Late-time evolution of charged gravitational collapse and decay of charged scalar hair. III. Nonlinear analysis

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We study the nonlinear gravitational collapse of a charged massless scalar field. We confirm the existence of oscillatory inverse power-law tails along future timelike infinity, future null infinity, and the future outer horizon. The nonlinear dumping exponents are in excellent agreement with the analytically predicted ones. Our results prove the analytic conjecture according to which a charged hair decays *slower* than a neutral one and also suggest the occurrence of mass inflation along the Cauchy horizon of a *dynamically* formed charged black hole. [S0556-2821(98)06214-6]

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I. INTRODUCTION

Linearized perturbation analysis has revealed important dynamical features of the gravitational collapse of massless neutral fields. In particular, according to the linearized perturbation theory there are two major features which characterize the late stages of the evolution: quasinormal ringing and inverse power-law tails (which follow the quasinormal ringing). These late-time tails are relevant to two major aspects of black-hole physics: The *no-hair theorem* of Wheeler and the *mass-inflation* scenario [1]. The mechanism responsible for the development of these neutral inverse power-law tails was first studied by Price [2]. This work was further extended by Gundlach, Price, and Pullin [3]. These authors have also shown that the nonlinear dumping exponents, describing the falloff of a massless neutral scalar field at late times, are in good agreement with the prediction of the linearized perturbation analysis [4].

The late-time evolution of a charged massless scalar field and the physical mechanism for the radiation of a charged hair were first studied analytically in [5,6], hereafter referred to as papers I and II, respectively. The main result presented there is the existence of oscillatory inverse power-law tails along the asymptotic regions of future timelike infinity, future null infinity, and the future outer horizon with *smaller* (compared to the neutral case) dumping exponents. These dumping exponents depend on the black hole's (or the star's) charge. In this paper we study numerically the fully nonlinear gravitational collapse of a massless charged scalar field. We confirm that the late-time behavior of the fully nonlinear evolution is in excellent agreement with the analytical predictions of the linearized analysis.

The plan of the paper is as follows. In Sec. II we give a short review of the linearized analytical results of papers I and II, on which we base our expectations for the fully nonlinear case. In Sec. III we describe our physical system, namely, the coupled Einstein-Maxwell charged scalar equations. In Sec. IV we study the late-time evolution of the nonlinear spherical charged gravitational collapse. In Sec. V we study the late-time evolution of nonspherical perturbations on a fixed Reissner-Nordström background and on a time-dependent background. In all cases we compare the nonlinear results with the predictions of the linearized analytical theory. We conclude in Sec. VI with a brief summary of our results and their physical implications.

II. REVIEW OF LINEARIZED ANALYTICAL RESULTS

We study numerically the late-time behavior of the fully nonlinear gravitational collapse of a massless charged scalar field. Our expectations are based on the linearized analytical results of papers I and II. Because of the *smallness* of the field's amplitude at late times, we expect these results to hold even in the fully nonlinear case. In other words, we expect that the late-time field may be regarded as a small perturbation. Thus, our expectations include the existence of inverse power-law tails along the three asymptotic regions: timelike infinity i_+, future null infinity $scri_+$, and the black-hole outer horizon H_+ (where the power law is multiplied by a periodic term). Quantitatively, in paper II it was found that the late-time behavior of linearized charged scalar perturbations (on a Reissner-Nordström background) is dominated by power-law tails of the form

$$\psi_m^l = A_{ti}(l,eQ) y^{\beta+1} t^{-(2\beta+2)} \tag{1}$$

at timelike infinity i_+,

$$\psi_m^l = A_{ni}(l,eQ) v_e^{-ieQ} u_e^{-(\beta+1-ieQ)} \tag{2}$$

at future null infinity $scri_+$ and

$$\psi_m^l = A_{eh}(l,eQ) e^{i(eQ/r_+)y} v_e^{-(2\beta+2)} \tag{3}$$

along the black-hole outer horizon H_+, where the charged scalar field ϕ is given by

$$\phi = e^{ie\Phi t} \sum_{l,m} \psi_m^l(t,r) Y_l^m(\theta,\varphi)/r \quad , \tag{4}$$

