^0+\cdots+k_{Pn}^0)(x_{Pn}^0)\,, \tag{2.3}$$

one obtains, formally, the result[3]

$$\tilde\tau(k)=\sum_P\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dq_1^0}{2\pi}\cdots\frac{dq_n^0}{2\pi}\langle\Omega|\,\tilde A(q_{Pn})\cdots\tilde A(q_{P1})\,|\Omega\rangle$$
$$\times\left(\frac{i}{E_{n-1}(P)-\omega_{n-1}(P)}\right)\left(\frac{i}{E_{n-2}(P)-\omega_{n-2}(P)}\right)\cdots\left(\frac{i}{E_1(P)-\omega_1(P)}\right)2\pi\delta(k_1^0+\cdots+k_n^0)\,, \tag{2.4}$$

where

$$\tilde A(q_i)\equiv\int d^4x_iA(x_i)e^{-iq_ix_i}\,, \tag{2.5}$$

$$E_j(P)\equiv k_{P1}^0+\cdots+k_{Pj}^0\,, \tag{2.6}$$

$$\omega_j(P)\equiv q_{P1}^0+\cdots+q_{Pj}^0\,, \tag{2.7}$$

$$\vec{\mathrm{q}}_i\equiv\vec{\mathrm{k}}_i\,, \tag{2.8}$$

and the pole singularities are resolved by the rule

$$\frac{i}{E_j(P)-\omega_j(P)}=\lim_{\epsilon_j\to0+}\frac{i}{E_j(P)-\omega_j(P)+i\epsilon_j}\,. \tag{2.9}$$

Use of the translation formula

$$A(x)=e^{iP\cdot x}A(0)e^{-iP\cdot x} \tag{2.10}$$

gives, for any two eigenstates $|Q_1\rangle$ and $|Q_2\rangle$ of the energy-momentum operator P, the identity

$$\langle Q_2|\tilde A(q_i)|\,Q_1\rangle=\langle Q_2|A(0)|\,Q_1\rangle(2\pi)^4\delta^4(Q_2-Q_1-q_i)\,. \tag{2.11}$$

The conservation-law δ function occurring here implies that the factor $\langle\Omega|\tilde A(q_{Pn})\cdots\tilde A(q_{P1})|\,\Omega\rangle$ in (2.4) vanishes unless for each $j=1,2,\ldots,n-1$ the vector

$$q_j(P)\equiv q_{P1}+q_{P2}+\cdots+q_{Pj} \tag{2.12}$$

lies in the physical energy-momentum spectrum. Thus the variables $\omega_j(P)$ in (2.4) range only over the values allowed by the spectral conditions, and in particular only over non-negative values, i.e., the integrand is zero whenever any $\omega_j(P)$ is negative.

The function $\tilde\tau(k)/(2\pi)^4\delta^4(\sum k_i)$ defined by (2.4), considered as a function of the n complex variables $(k_1^0,\ldots,k_n^0)\equiv k^0$, for fixed real $(\vec{\mathrm{k}}_1,\ldots,\vec{\mathrm{k}}_n)\equiv\vec{\mathrm{k}}$, has simple analyticity properties. It is formally analytic except where one or more of the quantities $E_j(P)$ is real and non-negative. That is, the function is analytic in the product of the cut $E_j(P)$ planes, where the cuts run along the positive real axes.

We consider only the subspace $\sum k_i=0$, take $\mathrm{Im}\vec{\mathrm{k}}=0$, and consider a fixed value of $\mathrm{Re}k$ such that $\mathrm{Re}E_j(P)\neq0$ for each j and P. These conditions define an $(n-1)$-dimensional subspace $(\mathrm{Im}k_1^0,\ldots,\mathrm{Im}k_n^0)/(\sum\mathrm{Im}k_i^0=0)$. This subspace is separated by the various surfaces $\mathrm{Im}E_j(P)=0$ into a set of disjoint "cells" Γ_λ^0 each of which lies on a well-defined side of each of the surfaces $\mathrm{Im}E_j(P)=0$. The function $\tilde\tau(k)/(2\pi)^4\delta^4(\sum k_i)$ is formally analytic inside each of the cells Γ_λ^0.

There are $n!(n-1)$ functions $E_j(P)$ $(j=1,\ldots,n-1)$, but they are not all different. The different ones correspond to the $N=2^{n-1}-1$ different partitions γ of the set of integers $(1,\ldots,n)$ into two disjoint, nonempty sets J_γ^+ and J_γ^-. The signs on J_γ^+ and J_γ^- are fixed here by the condition that the vector $k(J_\gamma^+)$ defined by

$$k(J) \equiv \sum_{j \in J} k_j \tag{2.13}$$

has a positive real energy part:

$$\mathrm{Re}k^0(J_\gamma^+) \equiv \mathrm{Re}E(J_\gamma^+) > 0\,. \tag{2.14}$$

The location of the cell Γ_λ^0 is specified by the set of signs

$$\sigma_\lambda \equiv (\sigma_{\lambda 1}, \ldots, \sigma_{\lambda\gamma}, \ldots, \sigma_{\lambda N}), \tag{2.15}$$

where $\sigma_{\lambda\gamma} = \sigma_\lambda(J_\gamma^+) = \sigma_\lambda(J_\gamma^-)$ signifies

$$\sigma_\lambda(J_\gamma^\pm) = \mathrm{sgn}\frac{\mathrm{Im}E(J_\gamma^\pm)}{\mathrm{Re}E(J_\gamma^\pm)} \text{ for } \mathrm{Im}k^0 \in \Gamma_\lambda^0\,. \tag{2.16}$$

The cell Γ_λ^0 lies in the upper half $E(J_\gamma^+)$ plane if $\sigma_{\lambda\gamma}$ is positive and lies in the lower half $E(J_\gamma^+)$ plane if $\sigma_{\lambda\gamma}$ is negative.

Although each cell Γ_λ^0 corresponds to a definite set of signs σ_λ, not every set of N signs corresponds to a cell. For the $N = 2^{n-1} - 1$ conditions on the $n-1$ variables are often incompatible.

There is, however, a cell corresponding to the set $\sigma_+ = (+, \ldots, +)$ consisting of all plus signs. This cell, Γ_+^0, is called the physical cell, because the physical scattering function is, formally at least, the limit from within this cell of the function

$$S(k) \equiv (-i)^n[(k_1^2 - m_1^2)\cdots(k_n^2 - m_n^2)]\bar{\tau}(k)\,. \tag{2.17}$$

That is, the scattering matrix element

$$S_{\alpha\beta} \equiv (\Phi_{\mathrm{out}}^\beta \,|\, \Phi_{\mathrm{in}}^\alpha) = \int d^4x_1 \cdots d^4x_n f_{\alpha_1}(x_1)\cdots f_{\alpha_m}(x_m) f^*_{\beta_1}(x_{m+1})\cdots f^*_{\beta_{n-m}}(x_n)(-i)^n K_1 \cdots K_n \tau(x_1, \ldots, x_n) \tag{2.18}$$

is given formally by

$$S_{\alpha\beta} = \lim_{(\eta \in \Gamma_+^0)\to 0} \int_{-\infty}^{\infty} \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4} \tilde{f}_{\alpha_1}(k_1)\cdots\tilde{f}_{\alpha_m}(k_m)\tilde{f}^*_{\beta_1}(k_{m+1})\cdots\tilde{f}^*_{\beta_{n-m}}(k_n) S(k + i\eta)\,, \tag{2.19}$$

where

$$\tilde{f}_{\alpha_i}(k_i) \equiv \int d^4x_i f_{\alpha_i}(x_i) e^{ik_i x_i}\,, \tag{2.19'}$$

$$\tilde{f}^*_{\beta_i}(k_i) \equiv \int d^4x_i f^*_{\beta_i}(x_i) e^{ik_i x_i}\,, \tag{2.19''}$$

and

$$K_i = -\Box_i - m_i^2 = -\left(\frac{\partial}{\partial t_i}\right)^2 + (\vec{\nabla}_i)^2 - m_i^2\,. \tag{2.19'''}$$

Within the Wightman framework of field theory the momentum-space results described above remain purely formal. This is because a product of distributions is not necessarily a distribution, and hence one does not know that the momentum-space functions $\bar{\tau}(k)$ exist. The problem is with the definition of the functions $\tau(x)$ at points where one or more arguments coincide. Since, however, the momentum-space functions are the ones of physical interest, and since the postulate that the Wightman functions $\langle\Omega|A(x_1)\cdots A(x_n)|\Omega\rangle$ are tempered distributions has, in any case, no physical basis, it is reasonable to take the momentum-space functions as the basic quantities, and to postulate for them the properties described above, which follow formally from the spectral conditions and the x-space support properties.

The same results can be obtained, alternatively, from the approach of Bros, Epstein, and Glaser,[4] who postulate the existence of tempered distributions corresponding to various generalized retarded functions, and assume that these distributions enjoy: (i) the algebraic properties that follow formally from the properties of the θ functions, and (ii) the support properties that follow formally from the commutation relations and θ function factors. It follows from these assumptions that for each cell Γ_λ^0 there is a function $S_\lambda(k)$ that is analytic in Γ_λ^0 (and in fact in a larger cell Γ_λ that will be described later) and that the limit of each function $S_\lambda(k)$ exists in the distribution sense on the various boundaries of Γ_λ^0. If one imposes, moreover, the spectral conditions in a natural manner, then the various functions $S_\lambda(k)$ are found to be analytic continuations of the single analytic function $S(k)$ discussed earlier. However, for the moment we shall consider the cell functions $S_\lambda(k)$ to be independent, each defined in its associated closed cell $\bar{\Gamma}_\lambda^0$. The analytic connection between them will follow from the discontinuity equation that is to be derived.

Within this framework, in which each cell Γ_λ^0 (or Γ_λ) is associated with a corresponding "cell function" $S_\lambda(k)$ analytic in Γ_λ^0 (or Γ_λ), we now describe the meaning of the discontinuity formula represented in Fig. 1.

A bubble with a set of signs σ_λ corresponding to a cell Γ_λ^0 represents the cell function $S_\lambda(k)$. This bubble has one line attached to its left-hand side for each variable k_i such that $\mathrm{Re}k_i^0 > 0$. And it has one line attached to its right-hand side for each variable with $\mathrm{Re}k_i^0 < 0$. The lines attached on the left- and right-hand sides correspond, in the physical limit, to the incoming and outgoing particles

of the scattering process, respectively. Shaded strips represent sets consisting of any number of lines.

Consider a partition γ of the set of lines associated with some process into two nonempty sets. This partition is indicated by the line γ in Fig. 1. The sets of lines corresponding to the index sets J_γ^+ and J_γ^- defined in (2.14) are labeled by $A = A' + A''$ and $B = B' + B''$, respectively. The arrow on the line γ indicates that energy flows from the set A to the set B, in accordance with (2.14). The sets A and B are broken into the incoming parts A' and B', which contain the lines with $\mathrm{Re}k_i^0 > 0$, and the outgoing parts, A'' and B'', which contain the lines with $\mathrm{Re}k_i^0 < 0$.

Let Γ_λ^{0+} and Γ_λ^{0-} be any two cells such that

$$\sigma_{\lambda^+\gamma} = +1,$$

$$\sigma_{\lambda^-\gamma} = -1,$$

and

$$\sigma_{\lambda^+\gamma'} = \sigma_{\lambda^-\gamma'} \equiv \sigma_{\lambda\gamma'} \quad \text{for all } \gamma' \neq \gamma. \tag{2.20}$$

Such cells are nearest neighbors in the sense that they have a common boundary face, $\mathrm{Im}E(J_\gamma^+) = 0$. The left-hand side of Fig. 1 represents $S_{\lambda^+} - S_{\lambda^-}$, evaluated on this common boundary face.

The set of common signs $\sigma_{\lambda\gamma'}$ in (2.20) can be separated into three subsets, called σ_A, σ_B, and σ_C. The set σ_A consists of the signs $\sigma_{\lambda\gamma'}$ for partitions γ' of $(A+B)$ into two nonempty subsets, one of which is a proper subset of A. The set σ_B consists of the signs $\sigma_{\lambda\gamma'}$ for partitions γ' of $(A+B)$ into two nonempty subsets, one of which is a proper subset of B. The set σ_C consists of the rest of the $\sigma_{\lambda\gamma'}$. The partitions γ' corresponding to σ_C are all "crossed" relative to partition γ:

$$\begin{aligned} &J_\gamma^+ \cap J_{\gamma'}^+ \neq \phi, \\ &J_\gamma^+ \cap J_{\gamma'}^- \neq \phi, \\ &J_\gamma^- \cap J_{\gamma'}^+ \neq \phi, \\ &J_\gamma^- \cap J_{\gamma'}^- \neq \phi. \end{aligned} \tag{2.21}$$

The sets of signs σ_A and σ_B are indicated in the bubble on the left-hand side of Fig. 1. The discontinuity formula will depend on these signs but, in accordance with the Steinmann relation,[3] will be independent of σ_C.

The right-hand side of Fig. 1 gives four equivalent formulas for the discontinuity $S_{\lambda^+} - S_{\lambda^-}$. The bubbles on the right-hand side represent the cell functions $S_{\lambda'}^{A+I}$ and $S_{\lambda'}^{B+I}$ for the processes $(A' \to A'' + I)$ and $(B' + I \to B'')$. The set I denotes an arbitrary number of lines, and there is a unitarity-type sum over all possible sets I.

