

Imposition of Cauchy data to the Teukolsky equation. II. Numerical comparison with the Zerilli-Moncrief approach to black hole perturbations

Manuela Campanelli*

Department of Physics, University of Utah, 201 JFB, Salt Lake City, Utah 84112

William Krivan†

*Department of Physics, University of Utah, 201 JFB, Salt Lake City, Utah 84112
and Institut für Astronomie und Astrophysik, Universität Tübingen, D-72076 Tübingen, Germany*

Carlos O. Lousto‡

*Instituto de Astronomía y Física del Espacio, Casilla de Correo 67, Sucursal 28, (1428) Buenos Aires, Argentina
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We reexamine the question of the imposition of initial data representing astrophysical gravitational perturbations of black holes. We study their dynamics for the case of nonrotating black holes by numerically evolving the Teukolsky equation in the time domain. In order to express the Teukolsky function Ψ explicitly in terms of hypersurface quantities, we relate it to the Moncrief waveform ϕ_M through a Chandrasekhar transformation in the case of a nonrotating black hole. This relation between Ψ and ϕ_M holds for any constant time hypersurface and allows us to compare the computation of the evolution of Schwarzschild perturbations by the Teukolsky and by the Zerilli and Regge-Wheeler equations. We explicitly perform this comparison for the Misner initial data in the close limit approach. We evolve numerically both the Teukolsky (with the recent code of Krivan *et al.*) and the Zerilli equations, finding complete agreement in resulting waveforms within numerical error. The consistency of these results further supports the correctness of the numerical code for evolving the Teukolsky equation as well as the analytic expressions for Ψ in terms only of the three-metric and the extrinsic curvature. [S0556-2821(98)07314-7]

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I. INTRODUCTION AND OVERVIEW

Binary black holes provide, in principle, one of the strongest sources of gravitational radiation to be observed by the detectors currently under construction. Black holes have proved to be very elusive to standard astronomical methods of detection. This is a consequence of the fact that astrophysical black holes, being one of the simplest objects in nature, interact only gravitationally with the rest of the universe. Thus, gravitational wave observatories are particularly suitable to detect them. The confirmation of the existence of black holes in nature can be considered to be as great a discovery as the confirmation of the existence of gravitational radiation.

This exciting new experimental-observational situation has taken theory a bit by surprise. In the early 1970s a great deal of effort was devoted to studying gravitational perturbations by means of the post-Newtonian and perturbative approaches. In 1979 Smarr [2] presented his pioneer work on the numerical approach to the full Einstein equations. Subsequently the field developed rather slowly, in part due to a shift of the research interests toward other fields and in part by the lack of concrete experimental support. The situation changed dramatically in the early 1990s with the launch of the interferometric detectors more or less simultaneously in

Europe and the U.S. Renewed efforts have been devoted to the full numerical approach in the form of the “Grand Challenge alliance” [3], to the post-Newtonian approximation [4], and the perturbative approach. It was the perturbative method which in 1994 led to the remarkable “close limit approximation” [5]. Its, perhaps unexpected, success in describing the final stage of binary black hole coalescence in terms of a single perturbed black hole was somewhat perturbing [6]. A complementary approach to this approximation is to consider not two black holes of similar masses (here the close limit gives its optimal approximation), but instead one much less massive than the other. This, again, allows us to treat the problem as a perturbation around a single black hole, although here the perturbation parameter is no longer the separation of the holes, but the mass ratio. This problem was recently reexamined and solved for a finite initial separation of the holes [7–9]. However, both the close and the particle limits have been studied as perturbations around nonrotating (Schwarzschild) black holes. We know that astrophysical black holes are most likely to be rotating, possibly with considerable angular momentum. This means that we should take as the background metric the Kerr rather than the Schwarzschild geometry.

Using the Newman-Penrose formalism, Teukolsky [10] succeeded in giving a decoupled equation for perturbations around a Kerr background. Notably, this equation can be separated in all its variables. The Teukolsky equation has been studied and solved in the *frequency* domain for gravitational radiation generated by a particle infalling from infinity or in a circular orbit around a Kerr hole [11]. Sasaki and

*Electronic address: manuela@mail.physics.utah.edu

†Electronic address: krivan@mail.physics.utah.edu

‡Electronic address: lousto@iafe.uba.ar

Nakamura [12] have given a modified version of the radial Teukolsky equation in the frequency domain that is better behaved (see Ref. [13] though) and suitable for numerical integration. Different trajectories of a particle infalling towards a Kerr black hole have been studied in this way [14]. The disadvantage of this approach, however, is that, in general, it is much more efficient to perform numerical integrations in the *time* domain. This was concretely our experience in Ref. [8], where in order to reproduce in the time domain the results of Ref. [7] obtained in the frequency domain, the running time of the program was reduced by three orders of magnitude.

