Constraints on the chaotic inflationary scenario with a nonminimally coupled ''inflaton'' field from the cosmic microwave background radiation anisotropy

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We investigate the possibility to restrict the chaotic inflationary scenario with the large nonminimally coupled inflaton field ϕ considered by Fakir and Unruh by means of the observed cosmic microwave background radiation (CMBR) anisotropy. This model is characterized by the condition $\xi>1$ and $\psi\equiv 8\pi G\xi\phi^2 \gg 1$ where ξ is the nonminimal coupling constant. We calculate the contributions of the long wavelength gravitational waves generated in the inflationary period to the CMBR anisotropy quadrupole moment. We obtain the constraint $\lambda/\xi^2=1.8\times10^{-8}/\psi_i$, where λ is the self-coupling and ψ_i means the initial value of ψ. Combining this with the previously obtained constraint $\sqrt{\lambda/\xi^2}\approx(\delta T/T)_{\rm rms}=1.1\times10^{-5}$, we conclude that the initial value has to be $\psi_i\approx1.6\times10^3$. If the self-coupling has reasonable values of order 10^{-2}, then $\xi\approx10^4$ and $\phi_i\approx10^{-1}m_{\rm Pl}$ where $m_{\rm Pl}$ is the Planck mass. [S0556-2821(98)05614-8]

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I. INTRODUCTION

Inflation is regarded as one of the most important conceptual problems in modern cosmology [1,2]. This concept gives us not only the solution of cosmological puzzles such as the horizon and flatness problem, but also the origin of cosmological perturbations. Among the various scenarios of inflation proposed so far, Linde's chaotic scenario seems to be a natural mechanism for the realization of inflationary expansion [3]. In spite of many attractive features, the original scenario still has the fine-tuning problem of the parameters. Namely, one has to choose an unreasonably small self-coupling constant, i.e., $\lambda<10^{-12}$, to have the correct magnitude of the density perturbation which is consistent with the observed tiny cosmic microwave background radiation (CMBR) fluctuations.

In order to overcome this difficulty, Fakir and Unruh proposed a new version of the chaotic model which has a *strong* nonminimal coupling, i.e., $\xi>1$, to the scalar curvature [4]. Note that we shall adopt in the present paper the sign convention for ξ such that the conformal coupling means $\xi=-1/6$. The chaotic inflationary scenario with nonminimal coupling has been investigated for various values of ξ by Futamase and Maeda [5]. Their result does not exclude the *strong* coupling model of Fakir and Unruh. They have shown that the larger value of ξ relaxes the fine-tuning problem of the self-coupling λ, and this conclusion has been asserted by Makino and Sasaki more rigorously [6]. Fakir, Habib, and Unruh also made a rigorous analysis of density perturbation [7].

Constraints on this scenario have been discussed by several authors (e.g., Refs. [8,9]). Here we shall investigate the Fakir-Unruh scenario from a different point of view. Namely, we shall study the effect of the scenario on the tensor CMBR anisotropy. It is known that the long wavelength gravitational waves (GWs) are generated during a period of inflationary expansion, and they make a significant contribution to CMBR anisotropy through the Sachs-Wolfe effect [10] on lower multipoles [11,12]. We will point out that the Fakir-Unruh scenario also has a non-negligible contribution to the spectrum of CMBR anisotropy by generated GWs, and then obtain the constraint on a certain combination of parameters by comparing the predicted quadrupole moment with the results of Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) observations.

This paper is organized as follows. We first review Fakir-Unruh scenario and present the self-consistent inflationary solutions in Sec. II. We then find the radiative modes of the metric perturbations in the spacetime generated by a nonminimally coupled inflaton field, and derive the resulting spectrum of GW in Sec. III. Based on the result in Sec. III, we shall calculate the power spectrum of CMBR anisotropy and compare it with the observations of COBE-DMR in Sec. IV. Finally some discussions are made in Sec. V. We shall follow Misner, Thorne, and Wheeler [13] for the definition of the Riemann tensor, Ricci tensor, and Ricci scalar.

II. BACKGROUND INFLATIONARY SOLUTIONS

As explained in the Introduction, in this paper we shall be interested in the chaotic inflationary scenario with large positive nonminimal coupling. Thus we shall take the following action as our model:

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2}+\frac{1}{2}\xi\phi^2R-\frac{1}{2}\phi_{,\mu}\phi^{,\mu}+V(\phi)\right], \tag{2.1}$$

where $\kappa^2\equiv8\pi G$. Our definition of ξ is the same as that of Fakir and Unruh, that is, conformal coupling means $\xi=-\frac{1}{6}$. Note that Futamase and Maeda used an opposite sign for ξ. Varying the action (2.1), we obtain the field equations

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$$\left(\frac{1}{\kappa^2}+\xi\phi^2\right)G_{\mu\nu}=(1+2\xi)\phi_{,\mu}\phi_{,\nu}-\left(2\xi+\frac{1}{2}\right)g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}-2\xi\phi(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)\phi+g_{\mu\nu}V(\phi), \tag{2.2}$$

$$\Box\phi+\xi R\phi+V_{,\phi}=0, \tag{2.3}$$

where $\Box\equiv-g^{-1/2}\partial_\alpha(-g^{1/2}g^{\alpha\beta}\partial_\beta)$ and $V_{,\phi}=\partial V/\partial\phi$. Greek and Latin indices take 0,1,2,3 and 1,2,3, respectively.