The integration over all mass-shell values and the normalization is exactly as in the unitarity equation. The plus and minus boxes in the last two lines represent S and its inverse S^{-1}, respectively.

The main content of these formulas resides in three rules given below. These rules specify precisely which cells $\Gamma_{\lambda'}^{0(A+I)}$ and $\Gamma_{\lambda'}^{0(B+I)}$ are to be used on the right-hand side. These cells are specified by giving the signs $\sigma_{\lambda'\gamma'}$ as functions of the signs $\sigma_{\lambda\gamma'}$ in σ_A and σ_B.

Instead of $\sigma_{\lambda'\gamma'}$ and $\sigma_{\lambda\gamma'}$ we shall write $\sigma_{\lambda'}(X)$ and $\sigma_\lambda(X)$, where X is either one of the two sets of lines defined by the partition γ'.

Rule 1. If X is a proper subset of A, or a proper subset of B, then $\sigma_{\lambda'}(X) = \sigma_\lambda(X)$. (A proper subset is not the whole set.)

Rule 2. If X is a subset of I then $\sigma_{\lambda'}(X) = \pm$, where $\pm$ is the sign appearing explicitly in the cut-out sector adjacent to I. For example, $\sigma_{\lambda'}^{A+I}(X)$ is

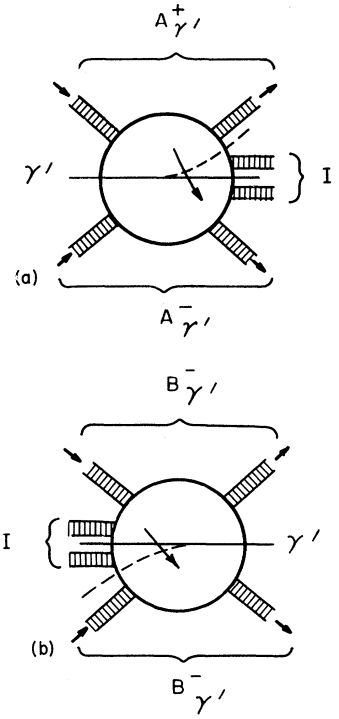

FIG. 2. (a) If the set I consists of outgoing lines, which carry energy out of the bubble, the partition γ' is shifted backwards, against the energy flowing across it, to the position indicated by the dotted line. The $\sigma_{\lambda'\gamma'}$ is the same as the sign corresponding to the shifted position, which is fixed by rule 1. (b) If the set I consists of incoming lines, which carry energy into the bubble, then the partition γ' is shifted forward, in the energy flow across it, to the position indicated by the dotted line. The sign $\sigma_{\lambda'\gamma'}$ is the same as the sign corresponding to the shifted position, which is fixed by rule 1.

plus in the first and fourth lines of the right-hand side of Fig. 1, and is minus in the second and third lines.

Rule 3. If $A^+_{\gamma'}$ is a proper subset of A, and $I^+_{\gamma'}$ is a proper subset of I, and the sum of the energies $\mathrm{Re}k^0_i$ of the lines of $X = A^+_{\gamma'} + I^+_{\gamma'}$ is positive, then $\sigma_{\lambda'}(X) = \sigma_\lambda(A^+_{\gamma'})$. If $B^-_{\gamma'}$ is a proper subset of B, and $I^-_{\gamma'}$ is a proper subset of I, and the sum of the energies $\mathrm{Re}k^0_i$ of the lines of $X = B^-_{\gamma'} + I^-_{\gamma'}$ is negative, then $\sigma_{\lambda'}(X) = \sigma_\lambda(B^-_{\gamma'})$.

Rule 3 is represented diagrammatically in Fig. 2 and is called the "back-up rule": It says that the sign of the imaginary part of the energy corresponding to any mixed set of internal and external lines is determined by the sign of the imaginary part of a corresponding "back-up energy" associated with a set of external lines. The first example of this rule was encountered by Hwa[5] in his study of the discontinuity function for the five-line function. Hwa traced out the motion of the triangle singularity and showed how it led to the determination of the sign corresponding to the mixed energy by the sign corresponding to the back-up energy.

Hwa's method is quite complicated even for this simplest case, and it depends on a knowledge of properties of singularity surfaces other than those associated with normal thresholds. It would be difficult to extend it to a general proof since the properties of these other singularity surfaces are quite complicated in general, and are not yet known. Also it depends on a knowledge of the singularity structure in some initial situation, from which the continuation starts, and upon a justification of both the continuation procedure and the procedure of considering the singularities individually, without regard to their possible interference with one another.

The proof to be given here avoids these difficulties. It refers explicitly only to the normal-threshold singularities themselves, and is not based on any continuation procedure. However, even though only normal-threshold singularities enter explicitly, there is no neglect of any other singularities. The proof is such that all possible singularities are taken into account, even though only the normal-threshold singularities are explicitly mentioned.

The cells Γ^0_λ defined above are restricted to the space $\mathrm{Im}\vec{k} = 0$, and the analyticity discussed was analyticity only in the energy components. These concepts are not Lorentz-invariant. Consideration of the Lorentz-invariant support properties in x space that follow from the local commutation relations leads to an extension of the analyticity in energy components in Γ^0_λ to analyticity in all variables k in the cell[3,4]

$$\Gamma_\lambda \equiv \bigcap_{\gamma=1}^{N} \{\mathrm{Im}k : \sigma_{\lambda\gamma}\,\mathrm{Im}k(J^+_\gamma) \in V^+\} \cap \{\textstyle\sum k_i = 0\}, \tag{2.22}$$

where $k(J^+_\gamma)$ is defined by (2.13) and V^+ is the forward light cone

$$V^+ \equiv \{v : v^2 > 0\,,\ v^0 > 0\}. \tag{2.23}$$

The cell Γ_λ is defined above as a set in the space of real $4n$ vectors $\mathrm{Im}k$, restricted by $\sum k_i = 0$. This cell was originally associated with some *fixed* value of $\mathrm{Re}k$, and the sets J^+_γ in (2.14) are defined with reference to that fixed value of $\mathrm{Re}k$. The discontinuity is to be evaluated at that fixed value of $\mathrm{Re}k$, or in some region around that value over which the sets J^+_γ do not change.

The set $\{k : \mathrm{Im}k \in \Gamma_\lambda, \sum k_i = 0\}$ is called the *tube* Γ_λ. This convex tube domain is the product of the convex cone Γ_λ in $\mathrm{Im}k$ with the entire space

$$\mathrm{Re}k \in R_{4n} / \textstyle\sum \mathrm{Re}k_i = 0\,.$$

The analytically continued $S_C(k)/(2\pi)^4(\sum k_i)$ is, according to field-theoretic principles,[3,4] analytic in each tube Γ_λ.

III. DISCUSSION OF THE FORMULA

The statement of the discontinuity formula in Sec. II takes no cognizance of any analyticity of any cell function $S_\lambda(k)$ outside the corresponding cell Γ_λ. In particular, $S_\lambda(k)$ is defined on the boundary $\partial\Gamma^0_\lambda$ of Γ^0_λ only in a distribution sense. However, the function $S_\lambda(k)$ is expected to be analytic almost everywhere on $\partial\Gamma^0_\lambda$. Consideration of perturbation theory, for example, leads to the expectation that the function $S_\lambda(k)$ will be analytic everywhere except on certain surfaces defined by the zeros of analytic functions.

We shall consider, in this section, points of the boundaries of Γ^0_λ, and will discuss the possibility that the $S_\lambda(k)$ are analytic at these boundary points except on certain surfaces defined by the zeros of analytic functions. This will lead to an understanding of the connection between our discontinuity formulas and singularity surfaces.

Consider then the surface $\phi(k) = 0$. A *regular* point of the surface $\phi(k) = 0$ is a point $\bar{k}$ such that $\phi(k)$ is analytic at $\bar{k}$, $\phi(\bar{k}) = 0$, and $\nabla\phi(\bar{k}) \neq 0$. [Here $\nabla\phi(\bar{k})$ is the $4(n-1)$-dimensional gradient vector: because of the constraint $\sum k_i = 0$ we shall, for any function $g(k)$, define the gradient ∇g by the equation

$$(\nabla g)_{i\mu} \equiv \partial(g - b_\nu \textstyle\sum_j k^\nu_j)/\partial k^\mu_i\,,$$

where b is fixed by the condition that $\sum_i (\nabla g)_{i\mu} = 0$ for all μ. This effectively reduces the gradient to a vector in the appropriate $4(n-1)$-dimensional space.]

A function $f(k)$ is said to be ϕ-*singular* at $\bar{k}$ if and only if: (i) The function $f(k)$ is not analytic at $\bar{k}$; (ii) $\bar{k}$ is a regular point of the surface $\phi(k)=0$; and (iii) the function $f(k)$ is, for some fixed real θ and real $r>0$, analytic in the intersection of some neighborhood of $\bar{k}$ with the set

$$\sigma(r,\theta)\equiv\{k:\phi(k)=r'\,e^{i\theta},\,0<r'<r\}. \tag{3.1}$$

Condition (iii) says that $f(k)$ is analytic at all points k near $\bar{k}$ that correspond to points on some open line segment in the ϕ plane that ends at $\phi=0$. This condition gives meaning to the idea that the singularity surface is $\phi(k)=0$, rather than some other surface that passes through the point $\bar{k}$. Note that (iii) is much weaker than the requirement that $f(k)$ be analytic in some full neighborhood of $\bar{k}$ except on the surface $\phi(k)=0$ and a trailing cut.

Analytic functions of several complex variables can have singularities on boundaries of domains of analyticity only under special conditions. These entail, in fact, that if a function $f(k)$ is ϕ-singular at $\bar{k}$, then it is singular at all points on the surface $\phi(k)=0$ in some neighborhood of $\bar{k}$. Thus if the function $S_\lambda(k)$ is ϕ-singular at $\bar{k}$ on $\partial\Gamma^0_\lambda$, then the surface $\phi(k)=0$ cannot enter the tube Γ_λ at $\bar{k}$.

This result is formalized and slightly extended in the following theorem, which is proved in Appendix A:

Theorem 3.1. The function $S_\lambda(k)$ is ϕ-singular at $\bar{k}$ on $\partial\Gamma^0_\lambda$ only if the set $\{k:(k-\bar{k})\cdot\nabla\phi(\bar{k})=0\}$ does not intersect the tube Γ_λ.

Convention. We shall always adjust the phase of $\phi(k)$ so that at least one nonzero component of $\nabla\phi(\bar{k})$ is real.

If $\nabla\phi(\bar{k})$ is not purely real then one can always find a solution of $(k-\bar{k})\cdot\nabla\phi(\bar{k})=0$ for any given value of $\mathrm{Im}(k-\bar{k})$. Thus Theorem 3.1 has the following corollary.

Corollary. The function $S_\lambda(k)$ is ϕ-singular at $\bar{k}$ on $\partial\Gamma^0_\lambda$ only if $\nabla\phi(\bar{k})$ is real.

Theorem 3.1 can be stated, alternatively, as a condition on the direction of the gradient vector $\nabla\phi(\bar{k})$. This statement involves the set $\{\gamma:\mathrm{Im}\bar{k}(J^+_\gamma)=0\}$, which is the set consisting of those γ's such that $\mathrm{Im}\bar{k}$ lies on the tip of the cone $\{\mathrm{Im}k:\sigma_{\lambda\gamma}\,\mathrm{Im}k(J^+_\gamma)\in V^+\}$. Let $\Gamma_\lambda(\bar{k})$ represent the intersection of these cones:

$$\Gamma_\lambda(\bar{k})\equiv\bigcap_{\{\gamma:\mathrm{Im}\bar{k}(J^+_\gamma)=0\}}\{\mathrm{Im}k:\sigma_{\lambda\gamma}\,\mathrm{Im}k(J^+_\gamma)\in V^+\}\cap\{\textstyle\sum k_i=0\}. \tag{3.2a}$$

This set coincides in some neighborhood of $\bar{k}$ with the set Γ_λ defined by (2.22), because the remaining conditions in (2.22) are satisfied at $\bar{k}$, and hence also in some neighborhood of $\bar{k}$.

The closure of $\Gamma_\lambda(\bar{k})$ is

$$\overline{\Gamma}_\lambda(\bar{k})=\bigcap_{\{\gamma:\mathrm{Im}\bar{k}(J^+_\gamma)=0\}}\{\mathrm{Im}k:\sigma_{\lambda\gamma}\,\mathrm{Im}k(J^+_\gamma)\in\overline{V}^+\}\cap\{\textstyle\sum k_i=0\}, \tag{3.2b}$$

where $\overline{V}^+$ is the closure of the forward light cone V^+. Define the corresponding positive polar

$$\overline{\Gamma}^+_\lambda(\bar{k})\equiv\{u:\mathrm{Im}(k-\bar{k})\cdot u\geq 0\ \text{ for all }\ \mathrm{Im}k\ \text{ in }\ \overline{\Gamma}_\lambda(\bar{k})\}\cap\{\textstyle\sum u_i=0\}. \tag{3.3}$$

Then we can state

Theorem 3.2. If the function $S_\lambda(k)$ is ϕ-singular at $\bar{k}$ on $\partial\Gamma^0_\lambda$, then the sign of the function $\phi(k)$ can be fixed so that

$$\nabla\phi(\bar{k})\in\overline{\Gamma}^+_\lambda(\bar{k}), \tag{3.4a}$$

or equivalently,

$$\nabla\phi(\bar{k})\in\Big\{u:u=\sum_{\{\gamma:\mathrm{Im}\bar{k}(J^+_\gamma)=0\}}\sigma_{\lambda\gamma}v_\gamma\cdot\nabla k(J^+_\gamma);\ v_\gamma\in\overline{V}^+\Big\}. \tag{3.4b}$$

Here $\nabla k(J^+_\gamma)$ is a $4\times4(n-1)$-dimensional vector, and each v_γ is a four-vector lying in the closure $\overline{V}^+$ of the forward light cone V^+.

