

Modifications of the BTZ black hole by a dilaton and scalar

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We investigate some modifications of the static BTZ black hole solution due to a chosen asymptotically constant dilaton and scalar. New classes of static black hole solutions are obtained. One of the solutions contains the Martinez-Zanelli conformal black hole solution as a special case. Using the quasilocal formalism, we calculate their mass for a finite spatial region that contains the black hole. Their temperatures are also computed. Finally, using some of the curvature singularities as examples, we investigate whether or not a quantum particle behaves singularly. [S0556-2821(97)01204-6]

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I. INTRODUCTION

Studies of lower dimensional gravitational theories continue to attract the attention of theorists for a number of reasons. It is always of interest to see how the dimensionality of spacetime affects the physical consequences of a given theory. For example, static spherically symmetric (SSS) vacuum solutions of general relativity (GR) in four or higher dimensions are Schwarzschild black holes. However, the same set of vacuum field equations yields a locally flat conical spacetime in three dimensions [1]. In other contexts, lower dimensional theories of gravity are widely used as “toy models” for studying various classical and quantum aspects of gravitation which are conceptually and mathematically difficult to handle in four or higher dimensions.

Many lower dimensional gravitational theories are not without consequence: For example, they admit nontrivial black hole or cosmological solutions (see, e.g., [2] for a review). The black hole conundrum has long been one of the most outstanding problems of modern physics. It has remained in focus as one of the potential testing grounds for quantum-gravitational phenomena for a long time. It is desirable to search for new black hole solutions in any space-time dimension.

We consider here dilaton and scalar-tensor (ST) theories as alternative theories of gravity in three spacetime dimensions. As previously mentioned three dimensional vacuum GR admits no black hole but rather a trivial locally flat (globally conical) solution. One has to either couple matter to GR (e.g., a cosmological constant [3] or dilaton as matter [4]), or consider alternative vacuum (or nonvacuum) gravitational theories in order to get black hole solutions. Motivated by this, we search for new black hole solutions in GR coupled to a self-interacting dilaton and vacuum ST theories. In particular, we will see that these black holes are modifications of the Banados-Teitelboim-Zanelli (BTZ) black hole by an asymptotically constant dilaton and scalar.

In three dimensions, dilaton and ST black holes were obtained in [4] and in [5,6], respectively. The most general action coupled to a dilaton and scalar can be written as [7]

$$S = \int d^3x \sqrt{-g} [C(\phi)\mathcal{R} - \omega(\phi)(\nabla\phi)^2 + V(\phi)], \quad (1)$$

where $\mathcal{R}$ is the scalar curvature, and $V(\phi)$ is a potential function for ϕ . $C(\phi)$ and $\omega(\phi)$ are collectively known as the coupling functions. For simplicity, we have restricted ourselves to one dilaton and scalar ϕ only. It is worthwhile noting that the two-dimensional version of Eq. (1) and generic black hole solutions of its field equations have attracted interest for a period of time (see, e.g., [8]).

Special cases to Eq. (1) in three dimensions were previously considered by a number of authors. The first example is the static BTZ black hole solution to $C(\phi)=1$, $\omega(\phi)=0$, and $V(\phi)=2\Lambda$ [3]. The second example corresponds to the same $C(\phi)$ as above, but with a nontrivial ϕ , $\omega(\phi)=4$, and $V(\phi)=2\Lambda e^{b\phi}$, for which static black hole solutions have been previously derived in [4]. These examples have the condition $C(\phi)=1$, for which the metric coupling to matter is the Einstein metric. This is no longer true when $C(\phi)\neq 1$, and the gravitational force is governed by a mixture of the metric and scalar fields; the class of actions (1) with a nontrivial $C(\phi)$ are called ST theories.