For the spacetime we assume a homogeneous and spatially flat Robertson-Walker universe:

$$ds^2=dt^2-a^2(t)\delta_{ij}dx^i dx^j. \tag{2.4}$$

Equation (2.2) yields the Hamiltonian constraint equation

$$H^2=\frac{\kappa^2}{3(1+\kappa^2\xi\phi^2)}\left[\frac{1}{2}\dot{\phi}^2+V(\phi)-6\xi H\phi\dot{\phi}\right], \tag{2.5}$$

where overdots denote time derivatives. The momentum constraint equation is

$$\dot{H}=\frac{\kappa^2}{1+\kappa^2\xi\phi^2}\left[-\frac{1}{2}\dot{\phi}^2+\xi H\phi\dot{\phi}-\xi\dot{\phi}^2-\xi\phi\ddot{\phi}\right]. \tag{2.6}$$

The equation of motion for the inflaton field is obtained from Eq. (2.3) as

$$\ddot{\phi}+3H\dot{\phi}-6\xi(\dot{H}+2H^2)\phi+V_{,\phi}=0. \tag{2.7}$$

Combining the above equations, we obtain

$$\ddot{\phi}+3H\dot{\phi}+\left[\frac{\kappa^2\xi\phi^2(1+6\xi)}{1+\kappa^2\xi\phi^2(1+6\xi)}\right]\frac{\dot{\phi}^2}{\phi}=\frac{1}{1+\kappa^2\xi\phi^2(1+6\xi)}[4\kappa^2\xi\phi V(\phi)-(1+\kappa^2\xi\phi^2)V_{,\phi}]. \tag{2.8}$$

Using ‘‘slow-roll approximations’’ such as

$$\left|\frac{\ddot{\phi}}{\dot{\phi}}\right|\ll H, \tag{2.9}$$

$$\left|\frac{\dot{\phi}}{\phi}\right|\ll H, \tag{2.10}$$

$$\frac{1}{2}\dot{\phi}^2\ll V(\phi), \tag{2.11}$$

the energy constraint and field equation take the following forms, respectively:

$$H^2\approx\frac{\kappa^2}{3(1+\kappa^2\xi\phi^2)}\left[V(\phi)-\frac{2\xi\phi}{1+\kappa^2\xi\phi^2(1+6\xi)}\times\{4\kappa^2\xi\phi V(\phi)-(1+\kappa^2\xi\phi^2)V_{,\phi}\}\right], \tag{2.12}$$

$$3H\dot{\phi}\approx\frac{1}{1+\kappa^2\xi\phi^2(1+6\xi)}[4\kappa^2\xi\phi V(\phi)-(1+\kappa^2\xi\phi^2)V_{,\phi}]. \tag{2.13}$$

These are the basic equations which determine the background solution. We shall consider the solutions of these equations in the case with the self-coupling constant and mass term, separately.

A. Chaotic inflation by a self-coupling

Let us consider the chaotic inflation generated by the self-coupling constant of the inflaton. Thus we take

$$V(\phi)=\frac{1}{4}\lambda\phi^4.$$

Then Eqs. (2.12) and (2.13) give, respectively,

$$H^2\approx\frac{\kappa^2\lambda\phi^4}{12(1+\kappa^2\xi\phi^2)}\left[1+\frac{8\xi}{1+\kappa^2\xi\phi^2(1+6\xi)}\right], \tag{2.14}$$

$$3H\dot{\phi}\approx-\frac{\lambda\phi^3}{1+\kappa^2\xi\phi^2(1+6\xi)}. \tag{2.15}$$

Let us now define our basic variable $\psi\equiv\kappa^2\xi\phi^2$ which turns out to be very useful in our analysis. Since we consider the *strong* curvature coupling, let us consider the situation where

$$\psi\gg1.$$

This condition simplifies significantly the above equations as

$$H^2\approx\frac{\lambda\psi}{12\kappa^2\xi^2}, \tag{2.16}$$

$$\frac{\dot{\psi}}{\psi}\approx-\frac{2\lambda}{3\kappa^2\xi(1+6\xi)}\frac{1}{H}. \tag{2.17}$$

Therefore we have the following solution:

$$\frac{\dot{\psi}}{H}\approx-\frac{8\xi}{1+6\xi}, \tag{2.18}$$

$$\psi\approx\psi_i-\frac{8\xi}{1+6\xi}\ln\left(\frac{a}{a_i}\right). \tag{2.19}$$