The condition (3.4a), together with the definition (3.3), asserts that the closed set $\overline{\Gamma}_\lambda(\bar{k})$ lies in the closed half-space $\{\mathrm{Im}k:\mathrm{Im}(k-\bar{k})\cdot\nabla\phi(\bar{k})\geq 0\}$. This follows immediately from Theorem 3.1, which asserts that the connected open set Γ_λ does not intersect the boundary of this half-space.

The equivalence of (3.4a) and (3.4b) is shown in Appendix A.

A simple but important application of Theorem

3.2 is to a point $\bar{k}$ that lies on a "face" of Γ^0_λ. A *face* of Γ^0_λ is a portion of the boundary $\partial\Gamma^0_\lambda$ that lies on only one of the surfaces $\mathrm{Im}k(J^+_\gamma)=0$. The discontinuity formula of Sec. II gives the difference of the functions $S_\lambda{}^+(k)$ and $S_\lambda{}^-(k)$ on coincident faces of $\Gamma^0_{\lambda+}$ and $\Gamma^0_{\lambda-}$.

If $\bar{k}$ lies on a face then the set $\{\gamma:\mathrm{Im}\bar{k}(J^+_\gamma)=0\}$ has just one element, say $\gamma=1$. It is then convenient to introduce a new set of variables $k'_j=\sum C_{ji}k_i$, where C_{ji} is a constant multiple of an orthogonal transformation and $k'_1=k(J^+_1)$, and to define $\phi'(k')$ by $\phi'(k')=\phi(k)$. Then (3.4b) shows that the k'-space gradient $\nabla'\phi'(\bar{k}')$ has the form (modulo $b\cdot\nabla\sum k_i$)

$$\nabla'\phi'(\bar{k}')=(v,0,\ldots,0), \tag{3.5a}$$

where v is a positive multiple of the four-vector $\sigma_{\lambda\gamma}v_\gamma$ for $\gamma=1$, i.e., $\sigma_{\lambda\gamma}v$ is nonzero and lies in $\overline{V}^+$. Equation (3.5a) is an abbreviation of

$$\partial\phi'(\bar{k}')/\partial k'_i{}^\mu=v_\mu\delta_{i1}. \tag{3.5b}$$

Equation (3.5) implies that $\partial\phi'(\bar{k}')/\partial k'_1{}^0\neq 0$. Thus the surface $\phi'(k')=0$, restricted to $\sum k_i=0$, can be written in some neighborhood of $\bar{k}'$ as[6]

$$\begin{aligned}k'_1{}^0&=f(k'_1{}^1,k'_1{}^2,k'_1{}^3,k'_2,\ldots,k'_{n-1})\\&=f(k'_1{}^i,k'_j)\quad(i\neq 0,\ j\neq 1).\end{aligned} \tag{3.6}$$

If $f(k'_1{}^i,\bar{k}'_j)$ is a real analytic function of the variables $k'_1{}^i$ then the intersection of the surface $\phi'(k'_1{}^i,\bar{k}'_j)=0$ with the real section $\{\mathrm{Im}k'_1{}^\mu=0\}$ of $k'_1{}^\mu$ space will be a three-dimensional surface in a four-dimensional space, and it will (near $\bar{k}'_1$) divide this real section into disjoint parts.

But if $f(k'_1{}^i,\bar{k}'_i)$ is not a real analytic function of the variables $k'_1{}^i$ then the intersection of the surface $\phi'(k'_1,\bar{k}'_j)=0$ with the real section of k'_1 space will have dimension less than three and will not divide the real section into disjoint parts. In this case singularity surface can be avoided, in the sense that one can locally continue around the singularity surface without ever leaving the real section. Thus the discontinuity function cannot change its analytic form as a consequence of such a singularity. Such singularities, if they exist at all, will thus not be important in the discussions that follow.

We shall consider, then, the case in which $f(k'_1{}^i,\bar{k}'_j)$ is a real analytic function of the variables $k'_1{}^i$, near $\bar{k}'_1{}^i$. In this case the surface $\phi'(k')=0$ near $\bar{k}'$ must be independent of the variables k'_j. This is the content of the following theorem.

Theorem 3.3. Suppose the following three conditions hold: (i) The function $S_\lambda(k)$ is ϕ-singular at k on $\partial\Gamma^0_\lambda$; (ii) $\mathrm{Im}\bar{k}(J^+_\gamma)=0$ for $\gamma=1$, but for no other γ; (iii) the function $f(k'_1{}^i,k'_j)$ defined by (3.6) when restricted to the set $k'_j=\bar{k}'_j$ for all $j\neq 1$ is a real analytic function of the remaining variables $k'_1{}^i$. Then the surface $\phi(k)=0$ coincides in a neighborhood of $\bar{k}$ with a surface $\hat{\phi}(k(J^+_1))=0$, where $\hat{\phi}(k'_1)$ is a real analytic function of the four-vector k'_1, $\nabla\hat{\phi}(\bar{k}'_1)\neq 0$, and $\nabla\hat{\phi}(\bar{k}'_1)\in\overline{V}^+$.

This theorem is proved in Appendix A.

In the following discussion the possible singularities that touch a face of Γ^0_λ but do not locally divide the real sector $\mathrm{Im}k(J^+_\gamma)=0$ into disjoint regions are ignored. They cannot separate the real sector into regions in which the discontinuity functions have different analytic forms, and hence are – if they exist at all – not important in the context of the discussion.

It is a consequence of the Lorentz-invariance requirement on the scattering functions that $\phi(k)$ must be a function of the (scalar and pseudoscalar) invariants of the process. But the only invariants that can be formed from a single four-vector $k(J^+_\gamma)$ are functions of the invariant $[k(J^+_\gamma)]^2=M_\gamma{}^2$. Thus the only singularity surfaces that can lie on a face lying in $\mathrm{Im}k(J^+_\gamma)=0$ are those that have the form $\phi(M_\gamma{}^2)=0$. These surfaces are surfaces that lie at fixed values of $M_\gamma{}^2$.

If one inserts this form into (3.4b) then the vector v_γ occurring there can be identified as a multiple of $k(J^+_\gamma)$. This means that the singularities must lie at $M_\gamma{}^2\geq 0$.

The discontinuity across a face lying on $\mathrm{Im}k(J^+_\gamma)=0$ vanishes if $M_\gamma{}^2$ is less than the square of the mass of the lowest-mass intermediate state in the channel specified by γ. And the only singularities that occur on these faces are those lying at constant values of $M_\gamma{}^2$. Thus the discontinuity is, in effect, a discontinuity around only these normal threshold-type singularities.

However, the two neighboring cells $\Gamma^0_{\lambda+}$ and $\Gamma^0_{\lambda-}$ have, in addition to the common boundary points lying on their common face, common boundary points that do not lie on this common face. They have, for example, a common boundary point at $\mathrm{Im}k=0$, which lies on all of the surfaces $\mathrm{Im}k(J^+_\gamma)=0$. At this real boundary point, which lies at the tip of both $\Gamma^0_{\lambda+}$ and $\Gamma^0_{\lambda-}$, the various cell functions can have non-normal-threshold singularities. For example, the function $S_\lambda(k)$ corresponding to the physical cell can have, and is expected to have, at $\mathrm{Im}k=0$ the singular positive-α Landau surfaces corresponding to triangle diagrams and box diagrams, etc. And the functions $S_\lambda(k)$ corresponding to nonphysical cells can have various nonpositive-α singularities at $\mathrm{Im}k=0$.

Theorem 3.1 ensures that if $S_\lambda{}^+(k)$ is ϕ-singular at $\mathrm{Im}k=0$, and $\phi(k)$ depends on variables other than $M_\gamma{}^2$, then the singularity at $\phi=0$ must move outside the closure of $\Gamma^0_{\lambda+}$ as $\mathrm{Im}k$ moves to a

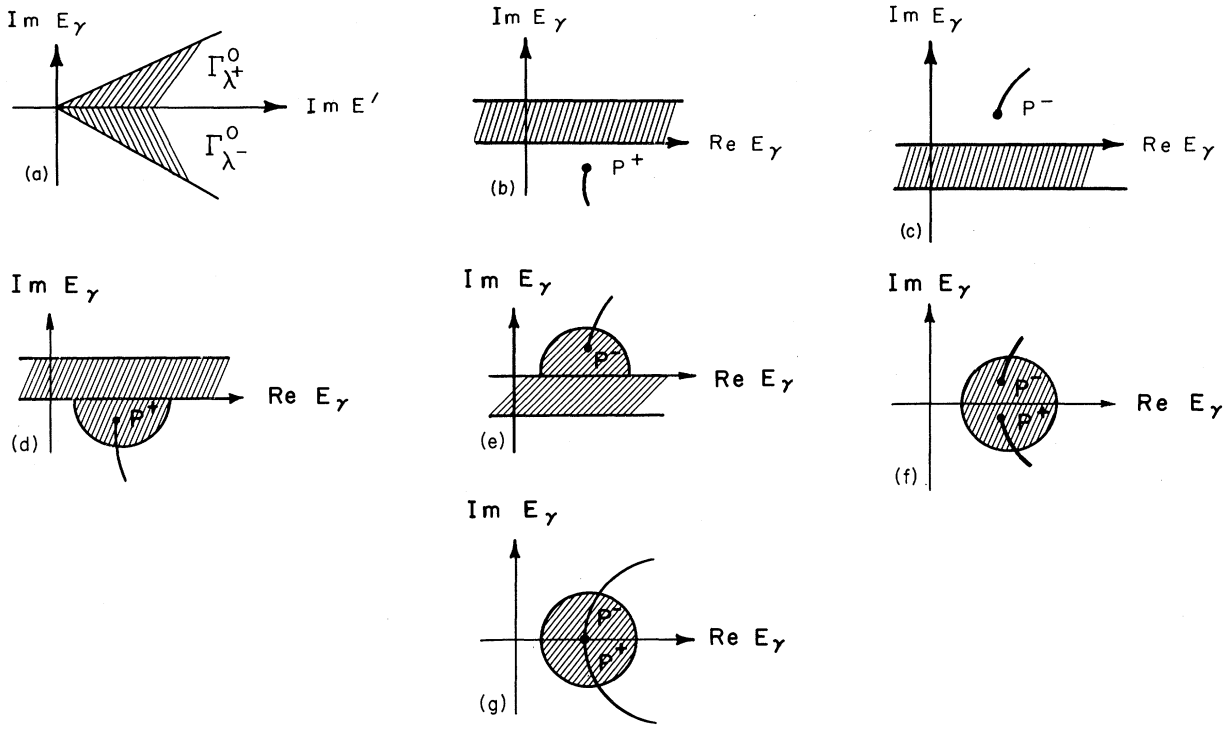

FIG. 3. (a) The cells $\Gamma^0_{\lambda+}$ and $\Gamma^0_{\lambda-}$ are neighboring cells with the common face $\mathrm{Im}E_\lambda = 0$. The variable E' represents the rest of the energies. ReE' will be held fixed. (b) The shaded strip is $\Gamma^0_{\lambda+}$ for some fixed $\mathrm{Im}E' > 0$. The point P^+ is a singularity that has moved outside $\overline{\Gamma}^0_{\lambda+}$. (c) The shaded strip is $\Gamma^0_{\lambda-}$ for some fixed $\mathrm{Im}E' > 0$. The point P^- is a singularity that has moved outside $\overline{\Gamma}^0_{\lambda-}$. (d) The domain $\Gamma^0_{\lambda+}$ is extended to reveal the singularity P^+ and its trailing cut. (e) The domain $\Gamma^0_{\lambda-}$ is extended to reveal the singularity P^- and its trailing cut. (f) A domain of analyticity of the discontinuity function $S_{\lambda+}(k) - S_{\lambda-}(k)$ for some $\mathrm{Im}E' > 0$. This domain is bounded in part by P^+, P^-, and their trailing cuts. (g) The domain of analyticity of the discontinuity function may be separated by the cuts trailing P^+ and P^- if these points coincide at $\mathrm{Im}E' = 0$.

point on the face of $\Gamma^0_{\lambda+}$ that lies in $\mathrm{Im}k(J^+_\gamma) = 0$. The analogous thing happens to the singularities of $S_{\lambda-}(k)$. This effect is shown in Fig. 3. [For brevity we write $k^0(J^+_\gamma) = E_\gamma$.]