where Φ is merely a gauge constant of the electromagnetic potential A_t, i.e., its value at infinity (A_t is given by $A_t = \Phi - Q/r$). The constant $\beta(l,eQ)$ is given by

$$\beta = \frac{-1+\sqrt{(2l+1)^2-4(eQ)^2}}{2} \quad . \tag{5}$$

The tortoise radial coordinate y is defined by $dy = dr/\lambda^2$, where $\lambda^2 = 1 - 2M/r + Q^2/r^2$. Here $v_e \equiv t + y$ and $u_e \equiv t - y$ are the ingoing and outgoing Eddington-Finkelstein null coordinates, respectively. The expressions for the coefficients $A(l,eQ)$ are given in paper II.

III. EINSTEIN-MAXWELL CHARGED SCALAR EQUATIONS

We consider a spherically symmetric self-gravitating charged scalar field ϕ. The system is described by the coupled Einstein-Maxwell charged scalar equations. This physical system was already described in a previous paper [7]. Here we give only a short description of the system and the final form of the equations studied. We express the metric of a spherically symmetric spacetime in the form [8,9]

$$ds^2 = -g(u,r)\bar{g}(u,r)du^2 - 2g(u,r)dudr + r^2 d\Omega^2 \quad , \tag{6}$$

in which u is a retarded time null coordinate and the radial coordinate r is a geometric quantity which directly measures the surface area. The coordinates have been normalized so that u is the proper time on the $r=0$ central world line. We use the auxiliary field $\tilde{\phi}$ defined by

$$\phi \equiv \frac{1}{r}\int_0^r \tilde{\phi} \; dr \quad , \tag{7}$$

in terms of which the Einstein equations yield

$$g(u,r) = \exp\left[4\pi \int_0^r \frac{(\tilde{\phi}-\phi)(\tilde{\phi}^*-\phi^*)}{r} dr\right] \tag{8}$$

and

$$\bar{g}(u,r) = \frac{1}{r}\int_0^r \left(1 - \frac{Q^2}{r^2}\right) g \; dr \quad . \tag{9}$$

The electromagnetic potential A_u is given by the Maxwell equations

$$A_u = \int_0^r \frac{Q}{r^2} g \; dr \quad , \tag{10}$$

where the charge $Q(u,r)$ within a sphere of radius r, at a retarded time u, is given by

$$Q(u,r) = 4\pi i e \int_0^r r(\phi^* \tilde{\phi} - \phi \tilde{\phi}^*) dr \quad . \tag{11}$$

The mass $M(u,r)$ within a sphere of radius r, at a retarded time u, is given by

$$M(u,r) = \int_0^r \left[2\pi \frac{\bar{g}}{g}(\tilde{\phi}-\phi)(\tilde{\phi}^*-\phi^*) + \frac{1}{2}\frac{Q^2}{r^2}\right] dr + \frac{1}{2}\frac{Q^2}{r} \quad . \tag{12}$$

Finally, the wave equation for the charged scalar field takes the form of a pair of coupled differential equations

$$\frac{d\tilde{\phi}}{du} = \frac{1}{2r}(g-\bar{g})(\tilde{\phi}-\phi) - \frac{Q^2}{2r^3}(\tilde{\phi}-\phi)g - \frac{ieQ}{2r} g\phi - ie\tilde{\phi}A_u \tag{13}$$

and

$$\frac{dr}{du} = -\frac{1}{2}\bar{g}. \tag{14}$$

We solve this system of equations numerically. A detailed description of our algorithm, numerical methods, discretization, and error analysis is given in Ref. [7].