Without loss of generality, $C(\phi)$ can be normalized to ϕ by a scalar field redefinition. Obviously, one can carry out a conformal transformation on the metric to Eq. (1) such that $C(\phi)$ is “rescaled” to 1, and the resulting action is three-dimensional GR coupled to dilaton matter [9]. Adopting the viewpoint that both scalar and metric fields govern the behavior of gravity, Eq. (1) with $C(\phi)=\phi$ is the natural choice. Our terminology is as follows: for $C(\phi)=1$, we call Eq. (1) dilaton gravity (GR coupled to a dilaton); otherwise it is referred to as the ST gravity if $C(\phi)=\phi$. The ST gravity can be interpreted as vacuum gravity [7]. Thus, any solution to it may be called a vacuum solution. In general, one may add a matter action to Eq. (1). For example, Verbin constructed a three-dimensional scalar-tensor gravity in [10]. It is dimensionally reduced from four-dimensional GR coupled to matter. The resultant action becomes Eq. (1) with $\omega(\phi)=V(\phi)=0$ but with an additional matter Lagrangian coupled to the scalar [10].

The third example, corresponding to a static, horizon-free solution to the choice $\omega(\phi)=\omega_0/\phi$ and $V(\phi)=0$, was obtained in [11]. This is a solution to the three-dimensional

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Brans-Dicke theory without a potential term. The fourth example is a three-dimensional ST black hole with the choice $\omega(\phi)=0$ and $V(\phi)=2\Lambda$ [5]. Recently, it was found that static black holes exist in three-dimensional Brans-Dicke theories with a variety of Brans-Dicke parameters and a potential term $V(\phi)=4\lambda^2\phi$ [6]; in particular, when $\omega(\phi)=-1/\phi$, the action becomes the simplest low energy string action. The only black hole solution to this string action is trivially obtained through the product of the two-dimensional black hole with S^1 [4,6]. A review of some of the above-mentioned black hole solutions may be found in [12]. More recently, a three-dimensional black hole solution in the conformal coupling case [$C(\phi)=1-\frac{1}{8}\phi^2$, $\omega=1$, and $V=2\Lambda$] was reported in [13]. In Sec. V, we will show that it is a special case of one of the solutions obtained in this paper. In other contexts, investigations of semiclassical properties of a massless conformally coupled scalar field in the BTZ background were carried out in [14].

We consider in this paper the extraction of new static circularly symmetric (SCS) solutions for the field equations derived from Eq. (1). We will see that these solutions are modifications of the BTZ spacetime in the sense that when the asymptotically constant dilaton and scalar is switched off, they all reduce to the BTZ case. It is impossible to get the most general exact solutions in terms of arbitrary $\omega(\phi)$ and $V(\phi)$, unlike the two-dimensional cases studied in [8]. One needs, for whatever motivation, to specify the choices for the coupling and potential functions, and then solve for exact solutions. Each possible choice of those coupling and potential functions may have interesting solutions. Here, we do not attempt to study the whole set of solutions of different possible choices. Rather, our potential functions are motivated by the forms previously considered in two and four dimensions.

The field equations for $C(\phi)=1$ and $C(\phi)=\phi$ will be derived in Sec. II. In Sec. III, we present the equations of quasilocal formalism developed in [15,16] for calculating mass for a finite spatial region that contains the black hole. In Sec. IV, we consider the dilaton gravity case and get three exact solutions for three different choices of the potential function $V(\phi)$. One solution has a naked timelike singularity, and the other two are black hole solutions with a ring curvature singularity (i.e., the Ricci scalar diverges at a finite circumference). We then investigate ST gravity in Sec. V. We get three exact black hole solutions for three sets of choices of coupling and potential functions. One of the solutions contains the Martinez-Zanelli conformal black hole as a special case. In Sec. VI, we compute the entropy and the temperature of these black holes. Finally, using the naked and ring singularities in the dilaton gravity as examples, we investigate whether or not a quantum particle behaves singularly in such spacetimes, following the approach in [17]. We summarize our results in the concluding section. Our conventions are as in Wald [18].