Figure 3(a) shows the two neighboring cells $\Gamma^0_{\lambda+}$ and $\Gamma^0_{\lambda-}$, and shows the "other" imaginary variables ImE'. Figure 3(b) shows the domain of analyticity $\Gamma^0_{\lambda+}$ for some fixed $\mathrm{Im}E' > 0$. The point P^+ is a singularity of the analytic continuation of $S_{\lambda+}(k)$. At $\mathrm{Im}E' = 0$ it reaches $\mathrm{Im}E_\gamma = 0$. For $\mathrm{Im}E' > 0$ it lies outside $\overline{\Gamma}^0_{\lambda+}$. Figure 3(c) is the analogous figure for $\Gamma^0_{\lambda-}$. Figures 3(d) and 3(e) show extensions of the domains $\Gamma^0_{\lambda+}$ and $\Gamma^0_{\lambda-}$. These extensions are bounded, in part, by P^+ and P^- and their trailing cuts. Figure 3(f) shows part of the domain of analyticity of the discontinuity function. Figure 3(g) shows how the domain of analyticity of the discontinuity function may be separated into two parts at $\mathrm{Im}E' = 0$, if the singularities P^+ and P^- are conjugate pairs that coincide at $\mathrm{Im}E' = 0$.

Polkinghorne[7] has pointed out that in the case of the discontinuity function that represents at $\mathrm{Im}k = 0$ (hence at $\mathrm{Im}E_\gamma = \mathrm{Im}E' = 0$) the inclusive cross section, the singularities corresponding to triangle diagrams and other singularities do occur in conjugate pairs, because of the special symmetry of the discontinuity function. The pinching of these singularities at $\mathrm{Im}E' = 0$ means that the *inclusive cross section* will not continue analytically around the various singularities that occur at $\mathrm{Im}k = 0$: The inclusive cross section at real points lying on one side of one of these singularity surfaces is not the analytic continuation of the function that represents the inclusive cross section at real points lying on the other side of the singularity surface. This nonanalyticity of the inclusive cross section is demanded by the fact that it is real, and is therefore not unexpected.

Note, however, that the discontinuity function that becomes the inclusive cross section at $\mathrm{Im}k = 0$ does not have this nonanalyticity at $\mathrm{Im}E' > 0$, except at the normal threshold-type singularities that lie at constant $M_\gamma{}^2$. Figure 3(f) shows how the singularities that depend on other variables retreat from the surface $\mathrm{Im}E^+_\gamma = 0$, and hence do not block the analytic continuation of the discontinuity

function.

We turn now to another question. The primitive domains of analyticity of the cell functions $S_\lambda(k)$ are the corresponding cells $\Gamma_\lambda(k)$. These cells are defined by conditions on the components of k, not by conditions on the scalar invariants. However, the Lorentz invariance of the scattering functions implies that the singularities must lie in surfaces defined by the vanishing of functions of scalar invariants. This poses the problem of expressing the domains of analyticity in terms of scalar invariants.

When restricted to $\mathrm{Im}\vec{k}=0$ the domains Γ_λ go to Γ^0_λ, which are bounded by surfaces of the form $\mathrm{Im}E_\gamma=0$. These surfaces coincide in the space $\mathrm{Im}\vec{k}=0$ with the sets $\mathrm{Im}S_\gamma=0$, where $S_\gamma\equiv M_\gamma{}^2=[k(J^+_\gamma)]^2$. This fact, together with prevailing ideas about the domain of analyticity of the $2\to 2$ scattering function, raises the hope that the scattering function $S_\lambda(k)$ may be analytic, at least near each real point k, in domains bounded by surfaces of the form $\mathrm{Im}S_\gamma=0$. Then the cut-plane analyticity in E space for $\mathrm{Im}\vec{k}=0$, would go over into local cut-plane analyticity in the space of invariants.

This hope is dashed by the counterexample provided by the simplest non-normal-threshold Landau singularity, namely the triangle-diagram singularity. The triangle-diagram singularity depends only on the invariants S_1, S_2, and S_3 associated with the three vertices of the diagram. The location of the singularity can be described by an equation $S_3=S_3(S_1, S_2)$, where S_3 is the square of the invariant energy associated with intermediate vertex, and S_1 and S_2 are the squares of the invariant energies associated with the initial and final vertices.

We consider a $3\to 3$ amplitude for equal-mass particles. Then S_3, and all other $1\to 1$ cross energies, will be negative and there will be no normal-threshold cuts associated with them. This remains true at nearby off-mass-shell points.

We take an energy less than the four-particle threshold. Then the only cross-energy normal-threshold singularities are pole singularities. We take a point k away from these poles. Then there will be, locally, no cross-energy normal-threshold cuts.

If only normal-threshold cuts $\mathrm{Im}S_i=0$ are present, then the physical cell can be extended to a cell bounded by the conditions $\mathrm{Im}S_i>0$, where the S_i run over S_1 and S_2; over the squares of all four other two-particle subenergies; over all six S_i associated with the six individual lines; and over the square of the total energy.

The invariant space cell defined by the above conditions contains singularities at points arbitrarily close to the positive-α triangle-diagram singularity at $\mathrm{Im}k=0$. To prove this, take any values of S_1 and S_2 in this cell and consider the point where $S_3=S_3(S_1, S_2)$. This point may not correspond to a point in the cell, because of the constraints other than $\mathrm{Im}S_1>0$ and $\mathrm{Im}S_2>0$. However, if we add a sufficiently large positive imaginary energy to the positive physical energy of the incoming line that is attached to the vertex associated with S_3, and add the same positive imaginary energy to the positive physical energy of the outgoing line that is attached to that vertex, then, without changing S_1, S_2, or S_3, we can shift the imaginary parts of all the other subenergies, and also the total energy, to positive values. And the $\mathrm{Im}S_i$ corresponding to the individual lines k_i can also be made positive. The triangle singularity will therefore be exposed: There are no normal-threshold cuts behind which it can go as it moves away from the singular point on the boundary of the physical cell.

The counterexample shows that one cannot in general define the domains of analyticity in invariant space, even near the physical points, by considering only normal-threshold-type cuts. It is k space that has this advantage.

The applications of the discontinuity formulas that we have in mind refer to the on-mass-shell scattering functions. The first complication that arises in trying to extend our formulas onto the mass shell is that the conditions on the cells include conditions on the individual lines, as well as on their sums: Each individual k_i is required to satisfy either $\mathrm{Im}k_i\in V^+$ or $\mathrm{Im}k_i\in V^-$. These conditions are incompatible with the mass-shell constraints $k_i{}^2=m_i{}^2>0$.

For each i the real mass-shell points $k_i{}^2=m_i^2$, $\mathrm{Im}k_i=0$, lie, however, on the common boundary of $\mathrm{Im}k_i\in V^+$ and $\mathrm{Im}k_i\in V^-$. Our discontinuity formulas give the difference between the limits of the functions in these two cones, evaluated on their common boundary, provided the difference is evaluated at a point k lying on the common face lying in $\mathrm{Im}k_i=0$ of two neighboring cells $\Gamma^0_\lambda{}_+$ and $\Gamma^0_\lambda{}_-$.

At the mass-shell point there is a contribution proportional to $\delta(k_i{}^2-m_i{}^2)$. However, there is also a factor $k_i{}^2-m_i{}^2$ coming from the definition of $S(k)$. The product $(k_i{}^2-m_i{}^2)\delta(k_i{}^2-m_i{}^2)$ is zero in the distribution sense. Thus the two boundary values are equal. But then the functions $S_{\lambda+}(k)$ and $S_{\lambda-}(k)$ are, by virtue of the generalized edge-of-the-wedge theorem,[8] equal and analytic in some neighborhood of the mass-shell point k, provided this point lies on a common face of $\Gamma^0_\lambda{}_+$ and $\Gamma^0_\lambda{}_-$ that lies on the surface $\mathrm{Im}k_i=0$.

The same procedure can be applied in turn to each k_i. Then using the method of analytic completion used in the proof of Theorem 3.1 (with disks in the energy variable) one can show that the

scattering function will be analytic in a neighborhood of each real mass-shell point that lies inside the "enlarged" cell. The enlarged cells are defined in the same way as the cells Γ_λ, but without the conditions on the individual vectors k_i.

We give no proof of this result. For even if it is granted a serious additional problem remains: There is a large physical region $\mathcal{R}$ such that the restriction of the enlarged physical cell to the mass shell does not come close to $\mathcal{R}$. This is true for all processes with three or more initial particles or three or more final particles.

To see this, note that if k_i is in the nonrelativistic domain $|\mathrm{Re}\vec{k}_i| \ll |\mathrm{Re}k_i^0|$, $|\mathrm{Im}k_i| \ll m_i$, then the surface $k_i^2 = m_i^2$ (which lies near $\mathrm{Im}k_i \cdot \mathrm{Re}k_i = 0$) must have $\mathrm{Im}k_i$ that are very spacelike: $|\mathrm{Im}k_i^0| \ll |\mathrm{Im}\vec{k}_i|$. If two such vectors add to give a vector in V^+ (or in V^-) then their vector parts must be nearly equal but opposite:

$$(\mathrm{Im}(k_i + k_j) \in V^+) \Rightarrow (\mathrm{Im}\vec{k}_i \approx -\mathrm{Im}\vec{k}_j). \tag{3.7}$$

But consider then a process with three or more initial particles, of momenta k_1, k_2, and k_3, and the corresponding conditions

$$\begin{aligned} &\mathrm{Im}(k_1 + k_2) \in V^+, \\ &\mathrm{Im}(k_2 + k_3) \in V^+, \\ &\mathrm{Im}(k_3 + k_1) \in V^+. \end{aligned} \tag{3.8}$$

Comparison with (3.7) gives an immediate contradiction. This shows that the enlarged physical cell restricted to the mass shell cannot enter the nonrelativistic domain for processes with three or more initial (or final) particles.

It might be hoped that there is some further enlargement of the physical cell such that the physical scattering amplitude is a limit from mass-shell points lying inside the enlargement. However, this hope also is dashed by a simple counterexample.[9] For the $4 \to 4$ scattering function one can find positive-α Landau curves that are tangent to each other, but have exactly opposite half spaces of analyticity, in the mass shell. Thus there can be no well-defined cone of analyticity such that the physical scattering function is the limit to the real boundary point from mass-shell points lying in this cone. The counterexample is shown in Fig. 4.

The counterexample exhibited in Fig. 4 entails that there is, in general, no curve $C(k)$ in the mass shell that (i) ends at the real point k; (ii) depends continuously on k; (iii) lies, apart from its end point, in the domain of analyticity of the scattering function; and (iv) has its end point at the physical point. Hence the physical scattering function cannot in general be defined as a limit from along a mass-shell curve that lies in the domain of analyticity of the scattering function and that depends continuously on the physical limit point. Continuity cannot be maintained at the point of contact of the two positive-α Landau surfaces shown in Fig. 4 because of the conflicting $i\epsilon$ rules of that point.

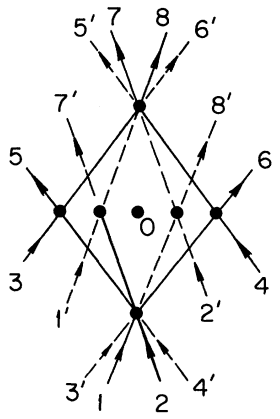

FIG. 4. Two Landau diagrams that correspond to positive-α Landau surfaces that are tangent at their point of intersection $k = (k_1, \ldots, k_8)$ but that have opposite half-spaces of analyticity. The first diagram consists of the outer solid lines $(1,2,3,4,) \to (5,6,7,8,)$. The second diagram consists of the inner dashed lines $(1',2',3',4') \to (5',6',7',8')$. The vector k_i equals k_i' for all i. Also $k_i = k_{9-i}$. The masses of the four particles whose lines intersect at any point on the horizontal axis are taken to be equal. The displacement vectors u_i that satisfy $u_i \cdot k_i = 0$ and that take lines i from initial positions passing through the origin 0 to the positions shown in the figure satisfy $u_i = -u_{9-i}$. The equivalence between the displacement vectors u_i and the gradients to the corresponding Landau surfaces established in Ref. 9 implies that the positive gradients to the Landau surfaces at the point k will be equal but opposite. Hence the $i\epsilon$ continuations around the surfaces are opposite.

The assumption upon which this conclusion is based is that the physical scattering amplitudes do have the positive-α Landau surfaces of singularities that are found in perturbation theory, and that can be derived directly from unitarity plus weak analyticity requirements.[10] The conclusion is that the mass shell does not in general contain an analog of the off-mass-shell physical cell: It does not in general contain a domain of analyticity such that the physical scattering function is the limit at a real boundary point of a function analytic in this domain.

This situation can be coped with in one of two ways. Within a field-theoretical framework one can simply regard the on-mass-shell physical scattering function as the limit of an off-mass-shell extension. This is the approach adopted here. In the general field-theoretical framework there is no need to define the scattering functions as a lim-

it from within the mass shell.

Within a strictly mass-shell S-matrix framework one accepts, at the basic level, only functions defined on the mass shell. In this framework one replaces the off-mass-shell concept of the physical cell, in which the scattering function is known to be analytic, and which contains the physical points on its boundary, by the concept of $i\epsilon$ rules.[11] The primitive domain of analyticity is the physical region itself, deprived of points on positive-α Landau surfaces. The $i\epsilon$ rules give the rules for continuing around these Landau surfaces. These rules are associated with the individual Landau surfaces; they do not ensure the existence of a universal physical cell of analyticity from which the physical limit is taken.