A successful code for the numerical integration of the Teukolsky equation in the time domain was formulated only very recently [1]. It has been used to study quasinormal mode excitation, superradiance, and the power law behavior of the (late time) gravitational radiation tails on the Kerr background. Also, consistency checks, similar to some of the material presented here, were performed during the development of the code. Given the success of the close limit approximation for the head-on collision of black holes, it is very interesting to study the same problem in the astrophysically more realistic situation of two merging black holes in an inspiral orbit. Presumably, this situation can be studied in terms of a single, highly rotating, perturbed Kerr black hole. To complete the solution of this problem, we have to provide the evolution code with consistent initial data. There are two kinds of problems that we have to address now. First, *what* initial data do we take? Here the problem is that, so far, all initial data available in the literature representing binary black holes assumed the initial three-geometry to be conformally flat. But since a $t = \text{const}$ slice of the Kerr geometry (at least in Boyer-Lindquist coordinates) is not conformally flat, we cannot consistently use those initial data. Only very recently a first step towards a different approach has been given by assuming the perturbed metric to have the Kerr-Schild form [15]. A second practical problem appears even if one had consistent initial data. *How* to impose them in terms of the initial Teukolsky waveform Ψ in order to start the evolution? We have begun to discuss this problem in Ref. [16]. The problem here is that Ψ is built up as a contraction of the perturbed Weyl tensor with the background (Kerr) null tetrad. This contraction, when written in terms of metric perturbations, depends explicitly not only on the hypersurface quantities (three-metric $^{(3)}g_{ij}$ and extrinsic curvature K_{ij}), but also on the lapse function N and the shift vector N^i . We then have to rewrite Ψ in terms of $^{(3)}g_{ij}$ and K_{ij} for $t=0$ (i.e., on the initial hypersurface) by making use of not only the constraints but also the evolution equations. In practice this task proved to be nontrivial. This is in contrast with the description, mostly after the work of Zerilli [17] and of Moncrief [18], of metric perturbations around a Schwarzschild hole. In this case, the Moncrief waveform ϕ_M depends explicitly on the perturbed three-geometry only and $\partial_t \phi_M$ can be written in terms of the extrinsic curvature only. (There are actually two waveforms representing even and odd parity perturbations and they correspond, respectively, to the real and imaginary parts of the Teukolsky waveform Ψ .) Our aim is to bring the Newman-Penrose-Teukolsky formulation to

an equally nice footing. One first step that can be done is, at least in the limit of a nonrotating background, to find a relation between Ψ and ϕ_M on the initial hypersurface. Such a relation was found in Ref. [16], making use of the simplifications introduced by choosing the Regge-Wheeler gauge and the possibility to restore the gauge invariance at the end of the calculation.

When one speaks of transformations between solutions of Teukolsky and Zerilli (or Regge-Wheeler) equations, one immediately associates with them the name of Chandrasekhar for his contributions to the understanding of these relations. In the next section, we extend Chandrasekhar's transformations to the time domain in order to relate Ψ to ϕ_M , and then be able to rewrite the Teukolsky waveform exclusively in terms of hypersurface quantities. We test this relation, which holds for every hypersurface, by numerically evolving initial data corresponding to the close limit in two independent ways: by the Teukolsky equation with the code detailed in Ref. [1] and by the Zerilli equation. We then compare waveforms, making use of the above relations, at later times, and check their agreement numerically. In turn, this provides further support about the correctness of the numerical code for evolving the Teukolsky equation. The numerical results are displayed in Sec. III. We conclude this paper with a brief discussion of our results and their generalization to the rotating background case.