II. FIELD EQUATIONS

We first derive the field equations for action (1) with the choice $C(\phi)=1$ and $\omega(\phi)=4$. Varying Eq. (1) with respect to the metric and dilaton fields, respectively, yields (after some manipulation)

$$\mathcal{R}_{\mu\nu}=4\nabla_\mu\phi\nabla_\nu\phi-g_{\mu\nu}V, \quad (2)$$

$$8\nabla^2\phi+\frac{dV}{d\phi}=0. \quad (3)$$

This set of field equations was considered in [4] for an exponential potential function. We wish to find exact SCS solutions with a different choice of $V(\phi)$. In 2+1 dimensions, the most general such metric has two degrees of freedom [19], and can be written in the form

$$ds^2=-f(r)dt^2+\frac{h(r)}{f(r)}dr^2+r^2d\theta^2. \quad (4)$$

With the metric (4), the equations of motion (2) and (3) are reduced to equations

$$\frac{f''}{h}-\frac{1}{2}\frac{f'h'}{h^2}+\frac{f'}{rh}=2V, \quad (5)$$

$$h=\exp\left(\int 8r(\phi')^2dr\right), \quad (6)$$

$$\frac{1}{h}\left(f'-\frac{h'f}{2h}\right)=rV, \quad (7)$$

$$\frac{8}{\sqrt{hr}}\left(\frac{fr}{\sqrt{h}}\phi'\right)'+\frac{dV}{d\phi}=0. \quad (8)$$

The prime denotes the ordinary derivative with respect to r . $\mathcal{R}_{tt}$ yields Eq. (5). $\mathcal{R}_{rr}$ and Eq. (5) together yield Eq. (6). $\mathcal{R}_{\theta\theta}$ gives Eq. (7). The last ϕ equation (8) follows from the local conservation law $\nabla^\mu T_{\mu\nu}=0$.

We now derive the field equations for ST theories. Varying Eq. (1) with respect to the remaining fields yields

$$\begin{aligned} G_{\mu\nu}=g_{\mu\nu}\frac{V}{2\phi}+\frac{\omega}{\phi}\left(\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right) \\ +\frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi-g_{\mu\nu}\nabla^2\phi), \end{aligned} \quad (9)$$

$$2\omega\nabla^2\phi+\omega(\nabla\phi)^2\frac{d}{d\phi}\ln(\omega)+\left(\mathcal{R}+\frac{dV}{d\phi}\right)=0, \quad (10)$$

where, without loss of generality, we have set $C(\phi)=\phi$ in Eq. (1). The local conservation equation holds due to the Bianchi identity $\nabla_\mu G^{\mu\nu}=0$ and it implies Eq. (10). We note that one can have $g_{tt}=-1/g_{rr}$ [i.e., $h(r)=1$ in Eq. (4)] without causing $\phi(r)$ to be trivial. For simplicity, we set $h(r)=1$. Now, using Eq. (4) with $h(r)=1$, Eq. (9) becomes

$$f''+\frac{f'}{r}=2\frac{V}{\phi}-\frac{1}{\phi}\left(f'\phi'+\frac{2}{r}(fr\phi')'\right), \quad (11)$$

$$\phi''+\omega(\phi')^2=0, \quad (12)$$

$$f'r=r^2\frac{V}{\phi}-\frac{1}{\phi}(fr^2\phi')'. \quad (13)$$

Similarly, Eq. (10) can be simplified using Eq. (4). It is easy to check that the ST solutions in [5,6,11] satisfy this system of differential equations.

The BTZ black hole solution to the system of differential equations (5)–(8) and (11)–(13) corresponds to $\phi = \text{const}$ and $V = 2\Lambda$ which imply $f = -M + \Lambda r^2$ and $h = 1$. If one constructs nontrivial $\phi(r)$ and $V(\phi)$ with the properties such that $\phi(\infty) \rightarrow \text{const}$ and $V \rightarrow 2\Lambda$, the resultant metrics should asymptotically approach the BTZ one.

In Sec. IV, we will solve the system of differential equations (5)–(8). We first assume a special form for $V(\phi)$ which was previously investigated in other spacetime dimensions. Then, we assume an asymptotically constant $\phi(r)$ to solve for $h(r)$ from Eq. (6). Finally, we use $h(r)$, $\phi(r)$, and the chosen $V(\phi)$ in Eqs. (5) and (7) to simultaneously solve for $f(r)$. Note that Eq. (6) implies that a nontrivial $\phi(r)$ will yield a nontrivial $h(r)$ and vice versa. One cannot have $g_{tt} = -1/g_{rr}$ when $\phi(r)$ is not a constant. Then, in Sec. V, we will solve Eqs. (11)–(13). Similarly, we first assume special forms of $V(\phi)$ and ϕ . By using Eq. (12), we obtain ω . Equations (11) and (13) are then solved consistently for $f(r)$.