Discontinuity formulas for all singularities that enter the physical region itself have already been derived within the mass-shell framework directly from unitarity.[12] Work is under way to derive by similar methods the restriction to the mass shell of the basic discontinuity equation described in Sec. II. That work will be reported later. But it will be mentioned here that the special status of the cell functions $S_\lambda(k)$ vis à vis the more general functions $S_\sigma(k)$, where σ is the set of *all* N signs, one for each normal threshold cut, without regard to compatibility of the corresponding conditions, appears to be lost. Within the framework of an $i\epsilon$ rule of definition of the functions, the preferred status of the cell functions $S_\lambda(k)$ over the σ functions $S_\sigma(k)$ seems to dissolve. We find that the basic discontinuity equation holds not only for the cell functions but also for many of the more general σ functions.

The contemplated applications of the discontinuity formulas stem from the work of Mueller.[1] Mueller proposed, in effect, that the discontinuity function that corresponds to the inclusive cross section should exhibit Regge asymptotic behavior in certain limits, and derived from this requirement a number of results that had been previously derived from special dynamical models, and which seem to agree with experiment. It has been suggested, moreover, by several authors that this Regge behavior of the discontinuity is a consequence of the Regge behavior of the $3 \to 3$ amplitude itself in the sectors that define the two terms that make up the discontinuity. The formalization and generalization of this suggestion is that all the cell functions (or at least their on-mass-shell real k limits) exhibit the Regge behavior. Consequently, *all* of the basic discontinuities should exhibit this behavior. These conditions would impose a large number of as yet unexploited theoretical constraints on the boundary values of the kind occurring in the Mueller analysis.

IV. PROOF OF THE FORMULA

Our proof is based on an equation of Bros, Epstein, and Glaser that reads[13]

$$r_\lambda{}^+(x) - r_\lambda{}^-(x) = \langle \Omega | [R^B_{\lambda'}(x^B), R^A_{\lambda''}(x^A)] | \Omega \rangle . \tag{4.1}$$

This x space formula was derived modulo the usual ambiguities associated with points where one or more arguments coincide. But following the lead of Bros, Epstein, and Glaser we shall assume that the momentum-space discontinuity formulas that follow from it do in fact hold, at least at finite values of the momenta.

The function $r_\lambda(x)$ is the Fourier transform of the boundary value at $\mathrm{Im}k = 0$ of the function $\tilde{r}_\lambda(k) = \tilde{\tau}_\lambda(k)$, where $\tilde{\tau}_\lambda(k)$ is related to $S_\lambda(k)$ via (2.17).[3] The quantity $R_\lambda(x)$ is a linear function of the operators $A(x_{P1}) \cdots A(x_{Pn})$ that satisfies

$$r_\lambda(x) = \langle \Omega | R_\lambda(x) | \Omega \rangle . \tag{4.2}$$

The operator $R_\lambda(x)$ is called the cell operator corresponding to the cell Γ_λ. The cell operators $R^A_{\lambda'}$ and $R^B_{\lambda''}$ occurring in (4.1) are the cell operators corresponding to the cells $\Gamma^A_{\lambda'}$ and $\Gamma^B_{\lambda''}$, respectively. These are the cells in the spaces of variables associated with the sets of lines A and B that are specified by the sets of signs σ_A and σ_B described in Sec. II.

The proof has five steps. First, complete sets of intermediate states are inserted into the right-hand side of (4.1). "In" states will lead to the first form of the result, and "out" states will lead to the second. Insertion of the identities

$$I = \sum | \text{out}, \beta \rangle S_{\alpha\beta} \langle \text{in}, \alpha |$$

and

$$I = \sum | \text{in}, \beta \rangle S^{-1}{}_{\alpha\beta} \langle \text{out}, \alpha |$$

will lead to the third and fourth forms. Next we use reduction formulas to express the matrix elements of $R^A_{\lambda'}$ and $R^B_{\lambda''}$ as vacuum expectation values. These vacuum expectation values are then shown to be the $r_{\lambda'}$ corresponding to the cells $\Gamma^{A+I}_{\lambda'}$ and $\Gamma^{B+I}_{\lambda''}$. These cells are identified and are shown to be just the ones specified by rules 1, 2, and 3 of Sec. II. Finally the momentum arguments k are allowed to be complex, and the resulting formulas are shown to give the discontinuity formula described in Sec. II.

The first step is the insertion into (4.1) of a complete set of "in" or "out" states, or one of the other two forms of unity mentioned above. In order to reduce the matrix elements of $R^A_{\lambda'}$ and $R^B_{\lambda''}$ to vacuum expectation values we introduce, following

Bros, Epstein, and Glaser, the operators $A_i\uparrow$ and $A_i\downarrow$ defined by

$$A_i\downarrow A_{P\,m}|A_{P(m-1)}\cdots A_{P1} = -\sum_{j=1}^{m}\theta(x^0_{Pj}-x^0_i)A_{P\,m}|\cdots[A_i,A_{Pj}]\cdots A_{P1} \tag{4.3a}$$

and

$$A_i\uparrow A_{P\,m}|A_{P(m-1)}\cdots A_{P1} = \sum_{j=1}^{m}\theta(x^0_i-x^0_{Pj})A_{P\,m}|\cdots[A_i,A_{Pj}]\cdots A_{P1}. \tag{4.3b}$$

Here A_j is an abbreviation for $A(x_j)$.

The operator defined by (4.3b) has two important properties: It vanishes in the limit $x^0_i\to-\infty$ with all other variables held fixed, and it becomes $[A_i, A_{P\,m}|\cdots A_{P1}]$ in the limit $x^0_i\to+\infty$ with all other variables held fixed. These are the properties needed to derive the reduction formulas. Following the Lehmann-Symanzik-Zimmermann (LSZ) argument[14] one obtains, in momentum space,

$$[\tilde A^{\rm an}_{i\,\rm out},\tilde R_{\lambda'}] = -i(k_i^2-m_i^2)\tilde A_i\uparrow\tilde R_{\lambda'} \quad \text{for } k_i^0<0 \tag{4.4a}$$

and

$$[\tilde R_{\lambda'},\tilde A^{\rm cr}_{i\,\rm out}] = -i(k_i^2-m_i^2)\tilde A_i\uparrow\tilde R_{\lambda'} \quad \text{for } k_i^0>0. \tag{4.4b}$$

And from the analogous properties of $A_i\downarrow R_{\lambda'}$ one obtains

$$[\tilde R_{\lambda'},\tilde A^{\rm cr}_{i\,\rm in}] = -i(k_i^2-m_i^2)\tilde A_i\downarrow\tilde R_{\lambda'} \quad \text{for } k_i^0>0 \tag{4.4c}$$

and

$$[\tilde A^{\rm an}_{i\,\rm in},\tilde R_{\lambda'}] = -i(k_i^2-m_i^2)\tilde A_i\downarrow\tilde R_{\lambda'} \quad \text{for } k_i^0<0. \tag{4.4d}$$

The vector k_i is a real mass-shell four-vector, and the factor $(k_i^2-m_i^2)$ on the right-hand side is expected to cancel a corresponding pole singularity in $\tilde A_i\uparrow\tilde R_{\lambda'}$ or $\tilde A_i\downarrow\tilde R_{\lambda'}$. The $\tilde A^{\rm an}_{i(\rm in,out)}$ and $A^{\rm cr}_{i\,(\rm in,out)}$ are, respectively, the annihilation and creation parts of the asymptotic field operators.[14] The quantities $\tilde A_i\uparrow\tilde R_{\lambda'}$ and $\tilde A_i\downarrow\tilde R_{\lambda'}$ are the Fourier transforms of $A_i\uparrow R_{\lambda'}$ and $A_i\downarrow R_{\lambda'}$, respectively.

By repeated use of the same argument, with $R_{\lambda'}$ replaced by operators of the form $A_{Pi}\downarrow A_{P(i-1)}\downarrow\cdots\times A_{P1}\downarrow R_{\lambda'}$ and $A_{Pi}\uparrow A_{P(i-1)}\uparrow\cdots A_{P1}\uparrow R_{\lambda'}$ for successively larger values of i, one may write the matrix elements of $R_{\lambda'}$ as vacuum expectation values. Introducing the abbreviations $\tilde A'_i=-i(k_i^2-m_i^2)\tilde A_i$, one obtains

$$\langle\Omega|\tilde R_{\lambda'}|\text{in},I\rangle = \langle\Omega|\tilde A'_{Pj}\downarrow\tilde A'_{P(j-1)}\downarrow\cdots\tilde A'_{P1}\downarrow\tilde R_{\lambda'}|\Omega\rangle \quad (\text{all } k^0_{Pi}>0), \tag{4.5a}$$

$$\langle\Omega|\tilde R_{\lambda'}|\text{out},I\rangle = \langle\Omega|\tilde A'_{Pj}\uparrow\tilde A'_{P(j-1)}\uparrow\cdots\tilde A'_{P1}\uparrow\tilde R_{\lambda'}|\Omega\rangle \quad (\text{all } k^0_{Pi}>0), \tag{4.5b}$$

$$\langle\text{in},I|\tilde R_{\lambda'}|\Omega\rangle = \langle\Omega|\tilde A'_{Pj}\downarrow\tilde A'_{P(j-1)}\downarrow\cdots\tilde A'_{P1}\downarrow\tilde R_{\lambda'}|\Omega\rangle \quad (\text{all } k^0_{Pi}<0), \tag{4.5c}$$

$$\langle\text{out},I|\tilde R_{\lambda'}|\Omega\rangle = \langle\Omega|\tilde A'_{Pj}\uparrow\tilde A'_{P(j-1)}\uparrow\cdots\tilde A'_{P1}\uparrow\tilde R_{\lambda'}|\Omega\rangle \quad (\text{all } k^0_{Pi}<0). \tag{4.5d}$$

The label I stands for the set of vectors $(k_{Pj},\ldots,k_{P1})$, which are the physical energy-momentum vectors in the case of the kets $|\text{in},I\rangle$ and $|\text{out},I\rangle$, and are minus the physical energy-momentum vectors for the case of the bras $\langle\text{in},I|$ and $\langle\text{out},I|$. The operators $\tilde A'_{Pj}\downarrow\cdots\tilde A'_{P1}\downarrow\tilde R_{\lambda'}$ and $\tilde A'_{Pj}\uparrow\cdots\tilde A'_{P1}\uparrow\tilde R_{\lambda'}$ are $(-i)^j(k_{Pj}^2-m_{Pj}^2)\cdots\times(k_{P1}^2-m_{P1}^2)$ times the Fourier transforms of $A_{Pj}\downarrow\cdots A_{P1}\downarrow R_{\lambda'}$ and $A_{Pj}\uparrow\cdots A_{P1}\uparrow R_{\lambda'}$, respectively.

Next it must be shown that $A_{Pj}\downarrow\cdots A_{P1}\downarrow R_{\lambda'}$ and $A_{Pj}\uparrow\cdots A_{P1}\uparrow R_{\lambda'}$ are cell operators. This follows from the following theorem.

Theorem. If $R_{\lambda'}$ is a cell operator then $A_i\downarrow R_{\lambda'}$ and $A_i\uparrow R_{\lambda'}$ are cell operators.

Proof. The proof rests on some results of Bros, Epstein, and Glaser. These authors prove the existence of a set of "tree functions" $f_\beta(x)$ that have the following properties:

(1) The Fourier transform $\tilde f_\beta(k)$ of $f_\beta(x)$ is analytic in a cone $\tilde C_\beta$ to be described below. If $\tilde f_\beta(k)$ is defined to be zero in the complement of the closure of $\tilde C_\beta$ (where it is otherwise ill defined), then for every λ and every k in Γ_λ

$$\tilde r_\lambda(k)=\sum_\beta\tilde f_\beta(k). \tag{4.6}$$

That is, each $\tilde r_\lambda(k)$ is equal inside of Γ_λ to the sum of the functions $\tilde f_\beta(k)$ that are nonzero there.

(2) There is a one-to-one correspondence $\tilde f_\beta(k)\leftrightarrow t_\beta$ between the tree functions $\tilde f_\beta(k)$ and the tree diagrams t_β of a certain class. The tree diagrams t_β of this class are constructed by first associating each variable of $k=(k_1,\ldots,k_n)$ with either a "dot" or a "cross" in all ways such that not all are dots and not all are crosses. There are 2^n-2 such sets of associations. For each such set of associations one forms all possible "tree" diagrams t_β (i.e., simply connected diagrams t_β) subject to the condition that each tree diagram t_β consists only of the dots and crosses and line segments that connect dots to crosses: i.e., no line segment that connects a dot to a dot or a cross to a cross is allowed. The cutting of any line δ of any tree diagram t_β separates the diagram into two parts, each of which is a connected diagram. The sets of indices corresponding to the vertices of these two parts are called $J^+_{\beta\delta}$ and $J^-_{\beta\delta}$, where $J^+_{\beta\delta}$ contains the index corresponding to the *cross* on one end of δ, and $J^-_{\beta\delta}$ contains the index corresponding to the *dot* on the other end

of δ, and $J^+_{\beta\delta}+J^-_{\beta\delta}=J_\beta$ is the full index set for t_β.