Note that this solution is consistent with the slow-roll approximations. In fact we have

$$\left|\frac{\dot{\psi}}{H\psi}\right| \approx \left|-\frac{8\xi}{1+6\xi}\frac{1}{\psi}\right| \ll 1. \tag{2.20}$$

From Eqs. (2.16) and (2.19), we obtain a differential equation for the scale factor:

$$\frac{\dot{a}}{a} = \sqrt{\frac{\lambda\psi_i}{12\kappa^2\xi^2}}\left[1-\frac{8\xi}{(1+6\xi)\psi_i}\ln\left(\frac{a}{a_i}\right)\right]^{1/2}. \tag{2.21}$$

Equation (2.21) can be easily integrated as

$$a(t) = a_i \exp[H_i t - \gamma(H_i t)^2]. \tag{2.22}$$

Thus we have the following Hubble parameter and scalar field:

$$a(t) = a_i \exp[H_i t - \gamma(H_i t)^2], \tag{2.23}$$

$$H(t) = H_i(1-2\gamma H_i t), \tag{2.24}$$

$$\psi(t) = \psi_i(1-2\gamma H_i t)^2, \tag{2.25}$$

where

$$H_i^2 = \frac{\lambda\psi_i}{12\kappa^2\xi^2}, \tag{2.26}$$

$$\gamma = \frac{2\xi}{(1+6\xi)\psi_i}. \tag{2.27}$$

These expressions indicate that the expansion rate is much more rapid than the rate of change in the Hubble parameter as well as the inflaton field. Since we are interested in the very long-wavelength tensor perturbations which give the largest contribution to the CMBR anisotropy today and such perturbations decouple at the early stage of inflation, we can approximate the above solution to the following simple forms:

$$a(t) \approx a_i e^{H_i t}, \tag{2.28}$$

$$H(t) \approx H_i, \tag{2.29}$$

$$\psi(t) \approx \psi_i. \tag{2.30}$$

Finally, we have to consider the amount of inflation sufficient to solve cosmological puzzles:

$$\frac{a(t)}{a_i} = \exp\left(\int H dt\right) = \exp\left(\int_{\psi_i}^{\psi_f}\frac{H}{\dot{\psi}}d\psi\right) \approx \exp\left(\int_{\psi_i}^{\psi_f} -\frac{1+6\xi}{8\xi}d\psi\right) = \frac{1+6\xi}{8\xi}(\psi_i-\psi_f) \geqslant 70. \tag{2.31}$$

Fakir and Unruh asserted that $\psi \approx 1$ signals the end of inflation [4]. Thus the *strong* coupling, i.e., $\xi > 1$, condition gives

$$\psi_i \gtrsim 80. \tag{2.32}$$

B. Chaotic inflation by a mass term

Next, we consider a massive free inflaton field with the following potential:

$$V(\phi) = \frac{1}{2}m^2\phi^2.$$

Then Eqs. (2.12) and (2.13) become, respectively,

$$H^2 \approx \frac{\kappa^2\xi m^2\phi^2}{6\xi(1+\kappa^2\xi\phi^2)}\left[1+\frac{4\xi}{1+\kappa^2\xi\phi^2(1+6\xi)} \times (1-\kappa^2\xi\phi^2)\right], \tag{2.33}$$

$$3H\dot{\phi} \approx \frac{m^2\phi(\kappa^2\xi\phi^2-1)}{1+\kappa^2\xi\phi^2(1+6\xi)}. \tag{2.34}$$

When $\psi \gg 1$, these equations may be approximated as

$$H^2 \approx \frac{m^2}{6\xi}\left(\frac{1+2\xi}{1+6\xi}\right), \tag{2.35}$$

$$\frac{\dot{\psi}}{\psi} \approx \frac{2m^2}{3(1+6\xi)}\frac{1}{H}. \tag{2.36}$$

We can find a scalar field grows exponentially so that the slow-roll approximations are not satisfied unless $\xi \ll 1$:

$$\left|\frac{\dot{\psi}}{H\psi}\right| \approx \left|\frac{4\xi}{1+2\xi}\right| < 2. \tag{2.37}$$

Thus, we conclude that mass-term inflation does not occur in the model with *strong* coupling. This result is consistent with Futamase and Maeda [5].

III. THE SPECTRUM OF GWs GENERATED DURING THE INFLATIONARY EXPANSION

In this section we shall calculate the spectrum of GWs generated during the period of inflationary expansion in the framework of the Fakir-Unruh scenario.

