

Null geodesic expansion in spherical gravitational collapse

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We derive an expression for the expansion of outgoing null geodesics in spherical dust collapse and compute the limiting value of the expansion in the approach to singularity formation. An analogous expression is derived for the spherical collapse of a general form of matter. We argue on the basis of these results that the covered as well as the naked singularity solutions arising in spherical dust collapse are stable under small changes in the equation of state. [S0556-2821(98)07714-5]

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I. INTRODUCTION

The spherical gravitational collapse of matter obeying an energy condition has been studied for various equations of state, including dust, perfect fluids and scalar fields. It is known that both black-hole and naked singularity solutions can result as the end state of spherical collapse, though the issue of the stability of these solutions remains an open one.

The expansion of a congruence of null geodesics, usually denoted by θ , can play a useful role in our understanding of the nature of the singularity that can form in gravitational collapse. With this in mind, we derive an expression for θ for spherical dust collapse and also for spherical collapse of a general form of matter. Using these expressions we argue for the stability of both the covered and naked singularity solutions under small perturbations of the equation of state.

II. GEODESIC EXPANSION IN THE DUST MODEL

In comoving coordinates the Tolman-Bondi metric, which describes spherical dust collapse, is given by

$$ds^2 = dt^2 - e^\omega dr^2 - R^2 d\Omega^2. \quad (1)$$

The field equations are

$$8\pi\rho = \frac{F'}{R^2 R'} \quad (2)$$

and

$$\dot{R}^2 = \frac{F}{R} + f. \quad (3)$$

The energy density $\rho(t, r)$ is the only non-vanishing component of the energy-momentum tensor. The function $f(r)$ above is defined by the relation

$$e^\omega = \frac{R'^2}{1+f} \quad (4)$$

and the function $F(r)$ is twice the mass within the comoving coordinate r [1]. A superscript dot and prime denote differentiation with respect to the comoving coordinates t and r respectively. We take $\dot{R}$ to be negative as we are considering a collapsing cloud.

Consider a congruence of outgoing radial null geodesics in this space-time, having the tangent vector $(K^t, K^r, 0, 0)$, where $K^t = dt/dk$ and $K^r = dr/dk$. The geodesic expansion θ is given by

$$\theta = K^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g} K^i), \quad (5)$$

which gives

$$\theta = \frac{\partial K^t}{\partial t} + \frac{\partial K^r}{\partial r} + \frac{K^t}{\sqrt{-g}} (\sqrt{-g})' + \frac{K^r}{\sqrt{-g}} (\sqrt{-g})'. \quad (6)$$

In order to compute the sum

$$\frac{\partial K^t}{\partial t} + \frac{\partial K^r}{\partial r} \quad (7)$$

we proceed by noting that

$$\frac{dK^r}{dk} = \frac{\partial t}{\partial k} \frac{\partial K^r}{\partial t} + \frac{\partial r}{\partial k} \frac{\partial K^r}{\partial r} \quad (8)$$

and, similarly,

$$\frac{dK^t}{dk} = \frac{\partial t}{\partial k} \frac{\partial K^t}{\partial t} + \frac{\partial r}{\partial k} \frac{\partial K^t}{\partial r}. \quad (9)$$

Dividing the first of these two relations by $\partial r/\partial k$ and the second by $\partial t/\partial k$, and after adding the two equations we get

$$\frac{\partial K^t}{\partial t} + \frac{\partial K^r}{\partial r} = \frac{1}{2} \left[\frac{1}{K^r} \frac{dK^r}{dk} + \frac{1}{K^t} \frac{dK^t}{dk} \right] + \frac{\dot{\omega}}{4} K^t - \frac{\omega'}{4} K^r. \quad (10)$$

In arriving at this equality we have used the fact that, for outgoing radial null geodesics,

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$$\frac{K^t}{K^r} = \frac{dt}{dr} = e^{\omega/2}. \quad (11)$$

Further, it follows from the geodesic equation that

$$\frac{dK^r}{dk} = -\Gamma_{rr}^r (K^r)^2 - 2\Gamma_{tr}^r K^t K^r \quad (12)$$

and

$$\frac{dK^t}{dk} = -\Gamma_{rr}^t (K^r)^2. \quad (13)$$

Together, these relations give the desired expression for θ :

$$\theta(t, r) = \frac{2R'}{R} \left(1 - \sqrt{\frac{f+F/R}{1+f}} \right) K^r. \quad (14)$$