II. CHANDRASEKHAR TRANSFORMATION IN THE TIME DOMAIN

A. Teukolsky equation

Let us briefly review the Newman-Penrose-Teukolsky description of perturbations around the Kerr metric. Gravitational perturbations with spin weight $s = \pm 2$ are compactly written in terms of contractions of the Weyl tensor

$$\Psi(t, r, \theta, \varphi) = \begin{cases} \rho^{-4} \Psi_4 \equiv -\rho^{-4} C_{n\bar{m}n\bar{m}} & \text{for } s = -2, \\ \Psi_0 \equiv -C_{lm\bar{l}\bar{m}} & \text{for } s = +2, \end{cases} \quad (2.1)$$

where an overbar means complex conjugation, $\rho \equiv 1/(r - ia \cos \theta)$, and we have considered Boyer-Lindquist (t, r, θ, ϕ) coordinates. This field represents either the outgoing radiative part of the perturbed Weyl tensor ($s = -2$) or the ingoing radiative part ($s = +2$). The components of the Kinnersley null tetrad [19] are given by

$$(l^a) = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right), \quad (2.2a)$$

$$(n^a) = \frac{1}{2(r^2 + a^2 \cos^2 \theta)} (r^2 + a^2, -\Delta, 0, a), \quad (2.2b)$$

$$(m^a) = \frac{1}{\sqrt{2}(r + ia \cos \theta)} (ia \sin \theta, 0, 1, i/\sin \theta). \quad (2.2c)$$

The Weyl scalars then satisfy the Teukolsky equation

$$\left\{ \left[a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right] \partial_{tt} - \frac{4Mar}{\Delta} \partial_{t\varphi} - 2s \left[(r + ia \cos \theta) - \frac{M(r^2 - a^2)}{\Delta} \right] \partial_t + \Delta^{-s} \partial_r (\Delta^{s+1} \partial_r) \right. \\ \left. + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \left(\frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \partial_{\varphi\varphi} + 2s \left[\frac{a(r-M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \partial_\varphi - (s^2 \cot^2 \theta - s) \right\} \Psi = 4\pi \Sigma T, \quad (2.3)$$

where M is the mass of the black hole, a its angular momentum per unit mass, $\Sigma \equiv r^2 + a^2 \cos^2 \theta$, and $\Delta \equiv r^2 - 2Mr + a^2$. The source term T is built up from the energy-momentum tensor [10].

B. Zerilli equation and Moncrief waveform

There is a historically independent way of computing gravitational perturbations around a Schwarzschild black hole (it can be likewise extended to any spherically symmetric background), developed mainly by Regge and Wheeler [20], Zerilli [17], and Moncrief [18]. In that formalism, the two degrees of freedom of the graviton are represented by two scalar quantities ϕ_M , the even and odd parity waveforms. After decomposition of the angular part in terms of spherical harmonics $Y_\ell^m(\theta, \phi)$, ϕ_M satisfies a wave equation

$$-\frac{\partial^2 \phi_M}{\partial t^2} + \frac{\partial^2 \phi_M}{\partial r^{*2}} - V_\ell(r) \phi_M = S_\ell(r, t). \quad (2.4)$$

Here $r^* \equiv r + 2M \ln(r/2M - 1)$, S_ℓ is the contribution of the source terms, and V_ℓ is the potential due to the curved background (slightly different for the even and odd parity waves).

The even and odd parity waveforms in terms of metric perturbations in the Regge-Wheeler notation [20] take the form

$$\phi_M(r, t) = \begin{cases} \frac{r}{\lambda + 1} \left[K + \frac{r - 2M}{\lambda r + 3M} \{H_2 - r \partial K / \partial r\} \right] + \frac{(r - 2M)}{\lambda r + 3M} (r^2 \partial G / \partial r - 2h_1), \\ \frac{1}{r} \left(1 - \frac{2M}{r} \right) \left[h_1 + \frac{1}{2} \left(\partial_r h_2 - \frac{2}{r} h_2 \right) \right], \end{cases} \quad (2.5)$$

where

$$\lambda \equiv (\ell + 2)(\ell - 1)/2, \quad (2.6)$$

and ℓ is the multipole index. The field ϕ_M explicitly depends only on the three-geometry. Likewise, one can write $\partial_t \phi_M$ exclusively in terms of the extrinsic curvature, using the same functional form of ϕ_M as above [21]:

$$\partial_t \phi_M = -2 \phi_M \{ (1 - 2M/r)^{1/2} \delta K_{ij}, \partial_r [(1 - 2M/r)^{1/2} \delta K_{ij}] \}. \quad (2.7)$$

C. Relation between waveforms

The Teukolsky function can be decomposed into angular modes [16]

$$\Psi_4(t, r, \theta, \phi) = \sum_m \Psi_4^\ell(t, r) {}_{-2}Y_\ell^m(\theta, \phi), \quad (2.8)$$

where we have made use of the fact that the spin-weighted spheroidal harmonics, in the case $a\omega = 0$, reduce to the spin-weighted spherical harmonics, i.e., $e^{im\phi} {}_{-2}S_\ell^m(\theta, a\omega = 0) = {}_{-2}Y_\ell^m(\theta, \phi)$.