A. Radiative solutions of metric perturbations

First we derive the equation for the GWs as the linear metric perturbation. The method is exactly the same as the case in general relativity (GR) [14]. Namely, we consider a small disturbance $h_{\mu\nu}$ of the spatially flat Friedmann-Robertson-Walker metric $\overline{g}_{\mu\nu}$

$$ds^2 = (\overline{g}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu. \tag{3.1}$$

The spatial flatness means that the metric perturbations can be expanded by a plane wave with a comoving wave number k

$$h^{\mu}_{\nu}(\boldsymbol{x},\tau)=\sum_{\lambda}\int d^3\boldsymbol{k}h_{\lambda}(\boldsymbol{k},\tau)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{\epsilon}^{\mu}_{\nu}, \tag{3.2}$$

where $\boldsymbol{\epsilon}^{\mu}_{\nu}$ is the polarization tensor, and $\lambda=+,\times$ is the mode of the polarization. We choose the synchronous gauge ($h_{00}=h_{0i}=0$), and use conformal time ($x^0=\tau$):

$$ds^2=a^2(\tau)d\tau^2-[a^2(\tau)\delta_{ij}+h_{ij}]dx^idx^j. \tag{3.3}$$

Then the linearized Einstein equations (2.2) lead to the radiative mode of the tensor perturbation,

$$h^k_k=h^k_{i,k}=\delta\psi=0, \tag{3.4}$$

$$h''_{\lambda}(\boldsymbol{k},\tau)+\left(2\frac{a'}{a}+\frac{\psi'}{1+\psi}\right)h'_{\lambda}(\boldsymbol{k},\tau)+k^2h_{\lambda}(\boldsymbol{k},\tau)=0, \tag{3.5}$$

where dashes denote conformal time derivatives. The other components of the Einstein equations (2.2) are trivially satisfied with the above conditions.

Writing each component $h_{\lambda}(\boldsymbol{k},\tau)$ of the GW perturbations as [15,16]

$$h_{\lambda}(\boldsymbol{k},\tau)\equiv\frac{1}{R(\tau)}\mu_{\lambda}(\boldsymbol{k},\tau)=\frac{1}{a\sqrt{1+\psi}}\mu_{\lambda}(\boldsymbol{k},\tau), \tag{3.6}$$

we have the following equation for μ_{λ}:

$$\mu''_{\lambda}(\boldsymbol{k},\tau)+\left(k^2-\frac{R''}{R}\right)\mu_{\lambda}(\boldsymbol{k},\tau)=0. \tag{3.7}$$

This is an ordinally ''Schrödinger-type'' equation. If $k^2\gg|R''/R|$, we can find a plane wave solution of h_{λ} with its amplitude scaled by $(a\sqrt{1+\psi})^{-1}$.

B. The spectrum of the GWs

Now we will calculate the spectrum of the GWs derived above. We follow previous works [11,17,18]. Let us choose the normalization of $a(\tau)$ as

$$a(\tau)=-\frac{1}{H_i\tau}\ \text{vacuum}\quad \tau\in(-\infty,-\tau_2), \tag{3.8}$$

$$=\frac{2\tau_1\tau}{\tau_0^2}\ \text{radiation}\quad \tau\in\left(\tau_2,\frac{\tau_1}{2}\right), \tag{3.9}$$

$$=\frac{\tau^2}{\tau_0^2}\ \text{matter}\quad \tau\in(\tau_1,\tau_0). \tag{3.10}$$

We assume that the transitions across each phase are instantaneous and we match $a(\tau)$ and $\dot{a}(\tau)$ at the transition points. This will be an accurate approximation for the long-wave tensor perturbations we are interested in. Note that τ is discontinuous across each transition. Of course, we could define continuous time across each phase (e.g., Refs. [19,20]), and we can obtain the same results as below.

We shall calculate the amplitude of the GWs generated as quantum noises. The following consideration will help us to choose a convenient field variable to quantize. Equation (3.7) shows that each polarization state of the waves behave as a scalar field φ with a normalization factor of $\sqrt{2\kappa^2_{\rm eff}}$:

$$h_{\lambda}=\sqrt{2\kappa^2_{\rm eff}}\varphi_{\lambda}=\sqrt{\frac{2\kappa^2}{1+\psi}}\varphi_{\lambda}. \tag{3.11}$$

We omit the subscript λ henceforth. This situation is analogous to the GR case [21]. We also neglect the other interactions of φ due to the mixture of ψ', which is a good approximation for our purposes. Thus we choose the scalar field φ to quantize as the same way in GR:

$$\varphi(\boldsymbol{x},\tau)=\int d^3\boldsymbol{k}[\hat{a}_{\rm phase}(\boldsymbol{k})\varphi_k(\tau)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}+\hat{a}^{\dagger}_{\rm phase}(\boldsymbol{k})\varphi^*_k(\tau)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}], \tag{3.12}$$

where $\hat{a}_{\rm phase}$ and $\hat{a}^{\dagger}_{\rm phase}$ are the annihilation and creation operator at each phase (vacuum dominated, radiation dominated, and matter dominated phases), and $\varphi_k(\tau)$ is the properly normalized solution of Eq. (3.5):

$$\varphi_k(\tau)=\frac{1}{\sqrt{2k}(2\pi)^{3/2}a(\tau)}e^{-ik\tau}\left(1-\frac{i}{k\tau}\right)\ \text{vacuum, matter} \tag{3.13}$$

$$=\frac{1}{\sqrt{2k}(2\pi)^{3/2}a(\tau)}e^{-ik\tau}\ \text{radiation.} \tag{3.14}$$