In order to compare our nonlinear results with the linearized analytical results of papers I and II, we use another spacetime coordinate, namely, the Bondi time $t_B \equiv t(u,\infty)$, which is the retarded time coordinate that agrees with time at infinity. The proper time $t(u,r)$ along an $r=$const trajectory is given by [4]

$$t(u,r) \equiv \int_0^u \sqrt{g(u',r)\bar{g}(u',r)} du' \quad . \tag{15}$$

The comparison with the analytical results also requires the use of the ingoing and outgoing Eddington-Finkelstein null coordinates v_e and u_e. In the asymptotic region $r \gg M_{BH}$, v_e is linear with r (along a $u=$const ray). In particular, $v_e \simeq 2y \simeq 2r$ along the $u = u_e = 0$ ray. In a similar way u_e is linear with t along a $v=$const ray, namely, $u_e = 2t - v_e$ (one may also use the asymptotic relation $u_e \simeq v_e - 2r$ along a $v=$const ray in order to evaluate u_e). The relation between A_u and A_t implies that $A_t(r_+) = 0$, i.e., $\Phi = Q/r_+$.

IV. NONLINEAR SPHERICAL COLLAPSE

In this section we present our results for the nonlinear spherically symmetric gravitational collapse of the self-gravitating massless charged scalar field. We have focused our attention on the behavior of the charged field ϕ along the three asymptotic regions: timelike infinity i_+, null infinity $scri_+$ and the black-hole outer horizon H_+. The late-time evolution of a charged scalar field is independent of the form of the initial data. The numerical results presented here have an initial profile of the form

$$\phi(u=0,r) = Ar^2 \exp\{-[(r-r_0)/\sigma]^2\} \quad , \tag{16}$$

where $r_0 = 2.5, \sigma = 0.5$ for the real part of the complex field and $r_0 = 3.0, \sigma = 0.5$ for the imaginary part. Figure 1 displays the time evolution of ϕ_R, the real part of the charged scalar field, along these asymptotic regions for a supercritical evolution ($A = 0.005, e = 0.85$) in which a black hole forms. The mass and charge of the formed black hole are $M_{BH} = 0.503$ and $Q_{BH} = -0.420$, respectively. The top panel shows the behavior of the field at a constant radius (here $r=10$) as a function of the Bondi time. The middle panel shows the be-

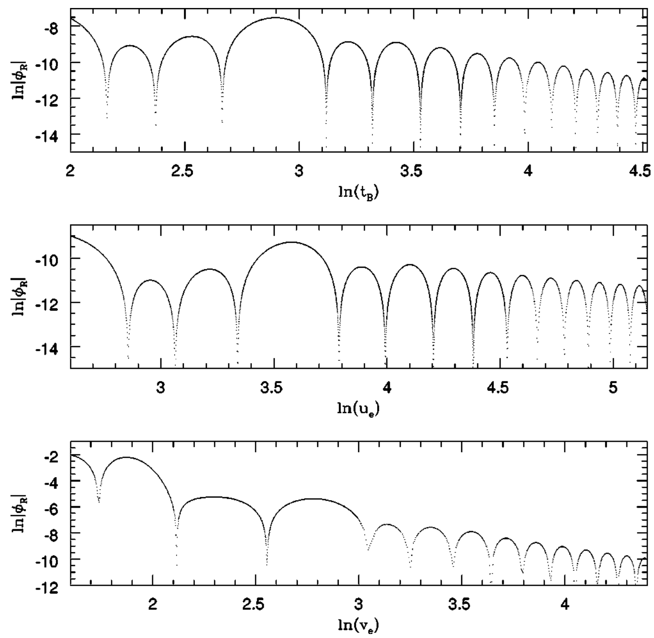

FIG. 1. Supercritical evolution of the charged field $|\phi_R|$ along the asymptotic regions of timelike infinity (top panel), null infinity (middle panel), and the black-hole outer horizon (bottom panel). The initial data are a Gaussian distribution (multiplied by an r^2 factor). The field's amplitude is $A=0.005$ and $e=0.85$. The mass and charge of the formed black hole are $M_{BH}=0.503$ and $Q_{BH}=-0.420$, respectively. A definite oscillatory power-law falloff is manifest at late times. The nonlinear power-law exponents are -1.78 at timelike infinity (here $r=10$), -0.86 along null infinity, and -1.96 along the black-hole outer horizon. These values are to be compared with the analytically predicted values of -1.70, -0.85, and -1.70, respectively. The oscillation frequency of the charged field ϕ agrees with the analytically predicted value to within 2%.