Chandrasekhar transformations provide a differential operator (first order in the frequency domain, but second order in the time domain) that links solutions of the Teukolsky equation with solutions of the Zerilli or Regge-Wheeler equations and vice versa. Chandrasekhar found his transformations in the frequency domain, but in the nonrotating

background case they can be easily extended to the time domain as [see Eqs. (3.353) and (3.345) of Ref. [22]]

$$\begin{cases} \text{Re } \Psi_4(t, r, \theta, \phi) \\ -\text{Im } \partial_t \Psi_4(t, r, \theta, \phi) \end{cases} = \sum_m \frac{C}{r} \left[-2 \partial_{tr^*} + 2 \partial_{r^* r^*} \right. \\ \left. - V^\pm(r) + W^\pm(r) (\partial_{r^*} \mp \partial_t) \right] \\ \times \phi_M {}_{-2}Y_\ell^m(\theta, \phi), \quad (2.9)$$

where $C = -\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}/16$. This equation has to be understood as the real part of Ψ_4 , being equal to the even parity counterpart (labeled as “+”), and the imaginary part of $-\partial_t \Psi_4$ equal to the odd parity terms (labeled as “−”). In the above equation $V^\pm(r)$ denote the Zerilli [17] and Regge-Wheeler [20] potentials, respectively, and

$$W^+(r) = 2 \frac{\lambda r^2 - 3\lambda M r - 3M^2}{r^2(\lambda r + 3M)}, \quad W^-(r) = \frac{2(r - 3M)}{r^2}. \quad (2.10)$$

Chandrasekhar also gives the inverse transformation for ϕ_M in terms of a differential operator acting on Ψ_4 [see Eqs. (3.319) of Ref. [22]].

We have checked analytically, in the close limit case (see below), that Eq. (2.9) leads to exactly the same relation as Eq. (2.8) of Ref. [16] for all times.

III. NUMERICAL TESTS

A. Close limit approximation

One of the more outstanding results of perturbation theory in the last years has been its application to the so called “close limit approximation.” The basic idea is to regard the collision of two black holes in its merger stage as a *single* perturbed black hole. When applied to two equal mass holes, this approximation gave excellent agreement with the full, nonperturbative, numerical computations, even going to not so small separations [5]. The simplest application of this method considers two black holes in head-on collision. The resulting final black hole will possess no spin and can then be studied as a perturbation around a Schwarzschild metric. For the two black holes starting from rest at a given separation, we have analytic expressions for the conformally flat initial geometry. In the Regge-Wheeler notation for the metric perturbations the only nonvanishing components of the three-metric, $H_2 = K$, are given by

$$H_2(t=0, r, \theta) = \left(\frac{2M/R}{1 + \frac{M}{2R}} \right) \sum_{\ell=2,4,\dots}^{\infty} \sqrt{\frac{4\pi}{2\ell+1}} (Z_0/R)^{\ell} Y_{\ell}^0(\theta), \quad (3.1)$$

where Z_0 is half the distance between the two (equal mass) holes in the conformal space. The above equation holds for the Brill-Lindquist data [23]. The corresponding analysis for the Misner data is equivalent to the one above replacing $(Z_0/M)^{\ell} \rightarrow 4\kappa_{\ell}$. The results of the Brill-Lindquist case [23] can be compared with those of Ref. [24], while for the case of Misner data [25] one can compare with Refs. [5,26]. For the case of two different masses, see Ref. [27].

To compute ϕ_M from Eq. (2.5) we further make the identification of the conformal radial coordinate with the Schwarzschild isotropic coordinate: i.e.,

$$R \equiv \frac{1}{4} (\sqrt{r} + \sqrt{r-2M})^2. \quad (3.2)$$

This allows us to compute the form of ϕ_M (here only even parity waves are generated) and from ϕ_M we compute Ψ_4 making use of the relations (2.9). Also, in this case,

$$\partial_t \Psi_4(t=0) = -\frac{2M}{r^2} \Psi_4(t=0). \quad (3.3)$$

We now have the explicit form of the initial data for evolving the Teukolsky equation.