III. QUASILOCAL MASS

We will use the quasilocal formalism to compute the mass of the solutions. If a SCS metric is written in the form

$$ds^2 = -f(r)dt^2 + W^{-2}(r)dr^2 + r^2 d\theta^2, \quad (14)$$

then the conserved quasilocal mass measured by a static observer at r (the radius of the spacelike hypersurface boundary) for dilaton gravity [$C(\phi) = 1$] is given by [15]

$$\mathcal{M}(r) = 2\sqrt{f(r)}[W_0(r) - W(r)]. \quad (15)$$

The terms inside the square brackets are identified as the quasilocal energy $E(r)$ which is the thermodynamic internal energy. Thus mass and energy differ by a lapse function. Here $W_0^2(r) = g_0^{rr}$ is a background solution which determines the zero of the energy. The background solution can be obtained simply by setting constants of integration of a particular solution to some special value that then specifies Ref. [16]. When the spacetime is asymptotically flat, the usual Arnowitt-Deser-Misner (ADM) mass M is the mass determined in Eq. (15) in the limit $r \rightarrow \infty$. For solutions with general asymptotic conditions and without any cosmological horizon or curvature singularity at spatial infinity, the large r limit of Eq. (15) is used to determine the analogous ADM mass parameter $M = \mathcal{M}(\infty)$.

For ST gravity, the quasilocal mass, according to Hawking and Horowitz prescription, is given by [16,20,21]

$$\mathcal{M}(r) = 2r\sqrt{f(r)}\left(\phi' + \frac{\phi}{r}\right)(W_0 - W). \quad (16)$$

We see that if $\phi(r) = 1$, then Eq. (16) reduces to Eq. (15). Both equations will be used to identify the analogous mass parameter M in our solutions. Note that the quasilocal mass is not necessarily positive definite for a spacetime [16]. For example, $\mathcal{M}(r)$ of a given solution may admit negative val-

ues for a certain range of r . We will consider that a solution is physical as long as the analogous ADM mass M is positive.

IV. EXACT SOLUTIONS IN DILATON GRAVITY

We begin by searching for solutions in the dilaton gravity case: $C(\phi) = 1$ and $\omega = 4$. We note that potential and/or coupling functions in terms of a polynomial of a certain function of ϕ with unspecified coefficients are not uncommon among field theories; for example, the four-dimensional bosonic action contains a coupling function in terms of a power series: $\sum_n c_n e^{2nk\phi}$, and the coefficients c_n are presently unknown [22]. There are a number of studies in two and four dimensions (see, e.g., [23–26]) in which the authors search for the forms of the potential or coupling functions which yield desired exact solutions. In particular, the idea of deriving the inflation potential in four dimensions by imposing particular theoretical or phenomenological requirements has been widely discussed in recent literature (e.g., [27]). In other contexts, for example, parameters in certain four-dimensional theories in early universe are unknown so that they can be adjusted to avoid certain cosmological difficulties such as the domain wall problem [28]. All the above-mentioned *ad hoc* approaches are justified by the fact that there are no *a priori* underlying principles that uniquely specify the potential or coupling functions. Here, we use the similar approach to obtain three-dimensional black hole solutions with an asymptotically constant dilaton and scalar. Motivated by this, we generalize our potential as a polynomial in certain function with unspecified coefficients which can be adjusted to yield exact solutions. We consider three individual cases in the following.