(3) The cone $\tilde{C}_\beta$ is

$$\tilde{C}_\beta=\bigcap_\delta\left\{\mathrm{Im}\sum_{j\in J^+_{\beta\delta}}k_j\in V^+\right\}\cap\left\{\sum_{j\in J_\beta}k_j=0\right\} \tag{4.7a}$$

$$=\bigcap_\delta\left\{\mathrm{Im}\sum_{j\in J^-_{\beta\delta}}k_j\in V^-\right\}\cap\left\{\sum_{j\in J_\beta}k_j=0\right\}, \tag{4.7b}$$

where the index δ runs over all lines of t_β. Note that (4.6) and (4.7) entail the Steinmann relations: No tree graph contains lines corresponding to two partitions that are "crossed" in the sense of (2.21), since for any two lines δ and δ' in t_β the set $J^+_{\beta\delta}$ lies in $J^+_{\beta\delta'}$ or in $J^-_{\beta\delta'}$. Thus the discontinuity associated with γ can have no discontinuity across a "crossed" partition γ'.

(4) Consider a linear space consisting of sums of tree diagrams t_β. Define ρ_λ to be the sum of trees t_β whose cones $\tilde{C}_\beta$ contain Γ_λ:

$$\rho_\lambda\equiv\sum_{\beta:\,\tilde{C}_\beta\supset\Gamma_\lambda}t_\beta. \tag{4.8}$$

Define $l(A_i\uparrow R_\lambda)$ to be the sum of tree diagrams formed by associating k_i with a dot and joining this dot in turn to each cross of each t_β of ρ_λ. And define $l(A_i\downarrow R_\lambda)$ to be the sum of tree diagrams formed by associating k_i with a cross, and joining this cross in turn to each dot in each t_β in ρ_λ. This mapping l from operators of a certain class to linear combinations of tree diagrams of a certain class has a well-defined inverse mapping l^{-1}, defined on this latter class. The inverse map l^{-1} is such that for any cell Γ_λ the corresponding cell operator R_λ is $l^{-1}(\rho_\lambda)$. This relationship in fact defines R_λ.

The result just stated reduces the problem of proving the theorem to the problem of proving that $l(A_i\downarrow R_{\lambda'})$ and $l(A_i\uparrow R_{\lambda'})$ are precisely the ρ_λ corresponding to two cells. These cells are denoted by $\Gamma^\downarrow_{\lambda'}$ and $\Gamma^\uparrow_{\lambda'}$, respectively, and the corresponding ρ_λ by $\rho^\downarrow_{\lambda'}$ and $\rho^\uparrow_{\lambda'}$.

$R_{\lambda'}$ is, by assumption, a cell operator. Thus it corresponds to some cell $\Gamma_{\lambda'}$. For each partition γ' of the set of indices $J_{\lambda'}$ corresponding to $\Gamma_{\lambda'}$ into two disjoint sets $J^+_{\gamma'}$ and $J^-_{\gamma'}$ the signs on $J^+_{\gamma'}$ and $J^-_{\gamma'}$ can be chosen so that

$$\Gamma_{\lambda'}=\bigcap_{\gamma'}\left\{\mathrm{Im}\sum_{j\in J^+_{\gamma'}}k_j\in V^+\right\}\cap\left\{\sum_{j\in J_{\lambda'}}k_j=0\right\} \tag{4.9a}$$

$$=\bigcap_{\gamma'}\left\{\mathrm{Im}\sum_{j\in J^-_{\gamma'}}k_j\in V^-\right\}\cap\left\{\sum_{j\in J_{\lambda'}}k_j=0\right\}. \tag{4.9b}$$

These sets $J^+_{\gamma'}$ and $J^-_{\gamma'}$ occur in the following definitions:

$$\Gamma^\downarrow_{\lambda'}\equiv\left\{\mathrm{Im}k_i\in V^+\right\}\cap_{\gamma'}\left\{\mathrm{Im}\sum_{j\in J^+_{\gamma'}}k_j\in V^+\right\}\cap\left\{k_i+\sum_{j\in J_{\lambda'}}k_j=0\right\}$$

$$=\left\{\mathrm{Im}k_i\in V^+\right\}\cap_{\gamma'}\left\{\mathrm{Im}\sum_{j\in J^+_{\gamma'}}k_j\in V^+\right\}\cap\left\{k_i+\sum_{j\in J_{\lambda'}}k_j=0\right\}$$

$$\times\bigcap_{\gamma'}\left\{\left(\mathrm{Im}k_i+\mathrm{Im}\sum_{j\in J^+_{\gamma'}}k_j\right)\in V^+\right\} \tag{4.10a}$$

and

$$\Gamma^\uparrow_{\lambda'}\equiv\left\{\mathrm{Im}k_i\in V^-\right\}\cap_{\gamma'}\left\{\mathrm{Im}\sum_{j\in J^-_{\gamma'}}k_j\in V^-\right\}\cap\left\{k_i+\sum_{j\in J_{\lambda'}}k_j=0\right\}$$

$$=\left\{\mathrm{Im}k_i\in V^-\right\}\cap_{\gamma'}\left\{\mathrm{Im}\sum_{j\in J^-_{\gamma'}}k_j\in V^-\right\}\cap\left\{k_i+\sum_{j\in J_{\lambda'}}k_j=0\right\}$$

$$\times\bigcap_{\gamma'}\left\{\left(\mathrm{Im}k_i+\mathrm{Im}\sum_{j\in J^-_{\gamma'}}k_j\right)\in V^-\right\}. \tag{4.10b}$$

The second lines of these equations, which follow trivially from the first lines, show that $\Gamma^\downarrow_{\lambda'}$ and $\Gamma^\uparrow_{\lambda'}$ are cells, provided they are nonempty.

The sets $\Gamma^\downarrow_{\lambda'}$ and $\Gamma^\uparrow_{\lambda'}$ are indeed nonempty. To see this let k' represent the set of vectors k_j with $j\in J_{\lambda'}$. (The sum of these k_j is not required to be zero.) And consider any point $\mathrm{Im}k'$ that lies inside the cell $\Gamma_{\lambda'}$. Let $n(\hat{k}')$ be the set of points $\mathrm{Im}k'$ such that, for all μ and $j\in J_{\lambda'}$, $|\mathrm{Im}k^\mu_j-\mathrm{Im}\hat{k}^\mu_j|<\epsilon$ for some small $\epsilon>0$. This ϵ is chosen small enough so that all the conditions

$$\mathrm{Im}\sum_{j\in J^+_{\gamma'}}k_j\in V^+ \tag{4.11a}$$

and

$$\mathrm{Im}\sum_{j\in J^-_{\gamma'}}k_j\in V^- \tag{4.11b}$$

occurring in (4.9) are satisfied for all points in $n(\hat{k}')$. Such a set $n(\hat{k}')$ exists because $\Gamma_{\lambda'}$ is open. Thus it is evident that

$$\hat{\Gamma}^\downarrow_{\lambda'}\equiv\left\{k=(k_i,k'):\mathrm{Im}k'\in n(\hat{k}');\,\mathrm{Im}\sum_{j\in J_{\lambda'}}k_j\in V^-;\,k_i=-\sum_{j\in J_{\lambda'}}k_j\right\} \tag{4.12a}$$

and

$$\hat{\Gamma}^\uparrow_{\lambda'}\equiv\left\{k=(k_i,k'):\mathrm{Im}k'\in n(\hat{k}');\,\mathrm{Im}\sum_{j\in J_{\lambda'}}k_j\in V^+;\,k_i=-\sum_{j\in J_{\lambda'}}k_j\right\} \tag{4.12b}$$

are nonempty sets that lie in $\Gamma^\downarrow_{\lambda'}$ and $\Gamma^\uparrow_{\lambda'}$, respectively:

$$\hat{\Gamma}^\downarrow_{\lambda'}\subset\Gamma^\downarrow_{\lambda'} \tag{4.13a}$$

and

$$\hat{\Gamma}^\uparrow_{\lambda'}\subset\Gamma^\uparrow_{\lambda'}. \tag{4.13b}$$

The properties of these sets $\hat{\Gamma}^{\downarrow}_{\lambda'}$ and $\hat{\Gamma}^{\uparrow}_{\lambda'}$ provide the basis for the rest of the proof.

First we must show that for every $t_{\beta'}$ in $l(A_i\downarrow R_{\lambda'})$ the corresponding $\tilde{C}_{\beta'}$ contains $\Gamma^{\downarrow}_{\lambda'}$. The rules for constructing $l(A_i\downarrow R_{\lambda'})$ from the trees t_β in the sum $\rho_{\lambda'}=l(R_{\lambda'})$ are such that the condition on $\tilde{C}_\beta$ in (4.7) corresponding to any line δ of t_β gets replaced in $\tilde{C}_{\beta'}$ by one or both of the conditions (4.11a) and (4.11b). Thus these conditions (4.11a) and (4.11b) on $n(\hat{k}')$ ensure that any k in $\hat{\Gamma}^{\downarrow}_{\lambda'}$ must satisfy all the conditions imposed by all the trees in $l(A_i\downarrow R_{\lambda'})$. All of these conditions are then satisfied also for all k in the whole cell $\Gamma^{\downarrow}_{\lambda'}$, because no cell is cut by any boundary of any cone $\tilde{C}_{\beta'}$. The argument holds also with $\uparrow$ in place of $\downarrow$.

It remains to show, conversely, that $l(A_i\downarrow R_{\lambda'})$ contains every t_β whose cone $\tilde{C}_\beta$ contains $\Gamma^{\downarrow}_{\lambda'}$, and that $l(A_i\uparrow R_{\lambda'})$ contains every t_β whose cone $\tilde{C}_\beta$ contains $\Gamma^{\uparrow}_{\lambda'}$.

For definiteness we take $l(A_i\downarrow R_{\lambda'})$ and $\Gamma^{\downarrow}_{\lambda'}$; the other case is completely analogous.

Let t_β be such that $\tilde{C}_\beta\supset\Gamma^{\downarrow}_{\lambda'}$. Then the condition $(\mathrm{Im}k_i\in V^+)$ in (4.10a) requires the vertex of t_β associated with k_i to be a cross. For (4.7b) ensures that if k_i is associated with a dot it satisfies $\mathrm{Im}k_i\in V^-$.

This cross associated with k_i cannot be connected to more than one line segment of t_β. For suppose it were connected to several. Then for each such line δ there would be, from (4.7b), a condition on $\tilde{C}_\beta$ of the form

$$k(J^-_{\beta\delta})\equiv\mathrm{Im}\sum_{j\in J^-_{\beta\delta}}k_j\in V^-. \tag{4.14}$$

These various sets $J^-_{\beta\delta}$ would be disjoint sets that, together with i, make up the whole set of indices associated with k. Thus

$$-\sum k(J^-_{\beta\delta})=k_i, \tag{4.15}$$

where the sum is over the several lines δ of t_β that connect to the dot associated with k_i.

The conditions (4.14) and (4.15) are incompatible with the requirement $\tilde{C}_\beta\supset\Gamma^{\downarrow}_{\lambda'}$. This requirement demands, by virtue of (4.13a), that $\tilde{C}_\beta\supset\hat{\Gamma}^{\downarrow}_{\lambda'}$. But if every point in $\hat{\Gamma}^{\downarrow}_{\lambda'}$ lies in $\tilde{C}_\beta$ then some point k with $|\mathrm{Im}k^\mu_j-\mathrm{Im}\hat{k}^\mu_j|\ll\epsilon$ for all μ and j lies in $\tilde{C}_\beta$. The conditions (4.14) and (4.15) then require that the vectors $-\mathrm{Im}k(J^-_{\beta\delta})\in V^+$ sum to $\mathrm{Im}k_i\in V^+$. But then, for any of these δ, $\mathrm{Im}k(J^-_{\beta\delta})+\mathrm{Im}k_i\in V^+$. Therefore the addition of $\mathrm{Im}k_i$ and $-\mathrm{Im}k_i$ to the imaginary parts of any two of the $k(J^-_{\beta\delta})$ will give a k in $\hat{\Gamma}^{\downarrow}_{\lambda'}$ that does not satisfy all the conditions (4.14). Thus the requirement $\tilde{C}_\beta\supset\Gamma^{\downarrow}_{\lambda'}$ is incompatible with the requirements (4.14) and (4.15), and hence with the possibility that more than one line of t_β connects to the cross associated with k_i.

Hence only one line of t_β connects to the cross associated with k_i.