havior of the field along null infinity (approximated by the null surface $v=v_{max}$, where v_{max} is the largest value of v on the grid) as a function of u_e. The bottom panel shows the behavior of the field along the black-hole outer horizon H_+ (approximated by the null surface $u=u_{max}$, where u_{max} is the largest value of u on the grid) as a function of v_e. Initially, the evolution is dominated by the prompt contribution and quasinormal ringing. However, at late times a definite oscillatory power-law falloff is manifest. The nonlinear power-law exponents (determined from the maximas of these oscillations) are -1.78 at timelike infinity, -0.86 along null infinity, and -1.96 along the black-hole outer horizon. These values should be compared with the analytically predicted values of -1.70, -0.85, and -1.70, [see Eqs. (1)–(3)], respectively. The theoretical values of the oscillation frequencies of the charged scalar field ϕ are eQ_{BH}/r_+ at timelike infinity and along the black-hole outer horizon and $eQ_{BH}/2r_+$ along null infinity [see Eq. (4) and the relation $\Phi=Q/r_+$]. These values agree to within 2% with the numerical estimates.

The numerical values of the dumping exponent at timelike infinity, null infinity, and along the horizon differ from the theoretical ones by 5%, 1%, and 15%, respectively. This follows, first, from the oscillations of the field, which force us to use only a few numerical points — the maxima of those oscillations — to estimate the decay exponents. This problem does not appear in the neutral case in which there are no oscillations. An additional difficulty arises at the horizon where the divergence of $g(u,r)$ and $\bar{g}(u,r)$ (which leads in our scheme to smaller and smaller du steps) limits our ability to approach the horizon. Consequently, the numerical values of the charged dumping exponents are determined only for finite values of t, u_e, and v_e while the tail phenomena is manifested only asymptotically. Given those difficulties we consider the numerical values of the dumping exponents as acceptable.

TABLE I. Dependence of the dumping exponents at timelike infinity i_+ on eQ.

e	Q_{BH}	$2\beta+2$	Nonlinear exponents
0.50	-0.241	-1.97	-1.99
0.70	-0.344	-1.88	-1.93
0.85	-0.420	-1.70	-1.78
0.90	-0.443	-1.60	-1.65

While the dumping exponents and the decay rate of neutral perturbations are independent of the spacetime parameters (M and Q), the analysis of paper II predicts that the charged dumping exponents depend on the black hole's (or the star's) charge (namely, on the dimensionless quantity $|eQ|$). In order to test this linearized analytical prediction, we have studied the dependence of the charged dumping exponents at timelike infinity on the parameter e. The initial form of the field is given by Eq. (16) and the amplitude is set to $A=0.005$. The nonlinear results are summarized in Table I. We find good agreement between the numerically measured exponents and the linearized predicted ones.

The analysis of paper I predicts the existence of charged power-law tails even in subcritical evolution when there is no collapse to a black hole (and all we have are imploding and exploding shells). This phenomenon is related to the fact that the late-time evolution of the field is dominated by the backscattering from asymptotically *far* regions, and it does *not* depend on the small-r details of the spacetime (this is also the situation for neutral massless perturbations [3]). Since the charge of the spacetime is a dynamical quantity, it is not clear which value of Q should be taken in calculating the value of the dumping exponent. However, since the charge of the spacetime falls to zero asymptotically, we expect the effective value of $|eQ|$ to be much *smaller* than unity. In this case the dumping exponents are expected to be $2l+2$ at timelike infinity and $l+1$ along null infinity (with small corrections of the order of $O[(eQ)^2]$). Figure 2 displays the time evolution of ϕ_R for a subcritical evolution ($A=0.002$, $e=3$). Shown is the behavior of the field at a constant radius (here $r=10$) as a function of Bondi time and along null infinity as a function of u_e. The nonlinear power-law exponents are -2.00 at timelike infinity and -1.00 along null infinity. These values are exactly equal to those predicted in paper I.