B. Numerical results

While gravitational perturbations can be described in terms of either sign of the spin-weight parameter s with $|s| = 2$, it was found more convenient from the numerical point of view to deal with the $s = -2$ waveform [1,28]. In particular, we make a further rescaling of Ψ_4 and mode decomposition in the azimuthal coordinate by defining

$$\Phi(t, r^*, \theta) \equiv \frac{e^{-im\tilde{\phi}}}{\rho^4 r^3} \Psi_4(t, r^*, \theta, \phi), \quad (3.4)$$

where

$$d\tilde{\phi} \equiv d\phi + \frac{a}{\Delta} dr, \quad (3.5)$$

$$\frac{dr^*}{dr} \equiv \frac{r^2 + a^2}{\Delta}. \quad (3.6)$$

The outgoing part of Φ satisfies the asymptotic conditions

$$\lim_{r^* \rightarrow +\infty} |\Phi| \sim 1, \quad (3.7)$$

$$\lim_{r^* \rightarrow -\infty} |\Phi| \sim 1, \quad (3.8)$$

while ingoing solutions, i.e., those propagating towards the black hole, are characterized by

$$\lim_{r^* \rightarrow +\infty} |\Phi| \sim 1/r^4, \quad (3.9)$$

$$\lim_{r^* \rightarrow -\infty} |\Phi| = 0. \quad (3.10)$$

In the following figures we display the results obtained from runs of the Teukolsky code described in Ref. [1] with grid spacings of $2\delta t = \delta r^* = 0.05, 0.1$, and $\delta\theta = \pi/64, \pi/32$. (We have set the mass of the Schwarzschild black hole to unity.) For the comparisons with the Teukolsky evolution, the (1+1) dimensional Zerilli equation [Eq. (2.4) with $\mathcal{S}_{\ell} \equiv 0$] was evolved with the same temporal and radial grid spacings as the Teukolsky equation. Previously, convergence tests have been performed by looking at the numerical errors for different steps of integration compared to exact solutions obtained by plugging a convenient analytic function into the Teukolsky equation and taking into account the resulting (artificial) source. In this way it is shown that the convergence is (almost) quadratic. The code can be used for any value of the multipole ℓ , but in practice we used it only for $\ell = 2$ and 4. This is so, because for the head-on collision of equal mass holes only even values of ℓ appear, and the mode $\ell = 2$

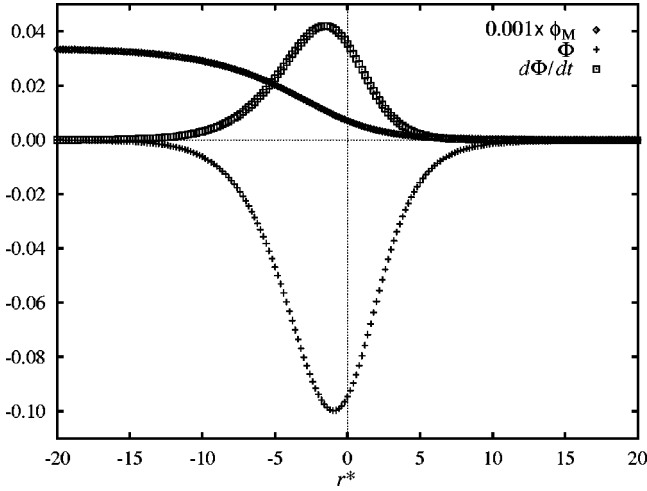


FIG. 1. The radial dependence of the initial data for the Zerilli equation, $0.001 \times \phi_M(t=0, r^*)$, and for the Teukolsky equation, $\Phi(0, r^*, \pi/2)$ and $\partial_t \Phi(0, r^*, \pi/2)$, for the $\ell=2$ mode.

already contributes with over 90% of the total radiated energy. If one considers black holes of unequal masses, one would obtain contributions from odd ℓ , but still the radiation is strongly dominated by the $\ell=2$ mode. It should be noted that this consideration of pure multipoles in the sense of spherical harmonics is only possible for nonrotating black holes: In the case of a rotating background, it is first of all impossible to generate initial data that represent a pure multipole specified by ℓ and m . But even if we started the evolution with a pure multipole, then other modes would be generated in the course of the evolution. Therefore different multipoles will be present in the evolution.

Figure 1 depicts the waveforms for $\ell=2$ at the initial time $t=0$. We show together the initial Moncrief waveform $0.001 \times \phi_M(t=0, r^*)$ for the evolution of the Zerilli equation and the initial data for the Teukolsky equation $\Phi(t=0, r^*, \pi/2)$ and $\partial_t \Phi(t=0, r^*, \pi/2)$. (The observer is located in the equatorial plane, i.e., $\theta = \pi/2$.) Here it is worth remarking on the fact that $\partial_t \Phi(t=0, r^*, \pi/2) \neq 0$ even for time-symmetric data [where $\partial_t \phi_M(t=0, r^*) = 0$]; i.e., see Eq. (3.3).