A. Hyperbolic potential: $\cosh(2\phi)$

In four dimensions, a form of potential $V(\phi) \propto (\mathcal{H}(\phi))^n$, where $\mathcal{H}(\phi)$ is a hyperbolic function, either “sinh” or “cosh,” and n is a real number, has been discussed by a number of authors in the context of massive dilaton black holes [29,30] and cosmologies [31]. We generalize our potential as $V(\phi) = \sum_n a_n \mathcal{H}^n$, a polynomial in $\mathcal{H}(\phi)$. Here, we are, in effect, taking the advantages of the arbitrariness in a_n in $V(\phi)$ and searching for exact solutions by adjusting them. We first choose a dilaton. Since $V(\phi)$ is in terms of a hyperbolic function, it is reasonable to set $\phi(r)$ to be a logarithmic function in order to simplify the field equations. We assume that

$$\phi = \frac{1}{4} \ln \left(1 - \frac{2B}{\sqrt{r^2 + B^2} + B} \right), \quad (17)$$

which is the choice made in four dimensions for asymptotically flat dilaton black holes [32,33]. With this choice of $\phi(r)$, Eq. (6) yields

$$h = \frac{r^2}{r^2 + B^2}. \quad (18)$$

Now using Eqs. (17) and (18) to simultaneously solve Eqs. (5) and (7) for the coefficients a_n and $f(r)$, we are unable to

get exact solutions unless $a_n=0$ ($n=0,2,3,4,\dots$), $a_1=2\Lambda$, and $\mathcal{H}=\cosh(2\phi)$, that is,

$$V=2\Lambda\cosh(2\phi), \quad (19)$$

and

$$f=\Lambda r\sqrt{r^2+B^2}. \quad (20)$$

Summing up, the Lagrangian and the corresponding solution are

$$\mathcal{L}=\mathcal{R}-4(\nabla\phi)^2+2\Lambda\cosh(2\phi), \quad (21)$$

$$ds^2=-(\Lambda r\sqrt{r^2+B^2})dt^2+\frac{rdr^2}{\Lambda(r^2+B^2)^{3/2}}+r^2d\theta^2, \quad (22)$$

$$\phi=\frac{1}{4}\ln\left(1-\frac{2B}{\sqrt{r^2+B^2}+B}\right).$$

The only curvature singularity is located at $r=0$ and it is naked and timelike. In Sec. VII, we will investigate whether or not a quantum particle still behaves singularly when it approaches near the singularity. Using Eq. (15), one can show that the mass parameter at $r\rightarrow\infty$ is $2M=-3\Lambda B^2$ for the choice of the background solution $M=0\Rightarrow W_0(r)=\sqrt{\Lambda}r$. One must have a negative mass solution for the correct signature of the metric. The solution (22) is for a non-vanishing B ; note that $B=0\Rightarrow\phi=0$ and $V=2\Lambda$. So $B=0$ yields a zero mass BTZ solution. As $r\rightarrow\infty$, $\phi\rightarrow 0$, and Eq. (22) asymptotically approaches the massless BTZ solution. Changing the choice of $\phi(r)$ may lead to new solutions but we are unable to get exact black hole solutions with the above hyperbolic potential.

B. First trigonometric potential: $\cot\phi$

We now turn our attention to trigonometric functions in $V(\phi)$. One example is the Sine-Gordon potential which is currently under investigation in two-dimensional curved spacetime (see, e.g., [34]). We are unable to get exact solutions for the Sine-Gordon potential either. Rather than assuming $V(\phi)$ in terms of Sine-Gordon form, we generalize $V(\phi)=\sum_n b_n \mathcal{T}^n(\phi)$, where $\mathcal{T}(\phi)$ is an unspecified trigonometric function. By following the approach similar to that demonstrated above, we find that an exact solution exists when $\mathcal{T}=\cot\phi$. The Lagrangian and solution are

$$\mathcal{L}=\mathcal{R}-4(\nabla\phi)^2+2\Lambda+3\Lambda\cot^2\phi+\left(\frac{4M}{27B^2}-\Lambda\right)\cot^6\phi, \quad (23)$$

$$ds^2=-\left[-M\left(1-\frac{B}{r}\right)+\Lambda r^2\right]dt^2+\frac{\left(1-\frac{3B}{2r}\right)^2}{-M\left(1-\frac{B}{r}\right)+\Lambda r^2}dr^2+r^2d\theta^2, \quad r=\frac{3B}{2}\sec^2(\phi). \quad (24)$$