We use the fact that $\psi\approx$const in the inflationary phase (2.30), and $\psi=0$ in the post inflationary phase. We also assume our universe approaches the conformal vacuum mode function as $\tau\to-\infty$, that is,

$$\varphi_{k,\rm VD}\to\frac{1}{\sqrt{2k}(2\pi)^{3/2}a(\tau)}e^{-ik\tau}\quad \text{for}\quad \tau\to-\infty. \tag{3.15}$$

Where the quantities with VD means the quantities evaluated at the vacuum dominated phase, namely, at the inflationary phase. The Bogoliubov coefficients relating operators at each phase are defined as

$$\hat{a}_{\rm RD}(\boldsymbol{k})=c_1(k)\hat{a}_{\rm VD}(\boldsymbol{k})+c^*_2(k)\hat{a}^{\dagger}_{\rm VD}(-\boldsymbol{k}), \tag{3.16}$$

$$\hat{a}_{\rm MD}(\boldsymbol{k})=c_3(k)\hat{a}_{\rm VD}(\boldsymbol{k})+c^*_4(k)\hat{a}^{\dagger}_{\rm VD}(-\boldsymbol{k}), \tag{3.17}$$

where RD and MD mean radiation dominated and matter dominated, respectively. Matching the field and its first derivative at the first transition,

$$\varphi_{k,\rm VD}(-\tau_2)=c_1\varphi_{k,\rm RD}(\tau_2)+c_2\varphi^*_{k,\rm RD}(\tau_2), \tag{3.18}$$

$$\varphi'_{k,\rm VD}(-\tau_2)=c_1\varphi'_{k,\rm RD}(\tau_2)+c_2\varphi^{*\prime}_{k,\rm RD}(\tau_2), \tag{3.19}$$

we obtain

$$c_1 = -\frac{H_i\tau_1}{(k\tau_0)^2} e^{2ik\tau_2}[1-2ik\tau_2-2(k\tau_2)^2], \tag{3.20}$$

$$c_2 = \frac{H_i\tau_1}{(k\tau_0)^2}. \tag{3.21}$$

We can see these coefficients satisfy the Bogoliubov relations required for the orthonormality of mode functions φ_k [22]:

$$|c_1|^2 - |c_2|^2 = 1. \tag{3.22}$$

Since we are interested in GWs which are still well outside the horizon at the time of matter-radiation equality and will give the largest contribution to the CMBR anisotropy today, we may use simpler results up to second order

$$c_1 \approx -\frac{H_i\tau_1}{(k\tau_0)^2}, \tag{3.23}$$

$$c_2 = \frac{H_i\tau_1}{(k\tau_0)^2}. \tag{3.24}$$

At the next transition the continuity conditions become

$$c_1\varphi_{k,\mathrm{RD}}\left(\frac{\tau_1}{2}\right) + c_2\varphi^*_{k,\mathrm{RD}}\left(\frac{\tau_1}{2}\right) = c_3\varphi_{k,\mathrm{MD}}(\tau_1) + c_4\varphi^*_{k,\mathrm{MD}}(\tau_1), \tag{3.25}$$

$$c_1\varphi'_{k,\mathrm{RD}}\left(\frac{\tau_1}{2}\right) + c_2\varphi^{*\prime}_{k,\mathrm{RD}}\left(\frac{\tau_1}{2}\right) = c_1\varphi'_{k,\mathrm{MD}}(\tau_1) + c_2\varphi^{*\prime}_{k,\mathrm{MD}}(\tau_1). \tag{3.26}$$

Thus we obtain

$$c_3 = \frac{H_i/k}{2(k\tau_1)(k\tau_0)^2}[\{1-2ik\tau_1-2(k\tau_1)^2\}e^{ik\tau_1/2} \times\{1-2ik\tau_2-2(k\tau_2)^2\}e^{2ik\tau_2} - e^{3ik\tau_1/2}], \tag{3.27}$$

$$c_4 = \frac{H_i/k}{2(k\tau_1)(k\tau_0)^2}[e^{-3ik\tau_1/2}\{1-2ik\tau_2-2(k\tau_2)^2\}e^{2ik\tau_2} - \{1+2ik\tau_1-2(k\tau_1)^2\}e^{-ik\tau_1/2}], \tag{3.28}$$

with

$$|c_3|^2 - |c_4|^2 = 1. \tag{3.29}$$

Up to second order, we get simple solutions again,

$$c_3 \approx -\frac{3iH_i}{2k(k\tau_0)^2}, \tag{3.30}$$

$$c_4 \approx -\frac{3iH_i}{2k(k\tau_0)^2}. \tag{3.31}$$

Since we assume each transition is instantaneous, the Universe will remain in the de Sitter vacuum. Thus we may be able to calculate the quantum mechanical two-point correlation function in the de Sitter vacuum:

$$\Delta^{(\mathrm{QM})} \equiv \frac{k^3}{(2\pi)^2}\int d^3\boldsymbol{x} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\langle 0|\varphi(\boldsymbol{x},\tau)\varphi(0,\tau)|0\rangle, \tag{3.32}$$

where

$$|0\rangle \equiv |\text{de Sitter vacuum}\rangle. \tag{3.33}$$

Note that this assumption makes us use the Heisenberg picture of quantum fields in which the states of vacuum do not evolve with time but the operators do [20]. For waves reentering the horizon at the matter dominated era, we have

$$\Delta^{(\mathrm{QM})} = \frac{H_i^2}{2(2\pi)^3}\left[\frac{3j_1(k\tau)}{k\tau}\right]^2, \tag{3.34}$$

where $j_1(k\tau)$ is a spherical Bessel function of order 1. According to our normalization, this is calculated as

$$\Delta^{(\mathrm{QM})}_{\mathrm{GW},\lambda\lambda'} = 2\kappa^2_{\mathrm{eff}}\delta_{\lambda\lambda'}\Delta^{(\mathrm{QM})} = \frac{\kappa^2 H_i^2}{(2\pi)^3}\left[\frac{3j_1(k\tau)}{k\tau}\right]^2\delta_{\lambda\lambda'}, \tag{3.35}$$

where we used the following facts:

$$\epsilon_{\mu\nu}(\lambda)\epsilon^{\mu\nu}(\lambda') = \delta_{\lambda\lambda'}, \tag{3.36}$$

$$\psi(\tau > \tau_2) = 0. \tag{3.37}$$

We match this result with our classical ensemble of GWs. The classical two-point function may be calculated by writing the classical GWs as

$$h_\lambda(\boldsymbol{k},\tau) = A(k)\chi_\lambda(\boldsymbol{k})\left[\frac{3j_1(k\tau)}{k\tau}\right], \tag{3.38}$$

where χ_λ is a random variable with statistical expectation value

$$\langle\chi_\lambda(\boldsymbol{k})\chi_{\lambda'}(\boldsymbol{k}')\rangle = \frac{1}{k^3}\delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda\lambda'}. \tag{3.39}$$

Then the classical two-point function is

$$\Delta^{(\mathrm{CL})}_{\mathrm{GW},\lambda\lambda'} = A^2(k)\left[\frac{3j_1(k\tau)}{k\tau}\right]^2\delta_{\lambda\lambda'}. \tag{3.40}$$

Comparing this with quantum two point function, we obtain

$$A^2(k) = \frac{\kappa^2 H_i^2}{(2\pi)^3} = \frac{8}{3\pi}v, \tag{3.41}$$

where

$$H_i^2 \equiv \frac{\kappa^2}{3} m_{\rm Pl}^4 v. \tag{3.42}$$

Since we know that $H_i^2 = \lambda \psi_i / 12\kappa^2 \xi^2$,

$$v = \frac{\psi_i}{(16\pi)^2}\left(\frac{\lambda}{\xi^2}\right). \tag{3.43}$$

In the end, we find that the existence of the nonminimal coupling only affects the amplitude of spectrum via Hubble constant.

IV. COMPARISON WITH OBSERVATIONS

We shall now make a comparison between the theoretical prediction derived in the previous section with the observation of CMBR anisotropy. Using the Sachs-Wolfe relation, one can calculate the power spectrum of CMBR anisotropy generated by the long-wavelength GW [18]:

$$\langle a_l^2 \rangle \equiv \left\langle \sum_m \left| a_{lm} \right|^2 \right\rangle = 36\pi^2 (2l+1) \frac{(l+2)!}{(l-2)!} \times \int_0^{2\pi/\tau_1} k dk A^2(k) |F_l(k)|^2, \tag{4.1}$$

where the angle brackets denote averages over statistical ensemble of a_l^2, and

$$F_l(k) \equiv \int_0^{\tau_0-\tau_1} dr \left[\frac{d}{d[k(\tau_0-r)]} \frac{j_1[k(\tau_0-r)]}{k(\tau_0-r)} \right] \times \left[\frac{j_{l-2}(kr)}{(2l-1)(2l+1)} + \frac{2 j_l(kr)}{(2l-1)(2l+3)} + \frac{j_{l+2}(kr)}{(2l+1)(2l+3)} \right]. \tag{4.2}$$

We shall use the following numerical value for $\langle a_2^2 \rangle$ evaluated by White [18]

$$\langle a_2^2 \rangle = 7.74 v. \tag{4.3}$$

On the other hand, the COBE-DMR group expresses this quantity in terms of $Q_{\rm rms-PS}$:

$$\langle a_2^2 \rangle \equiv 4\pi \left(\frac{Q_{\rm rms-PS}}{T_0} \right)^2. \tag{4.4}$$