The t_β must therefore consist of a subdiagram t'_β plus one line that connects a dot in t'_β to the cross associated with k_i. The requirement $\tilde{C}_\beta\supset\Gamma^{\downarrow}_{\lambda'}$ implies $\tilde{C}_\beta\supset\hat{\Gamma}^{\downarrow}_{\lambda'}$. But then $\tilde{C}_\beta$ contains points with $\mathrm{Im}k_i$ arbitrarily small and the remaining set of variables arbitrarily close to $\mathrm{Im}\hat{k}'$, which lies in $\Gamma_{\lambda'}$. The conditions on the $\tilde{C}'_\beta$ associated with the subgraph t'_β must therefore be compatible with the point $\mathrm{Im}\hat{k}'$. That is, $\tilde{C}'_\beta$ must contain $\mathrm{Im}\hat{k}'$, and hence the whole cell $\Gamma_{\lambda'}$. But then t'_β lies in $\rho_{\lambda'}$. Therefore t_β lies in $l(A_i\downarrow R_{\lambda'})$. That is, $[\tilde{C}_\beta\supset\Gamma^{\downarrow}_{\lambda'}]$ implies $[t_\beta\in l(A_i\downarrow R_{\lambda'})]$. Thus, in view of the earlier result that $[t_\beta\in l(A_i\downarrow R_{\lambda'})]$ implies $[\tilde{C}_\beta\supset\Gamma^{\downarrow}_{\lambda'}]$, and the definition (4.8) of ρ_λ, we have

$$\rho^{\downarrow}_{\lambda'}=l(A_i\downarrow R_{\lambda'}). \tag{4.16}$$

But then

$$\begin{aligned}A_i\downarrow R_{\lambda'}&=l^{-1}l(A_i\downarrow R_{\lambda'})\\&=l^{-1}(\rho^{\downarrow}_{\lambda'})\\&\equiv R^{\downarrow}_{\lambda'}.\end{aligned} \tag{4.17a}$$

Similarly,

$$A_i\uparrow R_{\lambda'}=R^{\uparrow}_{\lambda'}, \tag{4.17b}$$

and the theorem is proved.

Repeated application of the theorem shows that the operators occurring on the right-hand sides of Eqs. (4.5) are cell operators, apart from the factor $(-i)^j(k_{Pj}{}^2-m_{Pj}{}^2)\cdots(k_{P1}{}^2-m_{P1}{}^2)$. Their vacuum expectation values are, accordingly, cell functions, apart from these same factors.

The cells $\Gamma^{\downarrow}_{\lambda'}$ and $\Gamma^{\uparrow}_{\lambda'}$ corresponding to $A_i\downarrow R_{\lambda'}$ and $A_i\uparrow R_{\lambda'}$ were identified in the course of the proof. Repeated application of the result (4.10) shows that the cell corresponding to $A_{Pj}\downarrow\cdots A_{P1}\downarrow R_{\lambda'}$ is

$$\Gamma^{\downarrow\cdots\downarrow}_{\lambda'}=\bigcap_{i=1}^{j}\{\mathrm{Im}k_{Pi}\in V^+\}\bigcap_{\gamma'}\{\mathrm{Im}k(J^+_{\gamma'})\in V^+\}\cap\{\textstyle\sum k_i=0\} \tag{4.18a}$$

and that the cell corresponding to $A_{Pj}\uparrow\cdots A_{P1}\uparrow R_{\lambda'}$ is

$$\Gamma^{\uparrow\cdots\uparrow}_{\lambda'}=\bigcap_{i=1}^{j}\{\mathrm{Im}k_{Pi}\in V^-\}\bigcap_{\gamma'}\{\mathrm{Im}k(J^-_{\gamma'})\in V^-\}\cap\{\textstyle\sum k_i=0\}, \tag{4.18b}$$

where the sets $J^{\pm}_{\gamma'}$ are defined above (4.9), and the $\sum k_i$ runs over all the k_{Pi}, $1\leq i\leq j$, and over all the k_i with $i\in J_{\lambda'}$.

We must now show that the cells defined by (4.18) are the same as those identified by rules 1, 2, and 3 of Sec. II.

Consider first the case corresponding to Eq. (4.5a). This is the function associated with the

left-hand bubble of the first and fourth lines on the right-hand side of Fig. 1. Rule 1 follows from a comparison of the second set of conditions in (4.18a) with the conditions (4.9a) that define the cell $\Gamma_{\lambda'}$, which in our case would be the cell $\Gamma^B_{\lambda'}$ specified by the set of signs σ_B. The equivalence of these two sets of conditions ensures that rule 1 is satisfied. [Recall that the discontinuity equation is evaluated on the boundary $\mathrm{Im}E(J^+_\gamma)=0$. Thus $\mathrm{Im}\sum_{i=1}^{j} k_{Pi}=0$. Hence the conditions (4.9a) and (4.9b) are equivalent, even when the sets J^+_γ and J^-_γ are regarded as subsets of the larger set of indices corresponding to the larger process $B+I$.]

Rule 2 follows in this same case from the first set of conditions in (4.18a). These require that all combinations of the variables $(\mathrm{Im}k_{Pj},\ldots,\mathrm{Im}k_{P1})$ be evaluated as limits from the upper half cones: $\sum'\mathrm{Im}k_{Pi}\in V^+$, where $\sum'$ indicates any partial sum. Since the real energies $\mathrm{Re}k^0_{Pi}$ are all positive in this case the corresponding signs $\sigma_{\lambda'\gamma'}$ are all positive, as indicated in Fig. 1.

To prove rule 3 in this case suppose the sum of the energies $\mathrm{Re}k^0_i$ of the lines of $X\equiv B^-_{\gamma'}+I^-_{\gamma'}$ is negative, as specified by the rule. Then the sum of the energies $\mathrm{Re}k^0_i$ of the lines of $B^-_{\gamma'}$ is even more negative, since the contribution from $I^-_{\gamma'}$ is positive. Thus the rule says that the sign of the imaginary part is the same for $(B^-_{\gamma'}+I^-_{\gamma'})$ as it is for $(B^-_{\gamma'})$.

If this latter sign is positive then the result that the sign for $B^-_{\gamma'}+I^-_{\gamma'}$ is also positive follows directly from (4.18a). On the other hand, if the sign of the imaginary part is negative for $B^-_{\gamma'}$, then it is positive for $B^+_{\gamma'}$. But then (4.18a) implies that it is positive also for $(B^+_{\gamma'}+I^+_{\gamma'})$. But since the real energy of this set is also positive one has

$$\sigma(B^+_{\gamma'}+I^+_{\gamma'})=\sigma(B^-_{\gamma'}+I^-_{\gamma'})=\sigma(X)=\sigma(B^-_{\gamma'}) \tag{4.19}$$

since $\sigma(B^-_{\gamma'})$ is positive in this case. This confirms rule 3 for this case.

The proof of rules 1, 2, and 3 for the case (4.5b) proceeds in the same way, except that one uses (4.18b) instead of (4.18a). The proof for the cases (4.5c) and (4.5d) proceeds in the same way, except that the signs of the real parts of the energies are now reversed. This induces a reversal of the signs $\sigma_{\lambda'\gamma'}$. For example, the signs in the cut-out sector of the right-hand bubble in the first line of the right-hand side of Fig. 1 are negative because the imaginary parts in the corresponding expression (4.5c) are positive, by virtue of (4.18a), but the real parts are negative.

This completes the proof of rules 1, 2, and 3.

Bros, Epstein, and Glaser derive our initial equation (4.1) from the corresponding identity in the space of sums of tree diagrams. This identity, $\rho_\lambda{}^+-\rho_\lambda{}^-=[\rho^B_{\lambda'},\rho^A_{\lambda'}]$, expresses the difference between the sum of tree diagrams $\rho_\lambda{}^+$ and the sum of tree diagrams $\rho_\lambda{}^-$ as the sum of tree diagrams that are in $\rho_\lambda{}^+$ but not in $\rho_\lambda{}^-$, minus the sum of tree diagrams that are in $\rho_\lambda{}^-$ but not in $\rho_\lambda{}^+$. The vacuum expectation value of the image under l^{-1} of this tree diagram identity is (4.1). Thus each side of this equation represents the sum of the functions $f_\beta(x)$ such that $\tilde{f}_\beta(k)$ contributes to $\tilde{r}_\lambda{}^+(k)$, but not to $\tilde{r}_\lambda{}^-(k)$, minus the sum of the functions $f_\beta(x)$ such that $\tilde{f}_\beta(k)$ contributes to $\tilde{r}_\lambda{}^-(k)$, but not to $\tilde{r}_\lambda{}^+(k)$.

The Fourier transform of the right-hand side of (4.1) is defined only for points k that lie on the common boundary face of $\Gamma_\lambda{}^+$ and $\Gamma_\lambda{}^-$. For there is no θ function associated with the intermediate state γ that would lead to an analytic extension into $\mathrm{Im}k(J^+_\gamma)\neq 0$.

For any point k that lies on the common boundary face in $\mathrm{Im}k(J^+_\gamma)=0$ of $\Gamma_\lambda{}^+$ and $\Gamma_\lambda{}^-$, the function $\tilde{r}_\lambda{}^+(k)-\tilde{r}_\lambda{}^-(k)$ is defined to be the limit of $\tilde{r}_\lambda{}^+$ from points within $\Gamma_\lambda{}^+$, minus the limit of $\tilde{r}_\lambda{}^-$ from points within $\Gamma_\lambda{}^-$. This difference $\tilde{r}_\lambda{}^+(k)-\tilde{r}_\lambda{}^-(k)$ has no contribution from any $\tilde{f}_\beta(k)$ that contributes to both $\tilde{r}_\lambda{}^+(k)$ and $\tilde{r}_\lambda{}^-(k)$. For any such $\tilde{f}_\beta(k)$ will be analytic at k, and will contribute equally to both. Thus both sides of the equation

$$\tilde{r}_\lambda{}^+(k)-\tilde{r}_\lambda{}^-(k)=\langle\Omega|[\tilde{R}^B_{\lambda'}(k^B),\tilde{R}^A_{\lambda'}(k^A)]|\Omega\rangle \tag{4.20}$$

represent the same combination of functions $\tilde{f}_\beta(k)$.

The spectral condition that the energy of any physical state be non-negative, together with our original stipulation that the energy is flowing from set A to set B, entails that only one term of the commutator survive:

$$\tilde{r}_\lambda{}^+(k)-\tilde{r}_\lambda{}^-(k)=\langle\Omega|\tilde{R}^B_{\lambda'}(k^B)\tilde{R}^A_{\lambda'}(k^A)|\Omega\rangle\,. \tag{4.21}$$

This result combined with (4.5) and (4.18) gives the formula we seek, except in one special case.

This special case is the case in which one of the two sets A or B consists of a single line. This case is not covered by the work of Bros, Epstein, and Glaser. However, the basic discontinuity formula holds also in this case. This follows directly from (2.4). The relevant denominator is the one on one end or the other. The difference on the left-hand side of (4.21) gives a δ function that cancels the integration over dq^0_1 or dq^0_n, and converts the corresponding q^0_1 or q^0_n to k^0_1 or k^0_n, respectively. This yields (4.21) for this special case.

One could presumably derive the whole result from (2.4). We have used instead the beautiful work of Bros, Epstein, and Glaser, and have shown that the formula described in Sec. II is, apart from the trivial case, essentially the momentum-space form of a result obtained by them.

ACKNOWLEDGMENTS

One of us (K.E.C.) would like to thank the other (H.P.S.) for introducing him to this subject. He is also grateful to Professor Chew and Professor Jackson for extending to him the hospitality of LBL.

APPENDIX: PROOFS OF SOME THEOREMS

Proof of Theorem 3.1

Suppose the set $\{\mathrm{Im}k: (k-\bar{k})\cdot\nabla\phi(\bar{k})=0\}$ does intersect Γ_λ. Then, because of the conical shape of Γ_λ, there must be in any neighborhood of $\bar{k}$ points on the surface $\phi(k)=0$ that lie also in Γ_λ. The function $S_\lambda(k)$ is analytic at these points.

Because $\bar{k}$ is a regular point of the surface $\phi(k)=0$ one can introduce a set of local analytic coordinates[6]

$$(\phi, z)\equiv(\phi, z_1, \ldots, z_{4(n-1)-1}) \tag{A1}$$

such that $k(\phi, z)$ is analytic near the point (ϕ_0, z_0), where

$$\phi_0=0 \tag{A2}$$

and

$$z_0=(0, \ldots, 0) \tag{A3}$$

and

$$k(\phi_0, z_0)=\bar{k}. \tag{A4}$$

The coordinates z can be chosen so that

$$\{\mathrm{Im}k: k=k(\phi_0, z_1, 0, \ldots, 0),\ 0<z_1<a,\ a>0\}\subset\Gamma_\lambda. \tag{A5}$$

Consider then the set of one- (complex) dimensional analytic disks of the form

$$D(r')\equiv\{k: k=k(\phi, z_1, 0, \ldots, 0),\ \phi=r'e^{i\theta},\ 0\leq|z_1|<a'\}, \tag{A6}$$

where θ is the fixed angle occurring in (3.1). For sufficiently small $a'>0$ and $r'>0$ the function

$$S_\lambda(\theta, z)\equiv S_\lambda(k(\theta, z)) \tag{A7}$$

is analytic in $D(r')$ by virtue of (3.1). At the point $r'=0$ this function is analytic in $D(0)$ on the line $0<z_1<a$, by virtue of (A5). But then this function is analytic on the whole disk $D(0)$ by virtue of Bremermann's variation of the *Kontinuitätsatz.*[15] The center of the disk $D(0)$ is the image of $\bar{k}$ under the mapping $\theta(k), z(k)$. Since these functions are analytic near $\bar{k}$ the function

$$S_\lambda(k)=S_\lambda(\theta(k), z(k)) \tag{A8}$$

is analytic at $\bar{k}$. Thus it is false that the function $S_\lambda(k)$ is ϕ-singular at $\bar{k}$ on Γ^0_λ, and the theorem is proved.