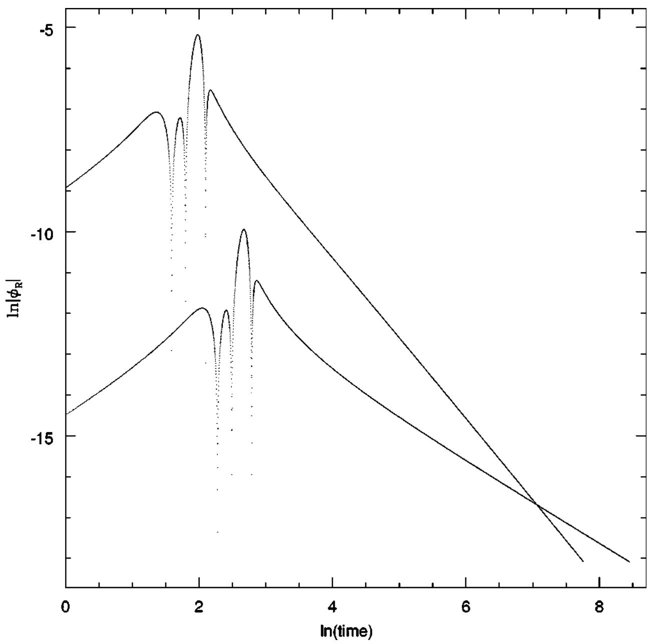

FIG. 2. Subcritical evolution of the charged field $|\phi_R|$. The initial form of the field is the same as in Fig. 1. The field's amplitude is $A=0.002$ and $e=3$. The field at future timelike infinity ($r=10$) is shown as a function of t_B. Along null infinity the field is shown as a function of u_e. A definite power-law falloff is manifest at late times. The nonlinear power-law exponents are -2.00 at timelike infinity and -1.00 along null infinity. These values are exactly the ones predicted by the linearized analytical approach.

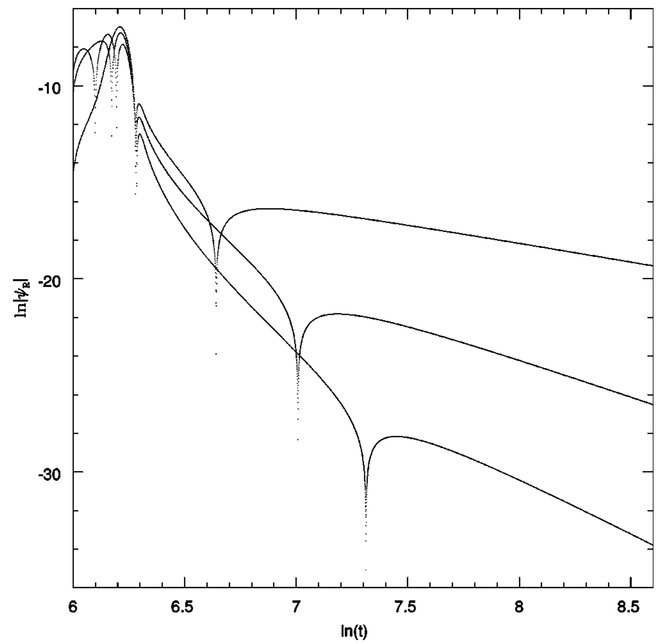

FIG. 3. Evolution of the charged field $|\psi_R(y=400,t)|$ on a fixed Reissner-Nordström background for different multipoles $l=0$, 1, and 2 (from top to bottom, respectively). The black-hole mass and charge are set equal to $M_{BH}=0.5$ and $Q_{BH}=0.45$, respectively. The field's charge is $e=0.01$. The initial data are a Gaussian distribution with $v_0=100$ and $\sigma=20$ (for both the real and the imaginary parts of the field). A definite power-law falloff is manifest at late times. The power-law exponents are -1.97, -3.94, and -5.75 for the $l=0$, 1, and 2 modes, respectively. These values are to be compared with the analytically predicted values of -2.0, -4.0, and -6.0.