In Fig. 2 we depict the initial configuration for the $\ell=4$ mode.

Figure 3 shows the early behavior, as a function of the radial coordinate r^* , of $\Phi(t=3, r^*, \pi/2)$ as computed in two different ways with $\delta r^* = 0.05$ and $\delta \theta = \pi/64$: (i) from the evolution of $\phi_M(t=0, r^*)$ with the Zerilli equation and then using relation (2.9) to build up Ψ_4 at the given time and (ii) by directly evolving $\Psi_4(t=0, r^*, \pi/2)$ with the Teukolsky equation using the code of Ref. [1] with the parameter a set to zero. There is an agreement between the two evolution methods for $\ell=2$ waveforms within 2% of the maximum amplitude (expected numerical error).

Figures 4 and 5 show the same comparison of waveforms at a later time ($t=10$ in units of $M=1$), for $\ell=2$ ($\delta r^* = 0.1$, $\delta \theta = \pi/32$) and $\ell=4$ ($\delta r^* = 0.05$, $\delta \theta = \pi/64$), respectively, in order to check the agreement of the two methods of computation for different multipoles. We observe

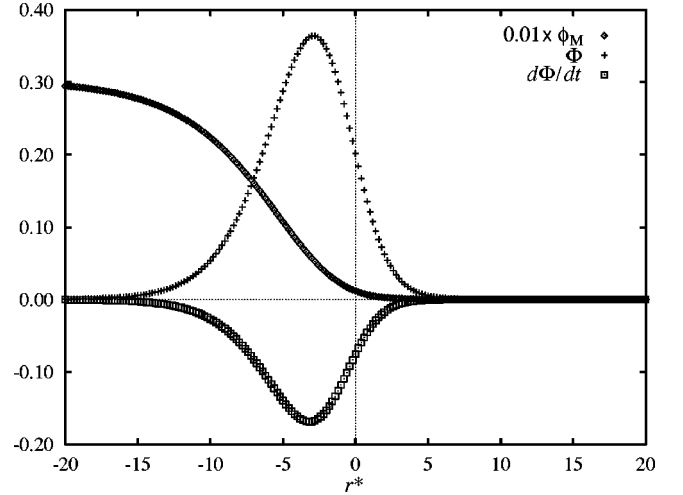


FIG. 2. The radial dependence of the initial data for the Zerilli equation, $0.01 \times \phi_M(t=0, r^*)$, and for the Teukolsky equation, $\Phi(0, r^*, \pi/2)$ and $\partial_t \Phi(0, r^*, \pi/2)$, for the $\ell=4$ mode.

again that the curves accord within 2% of the amplitude.

Figure 6 shows that the agreement of the waveforms evolved with the Zerilli and Teukolsky equations continues to hold until late times. (Here we display curves for $t=50$ obtained from an evolution with $\delta r^* = 0.1$ and $\delta \theta = \pi/32$.) Numerical errors are limited to 2% of the maximum amplitude of the waves.

These small errors have been obtained with short integration times and not so small steps in the coordinates. If we take them as representative of the errors in an astrophysically realistic computation, we confirm both the reliability and ef-

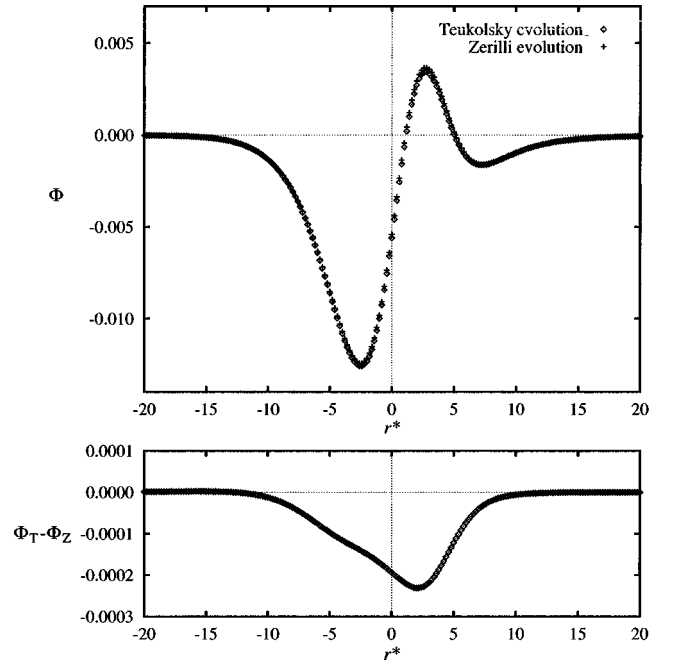


FIG. 3. Early radial dependence of $\Phi(t=3, r^*, \pi/2)$ for the $\ell=2$ multipole computed from the evolution of $\phi_M(t, r^*)$ via the numerical integration of the Zerilli equation and the transformations (2.9) and from the evolution of the Teukolsky code.