Proof of Theorem 3.2

What must be proved is the equivalence of (3.4a) and (3.4b). Let us define

$$C_{\lambda\gamma}\equiv\{v: v=\sigma_{\lambda\gamma}\nabla k(J^+_\gamma)\cdot v_\gamma,\ v_\gamma\in\overline{V}^+\}. \tag{A9}$$

This is the set of $(4n-4)$-dimensional vectors v generated as the four-dimensional vector v_γ ranges over the closure $\overline{V}^+$ of the forward light cone V^+.

The linearity of $k(J^+_\gamma)$ in the variables k_i implies that

$$\mathrm{Im}k\cdot\nabla k(J^+_\gamma)=\mathrm{Im}k(J^+_\gamma). \tag{A10}$$

Thus, if $v=\sigma_{\lambda\gamma}\nabla k(J^+_\gamma)$, then

$$\mathrm{Im}k\cdot v=\sigma_{\lambda\gamma}\,\mathrm{Im}k(J^+_\gamma)\cdot v_\gamma. \tag{A11}$$

Thus the set

$$C^+_{\lambda\gamma}\equiv\{\mathrm{Im}k: \mathrm{Im}k\cdot v\geq 0 \text{ for all } v \text{ in } C_{\lambda\gamma}\} \tag{A12}$$

is

$$C^+_{\lambda\gamma}=\{\mathrm{Im}k: \sigma_{\lambda\gamma}\,\mathrm{Im}k(J^+_\gamma)\cdot v_\gamma\geq 0 \text{ for all } v_\gamma \text{ in } \overline{V}^+\}. \tag{A13}$$

Define the set

$$\overline{\Gamma}_{\lambda\gamma}\equiv\{\mathrm{Im}k: \sigma_{\lambda\gamma}\,\mathrm{Im}k(J^+_\gamma)\in\overline{V}^+\}. \tag{A14}$$

Noting that the set of all four-vectors u such that $u\cdot v_\gamma\geq 0$ for all $v_\gamma\in\overline{V}^+$ is just the set $\overline{V}^+$ itself, one has

$$\overline{\Gamma}_{\lambda\gamma}=C^+_{\lambda\gamma}. \tag{A15}$$

The set occurring on the right-hand side of (3.4a) may now be written

$$\begin{aligned}\overline{\Gamma}^+_\lambda(\bar{k})&\equiv[\overline{\Gamma}_\lambda(\bar{k})]^+\\&\equiv\left[\bigcap_{\{\gamma:\,\mathrm{Im}\bar{k}(J^+_\gamma)=0\}}\overline{\Gamma}_{\lambda\gamma}\right]^+\\&=\left[\bigcap_{\{\gamma:\,\mathrm{Im}\bar{k}(J^+_\gamma)=0\}}C^+_{\lambda\gamma}\right]^+.\end{aligned} \tag{A16}$$

The condition $\mathrm{Im}k\cdot v\geq 0$ in (A12) can be replaced by $\mathrm{Im}(k-\bar{k})\cdot v\geq 0$ for all those $C^+_{\lambda\gamma}$ that occur in (A16). Then the connection between X and X^+ is expressed via $\mathrm{Im}(k-\bar{k})\cdot v\geq 0$ for all the sets X^+ occurring in (A16).

The set of vectors on the right-hand side of (3.4b) is

$$C_\lambda(\bar{k})\equiv\sum_{\{\gamma:\,\mathrm{Im}\bar{k}(J^+_\gamma)=0\}}C_{\lambda\gamma}. \tag{A17}$$

Thus the theorem can be proved by proving that

$$\left[\bigcap_{\{\gamma:\,\mathrm{Im}\bar{k}(J_\gamma^+)=0\}} C_{\lambda\gamma}^+\right]^+ = \sum_{\{\gamma:\,\mathrm{Im}\bar{k}(J_\gamma^+)=0\}} C_{\lambda\gamma}\,. \tag{A18}$$

It follows immediately from the definitions that

$$\bigcap_{\{\gamma:\,\mathrm{Im}\bar{k}(J_\gamma^+)=0\}} C_{\lambda\gamma}^+ = \left[\sum_{\{\gamma:\,\mathrm{Im}\bar{k}(J_\gamma^+)=0\}} C_{\lambda\gamma}\right]^+ = [C_\lambda(\bar{k})]^+\,, \tag{A19}$$

where

$$[C_\lambda(\bar{k})]^+ \equiv \{\mathrm{Im}k : \mathrm{Im}(k-\bar{k})\cdot v \geq 0 \text{ for all } v \text{ in } C_\lambda(\bar{k})\}\,.$$

For if $\mathrm{Im}k$ lies in the set on the left-hand side of (A19), and hence satisfies $\mathrm{Im}(k-\bar{k})\cdot v \geq 0$ for all v in each one of the sets $C_{\lambda\gamma}$, then it must satisfy this condition for any v that is a sum of such vectors. Hence it lies in the right-hand side. Conversely, if $\mathrm{Im}k$ is such that this condition $\mathrm{Im}(k-\bar{k})\cdot v \geq 0$ is satisfied for all v that can be written as sums of v's from the various sets $C_{\lambda\gamma}$, then this condition must hold for all v's in each of these sets $C_{\lambda\gamma}$. Hence $\mathrm{Im}k$ must lie in the set on the left-hand side of (A19). This confirms (A19).

The desired identity (A18) follows from (A19) and the identity

$$[C_\lambda(\bar{k})]^{++} = C_\lambda(\bar{k})\,. \tag{A20}$$

The set $[C_\lambda(\bar{k})]^{++}$ is

$$[C_\lambda(\bar{k})]^{++} = \{v : \mathrm{Im}(k-\bar{k})\cdot v \geq 0 \text{ for all } \mathrm{Im}k \text{ such that } \mathrm{Im}(k-\bar{k})\cdot v' \geq 0 \text{ for all } v' \text{ in } C_\lambda(\bar{k})\}\,. \tag{A21}$$

It is clear from this definition that if v is in $C_\lambda(\bar{k})$ then it is also in $[C_\lambda(k)]^{++}$:

$$C_\lambda(\bar{k}) \subset [C_\lambda(\bar{k})]^{++}\,. \tag{A22}$$

To prove the converse suppose v is not in $C_\lambda(\bar{k})$. We must find some real vector $\mathrm{Im}k$ such that $\mathrm{Im}(k-\bar{k})\cdot v' \geq 0$ for all v' in $C_\lambda(\bar{k})$, but $\mathrm{Im}(k-\bar{k})\cdot v < 0$. This will show that the point v is not in $[C_\lambda(\bar{k})]^{++}$.

The required $\mathrm{Im}k$ can be constructed as follows: Let v be the point not in $C_\lambda(\bar{k})$. Consider the set of closed balls (in the Euclidean metric) centered at v. Let $B(v)$ be the smallest one of these balls that intersects the convex set $C_\lambda(\bar{k})$. The tangent hyperplane at the unique point of contact divides the whole space of v's into two half-spaces, one of which, called $H(v)$, intersects $B(v)$ only at the point of contact.

The set $C_\lambda(\bar{k})$ must lie in $H(v)$. For if any point in the complement of $H(v)$ were in $C_\lambda(\bar{k})$ then the convexity property of $C_\lambda(\bar{k})$ would entail that some point in the interior of $B(v)$ would lie in $C_\lambda(\bar{k})$, contrary to the fact that $B(v)$ is the smallest ball centered at v that contains points of $C_\lambda(\bar{k})$. This means that $H(v)$ is one of the bounding half-spaces of $C_\lambda(\bar{k})$; i.e., the whole set $C_\lambda(\bar{k})$ lies in $H(v)$, and the boundary $\partial H(v)$ of $H(v)$ contains at least one point of $C_\lambda(\bar{k})$.

The boundary $\partial H(v)$ may contain more than one point of $C_\lambda(\bar{k})$. In fact, if it contains any interior point of any ray that lies in $C_\lambda(\bar{k})$ then it must contain the whole ray, since otherwise the whole set $C_\lambda(\bar{k})$ would not lie in $H(v)$. But the point of contact v_c between $B(v)$ and $C_\lambda(\bar{k})$ is either an interior point of some ray that lies in $C_\lambda(\bar{k})$ or it is the origin $v=0$. Thus the origin lies on $\partial H(v)$.

The point v_c lies at the point of contact of $B(v)$ and $H(v)$. Thus the half-space $H(v)$ is defined by

$$H(v) = \{v' : (v'\cdot(v_c - v))_E \geq 0\}\,, \tag{A23}$$

where

$$(v'\cdot(v_c-v))_E = \sum_{i=1}^{n}\sum_{n=0}^{3} v'_{i\mu}(v_c-v)_{i\mu}\,. \tag{A24}$$

Since v_c lies on $\partial H(v)$ one has

$$(v_c\cdot(v_c-v))_E = 0\,, \tag{A25}$$

and hence

$$(v\cdot(v_c-v))_E < 0\,. \tag{A26}$$

But then the required real vector $\mathrm{Im}k$ can be defined by

$$\mathrm{Im}(k-\bar{k})_i^\mu = (v_c-v)_{i\mu}\,. \tag{A27}$$

Proof of Theorem 3.3

The arguments given in the proof of Theorem 3.1 show that if the function $S_\lambda(k)$ is ϕ-singular at $\bar{k}$ on $\partial\Gamma_\lambda^0$, then the function $S_\lambda(k)$ is singular at all points on the surface $\phi(k)=0$ that lie in some neighborhood of $\bar{k}$. We shall prove the theorem by showing that the surface $\phi(k)=0$ either has the required form or penetrates into Γ_λ arbitrarily close to $\bar{k}$. The second possibility is ruled out by the fact that $S_\lambda(k)$ is analytic in Γ_λ.

Because $\partial\phi'(\bar{k}')/\partial k_1'^0 \neq 0$, the surface $\phi'(k')=0$ can be expressed near $\bar{k}'$ in the form[6]

$$k_1'^0 = f(k_1'^i, \bar{k}_j') + r(k_1'^i, k_j')\,, \tag{A28}$$

where $r(k_1'^i, k_j')$ is analytic in a neighborhood of $(\bar{k}_1'^i, \bar{k}_j')$, and vanishes at $k_j' = \bar{k}_j'$ (all $j \neq 1$). If $r(k_1'^i, k_j')$ is identically zero then the theorem is proved by defining the required function $\hat{\phi}(k_1')$ by

$$\hat{\phi}(k_1') = k_1'^0 - f(k_1'^i, \bar{k}_j')\,. \tag{A29}$$

To complete the proof we will show that if $r(k_1'^i, k_j')$ is not identically zero, then the surface $\phi(k)=0$ enters Γ_λ arbitrarily close to $\bar{k}$.

If $r(k_1'^i, k_j)$ is not identically zero then there must be some $j>1$, which we call $j=2$, some $\mu=0$, 1, 2, or 3, and some $n\geq 1$ such that

$$r(k_1'^i, k_j') = a(k_1'^i, k_j'')(k_2'^\mu - \bar{k}_2'^\mu)^n + r'(k_1'^i, k_j'), \tag{A30}$$

where k_j'' is the set of $k_j'^\nu$ for $j\neq 1$ and $(j,\nu)\neq(2,\mu)$; $a(k_1'^i, k_j'')$ is an analytic function of its arguments that is not identically zero near $(\bar{k}_1'^i, \bar{k}_j'')$; and $r'(k_1'^i, k_j')$ is either identically zero or is of order higher than n in $(k_2'^\mu - \bar{k}_2'^\mu)$. Since $a(k_1'^i, k_j'')$ is not identically equal to zero near $(\bar{k}_1'^i, \bar{k}_j'')$, one can find points arbitrarily close to $(\bar{k}_1'^i, \bar{k}_j'')$ such that $a(k_1'^i, k_j'')\neq 0$ at these points. Because $a(k_1'^i, k_j'')$ is analytic, these points can be taken to lie in the set $\mathrm{Im}k_1'^i = 0$.

For any neighborhood $\mathfrak{N}$ of $\bar{k}'$ we can therefore find a point k' with $\mathrm{Im}k_1'^i = 0$ such that $a(k_1'^i, k_j'')$ is nonzero and $(k_1'^i, k_2'^\mu, k_j'')$ lies in $\mathfrak{N}$ for $|k_2'^\mu - \bar{k}_2'^\mu|$ sufficiently small. Take $|k_2'^\mu - \bar{k}_2'^\mu|$ small enough so that the magnitude of the second term on the right-hand side of (A30) is very small compared to the first term. Then consider

$$\mathrm{Im}k_1'^0 = \mathrm{Im}f(k_1'^i, \bar{k}_j') + \mathrm{Im}[a(k_1'^i, k_j'')(k_2'^\mu - \bar{k}_2'^\mu)^n] + \mathrm{Im}r'(k'^i, k_j'). \tag{A31}$$

The first term on the right-hand side is zero, because $f(k_1'^i, \bar{k}_j')$ is by hypothesis a real analytic function of the arguments $k_1'^i$, and these are by construction real. The phase of $(k_2'^\mu - \bar{k}_2'^\mu)$ can be adjusted so that the second term times $\sigma_{\lambda\gamma}$ is positive and large compared to the last term. But then the point k' defined by (A31) lies in the k'-space image of Γ_λ. This gives the desired contradiction.

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*This work was supported in part by the U. S. Atomic Energy Commission.

†Participating guest, LBL.

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