V. NONSPHERICAL PERTURBATIONS OF SPHERICAL COLLAPSE

In order to study the dependence of the late-time dumping exponents on the multipole index l we have performed two series of numerical investigations. First, we have integrated the linearized charged scalar-field equation

$$\psi_{,tt}+2ie\frac{Q}{r}\psi_{,t}-\psi_{,yy}+V\psi=0 \quad , \tag{17}$$

where

$$V=V_{M,Q,l,e}(r) =\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}-\frac{2Q^2}{r^4}\right]-e^2\frac{Q^2}{r^2} \quad , \tag{18}$$

on a fixed Reissner-Nordström background. It is straightforward to integrate Eq. (17) using the method described in [3]. The late-time evolution of a charged scalar field is independent of the form of the initial data. The results presented here are of a Gaussian pulse on $u=0$,

$$\psi(u=0,v)=A\ \exp\{-[(v-v_0)/\sigma]^2\} \quad , \tag{19}$$

where the amplitude A is physically irrelevant due to the linearity of Eq. (17). It should be noted that the evolution equation (17) is invariant under the rescaling

$$r\to ar, \quad t\to at, \quad M\to aM, \quad Q\to aQ, \quad e\to e/a, \tag{20}$$

where a is some positive constant. The black-hole mass and charge are set equal to $M_{BH}=0.5$ and $Q_{BH}=0.45$, respectively. We have chosen $e=0.01$, $v_0=100$, and $\sigma=20$ (for both the real and the imaginary parts of the charged field). The numerical results for the $l=0$, 1, and 2 modes are shown in Fig. 3 (from top to bottom, respectively). This figure demonstrates the dependence of the late-time tails at i_+ (here $y=400$) on the multipole index l. A definite power-law falloff is manifest at late times. The numerical values of the power-law exponents, describing the falloff of the field at late times, are $-1.97, -3.94$, and -5.75 for $l=0$, 1, and 2, respectively. These values are to be compared with the analytically predicted values of $-2.0, -4.0$, and -6.0, respectively.

In a second series of numerical investigations, we have studied the evolution of a second perturbative charged scalar field ξ evolved on a time-dependent spacetime. This time-dependent spacetime is determined by the background solution ϕ, while we ignore the contribution of the field ξ to the energy-momentum tensor (this is analogous to the case studied in [4] for a neutral field). Resolving the field into spherical harmonics $\xi=\Sigma_{l,m}\xi^l_m(t,r)Y^m_l(\theta,\varphi)$ (and using the spherical symmetry of the time-dependent background) one obtains a wave equation for each multiple moment of the field ξ:

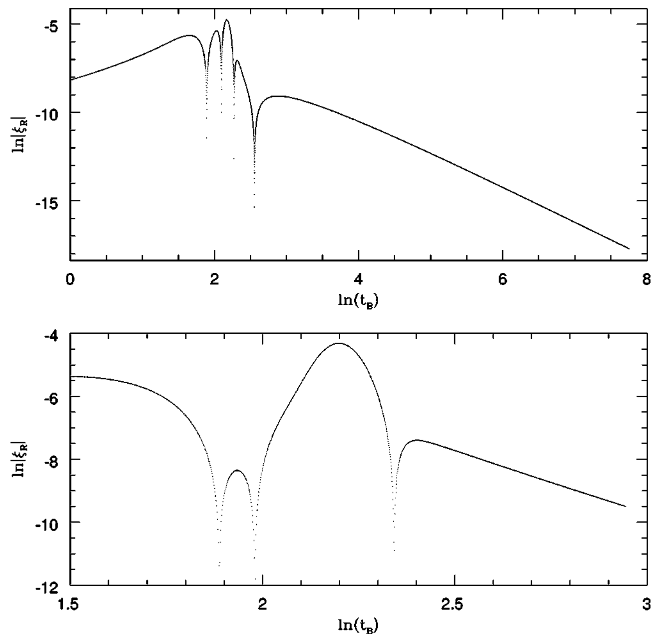

FIG. 4. Evolution of the charged test field $|\xi_R(r=10,t)|$ on a time-dependent spacetime. The initial data for the background field ϕ are those of Fig. 2 (noncollapsing case). The initial data for the test field ξ are a Gaussian distribution (multiplied by an r^2 factor). A definite power-law falloff is manifest at late times. The power-law exponents are -1.99 for the $l=0$ mode (top panel) and -4.16 for the $l=1$ mode (bottom panel). These values are to be compared with the analytically predicted values of -2.0 and -4.0, respectively.