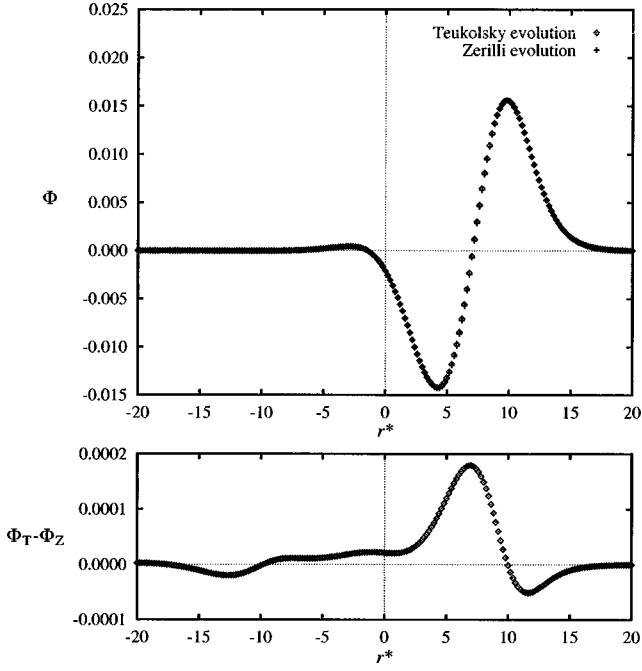


FIG. 4. Later radial behavior of $\Phi(t=10, r^*, \pi/2)$ for the $\ell=2$ multipole computed from the evolution of $\phi_M(t, r^*)$ via the numerical integration of the Zerilli equation and from evolving the Teukolsky code.

efficiency of the code we used for integrating the Teukolsky equation.

Finally, in order to have a complete description (in the context of the Teukolsky approach) of the close limit ap-

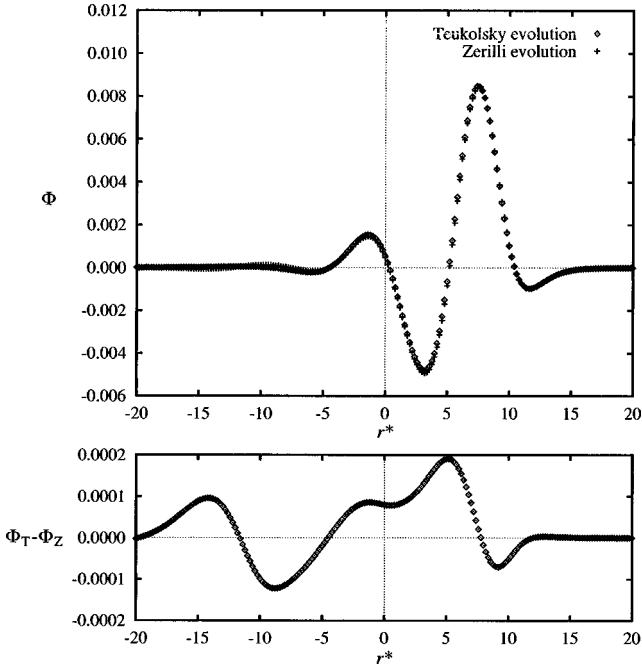


FIG. 5. Later radial behavior of $\Phi(t=10, r^*, \pi/2)$ for the $\ell=4$ multipole computed from the evolution of $\phi_M(t, r^*)$ via the numerical integration of the Zerilli equation and from the Teukolsky code evolution.