Note that since the quantity $Q_{\rm rms-PS}$ has already been handled statistically, we can make a direct comparison between this expression with the theoretical prediction $\langle a_l^2 \rangle$. Combining Eqs. (3.43), (4.3), and (4.4), we find

$$\frac{\lambda}{\xi^2} = \frac{(4\pi)^3}{38.7} \left(\frac{80}{\psi_i} \right) \left(\frac{Q_{\rm rms-PS}}{T_0} \right)^2. \tag{4.5}$$

According to the COBE 4 yr results [23,24],

$$T_0 = 2.728 \pm 0.004 \ \ {\rm K}, \tag{4.6}$$

$$Q_{\rm rms-PS} = 18 \pm 1.4 \ \ \mu{\rm K} \tag{4.7}$$

for the $n=1$ Harrison-Zel'dovich spectrum, thus we obtain the following constraint on the theory:

$$\frac{\lambda}{\xi^2} \approx 2.3 \times 10^{-9} \left(\frac{80}{\psi_i} \right), \tag{4.8}$$

where we do not take the errors of the observations (4.6) and (4.7) into account, because our result (4.8) mainly depends on the initial value of ψ_i. Furthermore, we have not considered the contribution from the scalar density perturbation which will be produced during early stage of the inflationary phase. Although a large fraction of the quadrupole anisotropy seems to be due to GW [25], we have to take that contribution into account for more precise discussions. Note that since the uncertainties just mentioned lead to higher estimates of λ/ξ^2, we may expect that the ''real value'' of λ/ξ^2 is slightly lower than the obtained result (4.8).

Makino and Sasaki [6] (hereafter, refered to as MS) derived the perturbation of spatial curvature $\mathcal{R}_m$ on the hypersurfaces orthogonal to the matter rest frame:

$$\mathcal{R}_m = \frac{N(t_k)}{\pi} \sqrt{\frac{\lambda}{3\xi(1+6\xi)}}, \tag{4.9}$$

where $N(t_k)$ is the e-fold scale which we are interested in. MS found that $\mathcal{R}_m$ for the superhorizon scale is expressed as follows:

$$\mathcal{R}_m = \Phi - \frac{H}{\dot{\phi}} X, \tag{4.10}$$

where Φ is the gauge-invariant metric perturbation in the longitudinal gauge condition, i.e.,

$$ds^2 = (1-2\Phi) dt^2 - a^2 (1+2\Phi) \delta_{ij} dx^i dx^j, \tag{4.11}$$

and X is the gauge-invariant scalar field perturbation. Several authors found that the canonical normal modes of perturbations which diagonalize the full action perturbed to second order in the conformal transformed Einstein frame (e.g., Refs. [6,26]). MS expressed Φ and X in terms of their canonical normal modes Q and π_Q (their notation) and found that $\mathcal{R}_m$ has the same amplitude in both of the Einstein and the original frame.

$\mathcal{R}_m$ is related to the gauge-invariant density perturbation as

$$\frac{\delta\rho}{\rho} = \frac{2(w+1)}{3w+5} \left(\frac{k}{Ha} \right)^2 \mathcal{R}_m, \tag{4.12}$$

where $w \equiv p/\rho$. Using our result (4.8), the density perturbation on the horizon scale ($k=aH$, $N=70$) at a matter-dominated phase ($w=0$) becomes

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm hor} \approx 2.1\sqrt{\frac{\lambda}{\xi^2}} \approx 1.0\times 10^{-4}\sqrt{\frac{80}{\psi_i}}. \qquad (4.13)$$

The density perturbation is also related to the rms temperature fluctuation of CMBR via the Sachs-Wolfe effect as

$$\left(\frac{\delta T}{T}\right)_{\rm rms} = -\frac{1}{3}\Phi. \qquad (4.14)$$

On the other hand, Φ is determined by the Poisson equation,

$$k^2\Phi = -4\pi G\rho a^2\left(\frac{\delta\rho}{\rho}\right) = -\frac{3}{2}(aH)^2\left(\frac{\delta\rho}{\rho}\right). \qquad (4.15)$$

Since our interest is the density perturbation on present horizon scale, i.e., $k=aH$, we have

$$\left(\frac{\delta T}{T}\right)_{\rm rms} = \frac{1}{2}\left(\frac{\delta\rho}{\rho}\right)_{\rm hor} \approx \sqrt{\frac{\lambda}{\xi^2}}. \qquad (4.16)$$

From the COBE-DMR result $(\delta T/T)_{\rm rms}=1.1\times 10^{-5}$ [24], we obtain the *initial condition of our universe* as

$$\psi_i \approx 1.6\times 10^3. \qquad (4.17)$$

We *must* notice again, however, that we have never paid attention to the ratio of the tensor contribution to the scalar one. Since a smaller tensor contribution leads to a smaller value of ψ_i by several factors, our discussion is reliable up to an order of magnitude.