Let us now examine the relation that θ bears with the formation or otherwise of a naked singularity in spherical dust collapse. As is well known, the spherical collapse of a dust cloud results in the formation of a shell-focussing curvature singularity which lies on the curve $R(t, r) = 0$. (We assume that the initial density monotonically decreases away from the center, so that no shell-crossing singularities form.) The central singularity, $R(t_0, 0) = 0$, is known to be (at least locally) naked for some initial data and covered for other initial data, whereas the non-central singularity $R(t, r \neq 0) = 0$ is necessarily covered. Consider first the simpler case of the non-central singularity; here the ratio F/R goes to infinity as the singularity is approached, while all other quantities in the expression for θ are finite. As a result, θ goes to negative infinity, which is consistent with the fact that the singularity is covered and not visible.

The peculiar nature of the central singularity has been analyzed in various previous papers [2]; in particular we draw attention to our analysis in [3]. There we check for the possible occurrence of a central naked singularity as follows. We define a quantity $X = R/u$, where $u = r^\alpha$, and $\alpha > 1$ is defined such that $R'/r^{\alpha-1}$ is a unique finite quantity in the limit of approach to $r = 0$. Then it is evident that in the approach to the singularity we can write the limiting value X_0 of X as

$$\begin{aligned} X_0 &= \lim_{R \rightarrow 0, r \rightarrow 0} X = \lim_{R \rightarrow 0, r \rightarrow 0} \frac{R}{u} = \lim_{R \rightarrow 0, r \rightarrow 0} \frac{dR}{du} \\ &= \lim_{R \rightarrow 0, r \rightarrow 0} \frac{1}{\alpha r^{\alpha-1}} \frac{dR}{dr} = \lim_{R \rightarrow 0, r \rightarrow 0} \frac{1}{\alpha r^{\alpha-1}} \left(R' + \frac{dt}{dr} \dot{R} \right) \\ &= \lim_{R \rightarrow 0, r \rightarrow 0} \frac{R'}{\alpha r^{\alpha-1}} \left(1 - \sqrt{\frac{f+F/R}{1+f}} \right). \end{aligned} \quad (15)$$

The quantities on the right of the final equality are written as functions of X and r ; that is, X is used as a variable instead of t . The occurrence or otherwise of a naked singularity depends on whether or not this equation has a positive

real root. More explicitly, by calculating the limiting value of R' it is shown that the singularity is naked if the equation

$$X = \frac{1}{\alpha} \left(X + \frac{\gamma}{\sqrt{X}} \right) \left(1 - \sqrt{\frac{f(0) + \Lambda(0)/X}{1+f(0)}} \right) \quad (16)$$

admits a positive root for X . The quantity γ is a known positive function of the initial data, and $\Lambda(r) \equiv F/r^\alpha$. It is seen from the above equation that whenever the singularity is naked the expression inside the second bracket is positive definite. Further, it is known that whenever the singularity is covered, this same expression is negative in the limit.

The connection with the limiting behavior of θ is seen as follows. By comparing the expression for θ with Eq. (16) above, we see that the limiting value of θ is simply $2\alpha/k$ if the singularity is naked. (We note that near $r = 0$, $K^r = dr/dk \approx r/k$.) Hence θ goes to positive infinity at the naked singularity. On the other hand, whenever the singularity is covered, θ goes to negative infinity. This kind of behavior of θ at a covered or naked singularity is of course to be expected. We merely intend to show that the occurrence of a central naked singularity is a result of the fact that for certain initial data the geodesic expansion for outgoing null geodesics continues to be positive even as the singularity is approached.

We now attempt to ask how the information about the formation or otherwise of a naked central singularity is contained in the initial distribution for θ . What we do know of course is that for some initial density and velocity distributions the resulting singularity is naked. We can expect that somehow this must be reflected in the initial profile for θ . As for the actual value of θ at any given point, that is naturally positive initially, for a congruence of outgoing geodesics—by itself this carries no information about the nature of the resulting singularity. Hence, we need to look at the initial distribution of $\theta(r)$ and how it changes from one point to another. Consider the expression (14) at a general time t for the marginally bound case ($f = 0$). We recall that the solution for the area radius is