$$\frac{d\tilde{\xi}^l_m}{du}=\frac{1}{2r}(g-\bar{g})(\tilde{\xi}^l_m-\xi^l_m)-\frac{Q^2}{2r^3}(\tilde{\xi}^l_m-\xi^l_m)g$$
$$-\frac{ieQ}{2r}g\xi^l_m-ie\tilde{\xi}^l_m A_u-\frac{1}{2r}l(l+1)g\xi^l_m \qquad (21)$$

[together with Eq. (14)], where

$$\xi^l_m\equiv\frac{1}{r}\int_0^r\tilde{\xi}^l_m dr \quad . \qquad (22)$$

The quantities g, $\bar{g}$, and Q are still determined by the background solution for the field ϕ. For the background field ϕ we choose the same initial form as in Sec. IV. The most interesting time-dependent backgrounds are the *noncollapsing* ones. Thus, we take the subcritical initial conditions $A=0.002$ and $e=3$ of Sec. IV. For the perturbation field ξ we choose initial data of the form (16) with $r_0=4.0$ and $\sigma=0.5$ for both the real and the imaginary parts of the field (again, the amplitude of the perturbation field is physically irrelevant). Figure 4 displays the time evolution of the perturbative charged scalar field ξ (its real part) at a constant radius (here $r=10$) as a function of Bondi time. It is interesting that even for time-dependent spacetimes the late-time behavior of charged fields is well described by an inverse power-law falloff. The numerical values of the power-law exponents are -1.99 for the $l=0$ mode (top panel) and -4.16 for the $l=1$ mode (bottom panel). These values should be compared with the fixed background analytically predicted values of -2.0 and -4.0, respectively.

VI. SUMMARY AND PHYSICAL IMPLICATIONS

We have studied the nonlinear gravitational collapse of a massless charged scalar field. Following the predictions of the linearized analytical theory (papers I and II) we have focused attention on the asymptotic late-time evolution of the charged field. Our main results and their physical implications are the following.

We have confirmed the existence of oscillatory inverse *power-law tails* in a *collapsing* spacetime along the asymptotic regions of future timelike infinity i_+, future null infinity $scri_+$ and the future outer horizon H_+. The *nonlinear* dumping exponents are in excellent agreement with the *analytically* predicted ones. In particular, we have verified the analytically conjectured dependence of the *charged* dumping exponents on the dimensionless quantity $|eQ|$. This dependence on the spacetime charge is contrasted to neutral perturbations, where the dumping exponents are fixed integers which do *not* depend on the spacetime parameters (namely, they are functions of the multipole index l only). We have confirmed the existence of power-law tails even in noncollapsing spacetimes (i.e., imploding and exploding shells).

We have also studied the late-time evolution of nonspherical perturbations (*charged* test fields) on a fixed Reissner-Nordström background and on a time-dependent background. In both cases we have found that the dumping exponents, describing the falloff of the charged field at late times, are in good agreement with the predictions of the linearized analytical theory. (In particular, we have verified the functional dependence of the dumping exponents on the multipole index l.)

Our nonlinear results verify the analytic conjecture that a *charged* hair decays *slower* than a neutral one. Furthermore, the existence of oscillatory inverse power-law tails along the black-hole outer horizon suggests the occurrence of the well-known phenomenon of *mass inflation* [1] along the Cauchy horizon of a *dynamically* formed charged black hole.

In a forthcoming paper we study the *fully nonlinear* gravitational collapse of a charged scalar field, using a different numerical scheme which is based on *double* null coordinates. This scheme allows us to start with *regular* initial conditions (at approximately past null infinity), calculating the *formation* of the regular black hole's event horizon, and continue the evolution all the way *inside* the black hole. Thus, this numerical scheme makes it possible to test the mass-inflation conjecture during the gravitational collapse of a *charged* scalar field. To our knowledge, the mass-inflation scenario has *never* been demonstrated explicitly before in *collapsing* situations. (The numerical work of Brady and Smith [10] begins on a Reissner-Nordström spacetime and the black-hole formation itself was *not* calculated there.)

ACKNOWLEDGMENTS

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