Finally, let us consider the model of $\lambda=10^{-2}$, which would be a reasonable value for some of the particle physics models. Then the nonminimal coupling constant and the initial value of the inflaton field are constrained as

$$\xi \approx 10^4, \qquad (4.18)$$

$$\frac{\phi_i}{m_{\rm Pl}} \approx 10^{-1}. \qquad (4.19)$$

As Makino and Sasaki commented [6], the relaxation of fine-tuning for λ is nothing but a restatement of the issue of a large values of ξ. According to our results, the initial inflaton field has to be smaller by one order of magnitude than that of chaotic scenario in framework of GR. Note that the smaller contribution of tensor perturbation leads to smaller ψ_i and then smaller $\phi_i/m_{\rm Pl}$.

V. CONCLUSIONS

We calculated the spectrum of GWs generated in the chaotic inflationary scenario with a nonminimally coupled inflaton under the condition where $\psi\equiv\kappa^2\xi\phi^2\gg 1$, and their contributions to the CMBR anisotropy via the Sachs-Wolfe effect. Comparing the COBE-DMR 4 yr observations with the predicted power spectrum of CMBR, we find the initial value $\psi_i\approx 1.6\times 10^3$. Our result decreases a number of undetermined parameters in the theory, i.e., ψ_i, and indicates that if we choose $\lambda=10^{-2}$, $\xi\approx 10^4$, and $\phi_i/m_{\rm Pl}\approx 10^{-1}$ are required. In the chaotic inflationary scenario with minimally coupled inflaton, an argument based on the comparison between the energy content of the inflaton field and planck energy density places the upper bound of the initial value of the field. It would be hoped in the nonminimally coupled case that the same sort of argument will place an upper bound on the initial value for ψ_i. However, it is the nonminimality that prevents us from having such a simple argument. In this context Hochberg and Kephart made some discussion on the energy density of a nonminimally coupled scalar field [27], but their result seems to be not available in the present context because of the existence of self-coupling. It should be mentioned that the condition for the Hubble constant to be smaller than the inverse of the planck time is easily satisfied in the above choice of parameters. In this sense the Fakir and Unruh scenario is not excluded as a good candidate of the inflationary scenario.

Finally we mention the chaotic inflationary scenario by induced gravity [28,29]. The above analysis may be applied in exactly same way and one has the same constraint of parameters when $\psi\gg 1$. However, this case has also another constraint on ξ coming from solar system experiment on the omega parameter of Brans-Dicke theory since the theory is translated into Brans-Dick type. These two constraints are incompatible with each other so that the chaotic scenario by induced gravity with *strong* coupling seems not to work.

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[1] K. Sato, Mon. Not. R. Astron. Soc. **195**, 467 (1981); Phys. Lett. **99B**, 66 (1981).
[2] A. Guth, Phys. Rev. D **23**, 347 (1981).
[3] A. D. Linde, Phys. Lett. **129B**, 177 (1983).
[4] R. Fakir and W. G. Unruh, Phys. Rev. D **41**, 1783 (1990).
[5] T. Futamase and K. Maeda, Phys. Rev. D **39**, 399 (1989).
[6] N. Makino and M. Sasaki, Prog. Theor. Phys. **86**, 103 (1991).
[7] R. Fakir, S. Habib, and W. G. Unruh, Astrophys. J. **394**, 396 (1992).
[8] D. S. Salopek, Phys. Rev. Lett. **69**, 3602 (1992).
[9] D. I. Kaiser, Phys. Rev. D **52**, 4295 (1995).
[10] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967).
[11] L. F. Abbott and Mark B. Wise, Nucl. Phys. **B244**, 541 (1984).
[12] M. Sasaki, Prog. Theor. Phys. **89**, 1183 (1993).
[13] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[14] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[15] M. Gasperini and M. Giovannini, Phys. Rev. D **47**, 1519 (1993).
[16] John D. Barrow, José P. Mimoso, and Márcio R. de Garcia Maia, Phys. Rev. D **48**, 3630 (1993).

[17] L. F. Abbott and D. D. Harari, Nucl. Phys. **B264**, 487 (1986).
[18] M. White, Phys. Rev. D **46**, 4198 (1992).
[19] L. Grishchuk and Y. Sidorov, Phys. Rev. D **42**, 3413 (1990).
[20] B. Allen and S. Koranda, Phys. Rev. D **50**, 3713 (1994).
[21] L. P. Grishchuk, Zh. Éksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys. JETP **40**, 409 (1975)]; L. P. Grishchuk, Ann. (N.Y.) Acad. Sci. **302**, 439 (1977).
[22] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[23] D. J. Fixsen *et al.*, Astrophys. J. **473**, 576 (1996).
[24] C. L. Bennett *et al.*, Astrophys. J. **464**, L1 (1996).
[25] L. M. Krauss and M. White, Phys. Rev. Lett. **69**, 869 (1992).
[26] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[27] D. Hochberg and T. W. Kephart, Phys. Rev. D **51**, 2687 (1995).
[28] R. Fakir and W. G. Unruh, Phys. Rev. D **41**, 1792 (1990).
[29] F. S. Accetta, D. J. Zoller, and M. S. Turner, Phys. Rev. D **31**, 3046 (1985).