$$R^{3/2} = r^{3/2} - \frac{3}{2} \sqrt{F(r)} t \quad (17)$$

where the scaling $R = r$ at the initial epoch $t = 0$ is assumed. Let us also recall from [3] that the series expansion for $F(r)$ near $r = 0$ is written as

$$F(r) = F_0 r^3 + F_q r^{q+3} + \dots \quad (18)$$

where it is assumed that the first non-vanishing derivative in an expansion for the density near $r = 0$ is the q th one. Using this, we can write the expression (14) for θ near $r = 0$, to leading order, as

$$\theta(t, r) = \frac{2R'}{R} \left[1 - \left(\frac{F_0^{3/2} r^3}{1 - \frac{3}{2} \sqrt{F_0} t - \frac{3F_q}{4F_0} r^q t} \right)^{1/3} \right] K^r. \quad (19)$$

We concentrate on the behavior of the expression inside the second brackets:

$$\psi(t, r) \equiv \left(\frac{F}{R} \right)^{3/2} = \frac{F_0^{3/2} r^3}{1 - \frac{3}{2} \sqrt{F_0} t - \frac{3F_q}{4F_0} r^q t}. \quad (20)$$

This expression already contains information as to whether a naked singularity will result or not, as we now elaborate. The term which carries information about the inhomogeneity in the distribution is the last term in the denominator: $[(-3F_q/4F_0)r^q t]$. The remaining terms in $\psi(t, r)$ are exactly as they would be for a homogeneous cloud. In the approach to the time of formation of the central singularity, which is equal to $2/3\sqrt{F_0}$, the inhomogeneous term causes $\psi(t, r)$ to go to zero or infinity depending on whether q is less than three or greater than 3. As a result, θ goes to a positive limit if q is less than 3 and to a negative limit if q is greater than 3. This is consistent with the fact that the resulting singularity is naked for $q < 3$ and covered for $q > 3$. The case $q = 3$ is, however, described satisfactorily by ψ only to an extent: θ is negative in the limit for $\zeta \equiv F_3/F_0^{5/2} < -4/3$, whereas it is known that a naked singularity results only for $\zeta < -25.9904$.

How can we look at the initial data for θ and decide whether or not a naked singularity will result? We propose to look at the expression for $1/\psi(t, r)$, which is

$$\frac{1}{\psi(t, r)} = \left(\frac{R}{F} \right)^{3/2} = \frac{1 - \frac{3}{2} \sqrt{F_0} t}{F_0^{3/2} r^3} - \frac{3F_q r^q t}{F_0^{3/2} r^3}. \quad (21)$$

If we look at the inhomogeneous contribution, given by the second term, at an epoch t just after the start of evolution, we notice that this contribution diverges at $r = 0$, if $q < 3$, and goes to zero for $q > 3$. (We do not consider the transition case $q = 3$.) In fact this continues to be so at all times t . Hence one could suggest that if the inhomogeneous contribution initially diverges at the center, the singularity will be naked, and if it converges to zero, the singularity will be covered. Physically this means that at the initial epoch, the inhomogeneity can cause the geodesic expansion to either decrease or increase as one moves away from the center. The former case results in a naked singularity, and the latter in a covered singularity. The same set of arguments also hold for the non-marginally bound case ($f \neq 0$).

III. GEODESIC EXPANSION IN THE GENERAL SPHERICAL CASE

We now show that the expression for θ in the case of general spherical collapse can be cast in exactly the same form as for dust collapse. To begin with, we write the Einstein equations for the general spherical case in comoving coordinates. The metric, in comoving coordinates, is

$$ds^2 = e^\sigma dt^2 - e^\omega dr^2 - R^2 d\Omega^2 \quad (22)$$

and the energy-momentum tensor is $T_k^i = \text{diag}(\rho, p_r, p_T, p_T)$. The field equations for this system are

$$\rho = \frac{F'}{R^2 R'}, \quad (23)$$

$$\dot{F} = -p_r R^2 \dot{R}, \quad (24)$$

$$\sigma' = -\frac{2p_r'}{\rho + p_r} + \frac{4R'}{R(\rho + p_r)}(p_T - p_r), \quad (25)$$

$$\dot{\omega} = -\frac{2\dot{\rho}}{\rho + p_r} - \frac{4\dot{R}(\rho + p_T)}{R(\rho + p_r)}, \quad (26)$$

and

$$e^{-\sigma} \dot{R}^2 = \frac{F}{R} + f. \quad (27)$$

The function $F(t, r)$ is equal to twice the mass inside the comoving coordinate r , and as in the dust case, the function $f(t, r)$ is defined by the relation (4). The difference from the dust case is that now the functions F and f depend on time as well. (It is only for the dust equation of state that both these functions are time independent.) When one considers the special case of dust, the above five equations behave as follows. Equation (23) holds as such, while Eq. (27) reduces to Eq. (3). Equations (24) and (25) are trivially satisfied, whereas Eq. (26) reduces to an identity.