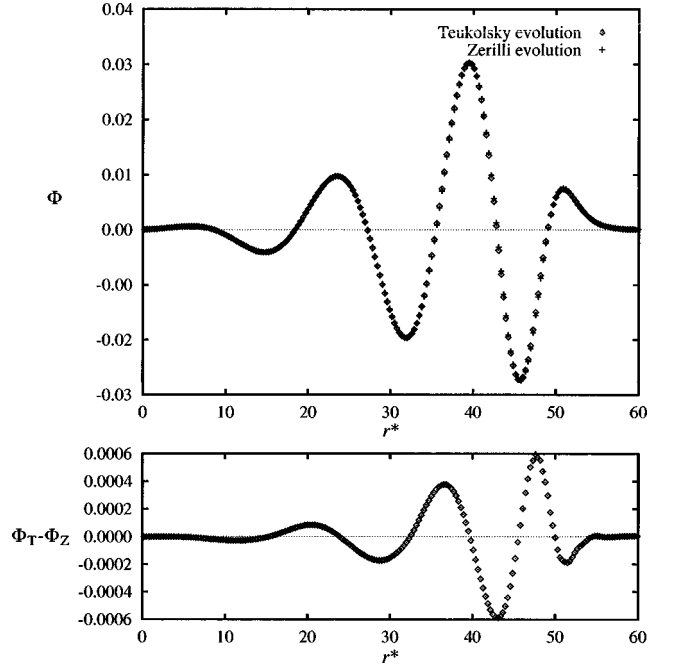


FIG. 6. Late radial behavior of $\Phi(t=50, r^*, \pi/2)$ for $\ell=2$ as computed through the integration of the Zerilli equation and through the Teukolsky equation.

proach for head on collisions, we show in Fig. 7 the time dependence of Φ for $\ell=2$ as it would be seen by an observer located at $r^*=50$, $\theta=\pi/2$.

IV. DISCUSSION

In this paper we reviewed the question of the imposition of Cauchy data for the Teukolsky equation. In order to write the Weyl scalar Ψ_4 entering into this wave equation in terms of pure hypersurface quantities, we related Ψ to the Moncrief waveform ϕ_M , which already possesses the property of depending on the three-metric and the extrinsic curvature

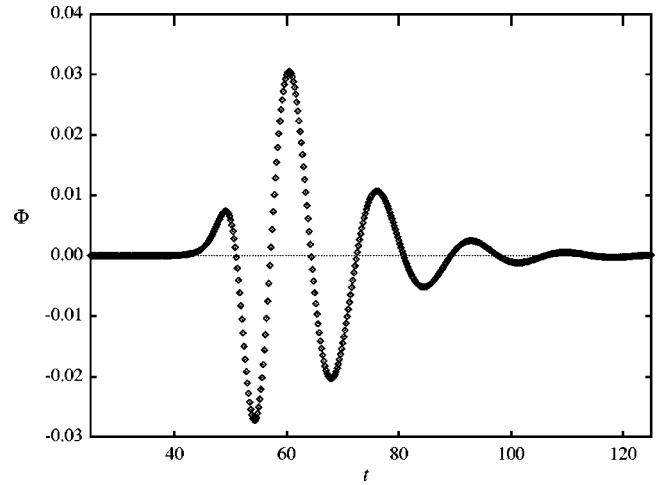


FIG. 7. Waveform for $\ell=2$. The function Φ is plotted as a function of the coordinate time t , for a detector located at $r^*=50$, $\theta=\pi/2$.

only. This relation was established via a Chandrasekhar transformation extended to the time domain. As the Chandrasekhar transformation never makes use of any gauge fixing, the transformation between waveforms (already gauge invariant) and the subsequent results are gauge invariant. We checked analytically that these results are equivalent to those obtained in Ref. [16]. Using this relation to impose initial data we have been able to reproduce the results of the close limit approximation [5], for the first time, by integrating the Teukolsky equation instead of the Zerilli one. In the astrophysically interesting case of the close limit, we have also been able to check numerically that Chandrasekhar transformations give the same relations as found in Ref. [16]. Furthermore, since these relations hold for any hypersurface, we could test their correctness at any time by evolving the Zerilli and Teukolsky equations independently. We thus gained reliability about the final expressions obtained in both this paper and Ref. [16]. In turn, we also gained further confidence in the numerical code developed in Ref. [1], which focused on the late time behavior of the radiation. Here the code is tested in the regime where the initial data influence is still of some importance as opposed to previous studies of gravitational tails.

All the tests that we performed have been made for perturbations around a Schwarzschild, i.e., nonrotating, black hole. This allowed us to make a comparison with the Zerilli-Moncrief formalism. In the general case of non-negligible rotation of the background hole, we do not have this counterpart to compare with and one has no other option than to evolve the Teukolsky equation. However, as we stated in the Introduction, to the present day, we do not know explicitly what initial data to evolve or how to impose them to build up Ψ and $\partial_t \Psi$. There is perhaps an intermediate step we can study with the present techniques which is the case of slowly rotating holes that can still be studied as perturbations of the Schwarzschild metric [29].

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