The calculation of θ proceeds exactly as in the case of dust, and we get again the expression in Eq. (6). In order to calculate the sum (7) we use the relations (8) and (9) and we note that now for outgoing null geodesics we have

$$\frac{K^t}{K^r} = \frac{dt}{dr} = e^{(\omega - \sigma)/2}. \quad (28)$$

Hence we get, by proceeding as in the dust case,

$$\begin{aligned} \frac{\partial K^t}{\partial t} + \frac{\partial K^r}{\partial r} &= \frac{1}{2} \left[\frac{1}{K^r} \frac{dK^r}{dk} + \frac{1}{K^t} \frac{dK^t}{dk} \right] + \frac{\dot{\omega} - \dot{\sigma}}{4} K^t \\ &\quad - \frac{\omega' - \sigma'}{4} K^r. \end{aligned} \quad (29)$$

By substituting for dK^r/dk and dK^t/dk from the geodesic equation we again find that θ is given by

$$\theta(t, r) = \frac{2R'}{R} \left(1 - \sqrt{\frac{f + F/R}{1 + f}} \right) K^r, \quad (30)$$

which is the same expression as in the dust case, except that now F and f are functions of t as well as r . We have also utilized the field equation (27) which in fact is the only field equation used in writing the above expression for θ .

IV. DISCUSSION

We note that θ for dust collapse has a peculiar behavior in the limit of approach to the singularity curve. As regards the points other than the center (i.e., those with $r \neq 0$), they first get trapped (i.e., θ becomes zero), and then singular. This is consistent with the singularity theorems—the formation of a trapped surface at a given r is followed by the formation of a singularity for this value of r . However, $r=0$ is a very special point: the apparent horizon curve begins to form at $r=0$ simultaneously with the occurrence of the singularity. As a result, it would appear that one could not directly appeal to a singularity theorem to predict the formation of the central singularity, even though an explicit calculation shows that a singularity does form at $r=0$. Similarly, one cannot predict *a priori* that the central singularity will necessarily be covered, and explicit calculation actually shows it to be otherwise.

We would like to argue here, judging by the expression for θ , that both the black-hole and naked singularity solutions arising in spherical dust collapse are stable under sufficiently small changes of equation of state, for a fixed initial data. This is expected to hold for all initial data, except near the transition region (i.e., the case $q=3$) in the above discussion, and the reasoning is as follows. Consider an initial density and velocity distribution in spherical dust collapse which results in a covered singularity. As discussed above, this covered singularity results because for these initial data, the ratio F/R goes to positive infinity in the limit of approach to the singularity. We have noticed, in Eq. (21), that the initial data already contain the information that F/R will behave in this particular manner in the approach to the singularity, thereby making it a covered singularity. Now, if we

keep the initial density and velocity distribution fixed, and make a sufficiently small change in the equation of state, the initial behavior of F/R will be arbitrarily close to that in the dust case. This is to be expected from the stability of solutions of Einstein equations under a small change in the equation of state. Hence the evolution will still be such that F/R will again go to positive infinity as the singularity is approached, which makes θ negative, and hence the singularity continues to be covered. [The function $f(r)$ plays an insignificant role in this argument: one only has to make the plausible demand that $f(0)$ be finite during the evolution.] A similar argument ensures that the dust naked singularity solutions will be stable under perturbation of the equation of state (in this case F/R goes to zero in the limit and θ is positive). It is only in the transition region ($q=3$) that both the naked and covered solutions could be unstable: one kind of solution could go into the other kind.

From an argument of this nature it also follows that if for any given equation of state some initial data lead to a covered (naked) singularity, the nature of the singularity will be left unchanged if the equation of state is perturbed while keeping the initial conditions fixed. The same conclusion has also been arrived at earlier in [4] by independent means. It appears as if both the covered and naked solutions arising in spherical collapse must be treated on the same footing, insofar as stability under a change of equation of state is concerned.

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