We first comment on the status of constants B and M which appear explicitly in the Lagrangian. (Λ is the analogous cosmological constant.) There are three possible situations. First, both B and M are fixed coupling constants. Black holes for this action are like discrete bound state solutions since their metrics depend only on coupling constants. Examples of such black holes are not uncommon and can be found in two [25] and four [26,35] dimensions, respectively. Second, both B and M are integration constants. In this case, it is only the combination $M/B^2=b^2$ which appears in the Lagrangian (23) and which should be considered as a coupling constant. Now, replacing $B=\sqrt{M}/b$ in the metric (24), we can regard M as an integration constant in the metric. When using Eq. (15) to compute the mass at spatial infinity when $\Lambda>0$ (no cosmological horizon) by setting $M=0$ in the background solution, one finds that it is diverging. The third case is to consider M as an integration while B as a coupling constant by introducing a three-form field in the action. In Sec. VI, we will illustrate how this can be done. Now using Eq. (15), one finds that $M=\mathcal{M}(\infty)$ is the finite mass parameter when the background $M=0$ is chosen. Although the metric is the same in the second and third situations, it is not asymptotically flat, and therefore, no unique background metric exists. Different choices of integration constants may lead to different backgrounds. As a result, one may get different values of mass for a given r . From now on, we will consider only the third situation.

The dilaton in Eq. (24) implies that $B>0$. Different relative signs and/or magnitudes between B and Λ may lead to solutions with or without event or cosmological horizons. When $\Lambda=0$, the solution is asymptotically flat but M shown in Eq. (24) is no longer the total mass measured at spatial infinity. Instead, one has to use Eq. (15) again to identify the total mass. When $\Lambda=0$, we replace M by C in Eq. (24). Using Minkowski metric as the background we find that the mass is given by $2(1-\sqrt{C})=M$. When we further set $B=0$ and $\phi=\pi/2$, then the metric reduces to the locally flat conical spacetime in three dimensions obtained in [1].

For the $\Lambda\neq 0$ solution, there are several properties.

(i) As $r\rightarrow\infty$, $\phi\rightarrow\pi/2$ in Eq. (24), and $V\rightarrow 2\Lambda$ in Eq. (23). The asymptotic form of the metric is like the BTZ solution. Solution (24) indicates that when $B=0$, one must have $\phi=\pi/2$ and, as a consequence, $V=2\Lambda$. As a result, one gets back the BTZ black hole solution. In that sense, the solution (24) is a result from the modification of the BTZ black hole because of the dilaton given in Eq. (24).

(ii) There is a ‘‘ring’’ curvature singularity at $r_s=3B/2$. If $\Lambda>0$, then the range of r is $\infty>r>3B/2$. One can imagine that there is a special collapsing dilaton fluid which collapses into a ring rather than a point due to a strong enough outward radial pressure at r_s . Note that a similar solution can be found in four dimension: The string magnetic black hole admits a singular two-sphere [32].

(iii) $-g_{tt}=0$ signals the location(s) of event or cosmological horizon(s). Recall that both conditions $M<0$ and $B<0$ are disregarded. A simple graphical analysis indicates that there is only one turning point (a minimum) for $-g_{tt}$ if $\Lambda>0$. The condition $\Lambda>0$ yields at most two real positive roots which are the outer and inner horizons. Since $-g_{tt}=0$ is a cubic equation, the roots are given by

$$r_k = -\sqrt{\frac{4}{3}} \frac{M}{\Lambda} \cos \delta_k \quad \text{and} \quad \cos 3\delta_k = \frac{BM\sqrt{\Lambda}}{2} \left(\frac{3}{M}\right)^{3/2}. \quad (25)$$

The two real positive roots of Eq. (25) are the outer and inner horizons $r_{\pm}$. Concerning the $\pm$ sign of the horizon, from now on it is understood that when there is only one event horizon the $+$ sign is chosen. As long as $4M \geq 27B^2\Lambda$, the outer and inner horizons do not vanish. As long as r_+ (or r_-) is greater than $r_s = 3B/2$, the ring curvature singularity is not naked. The extremal condition is $4M = 27B^2\Lambda$ and, consequently, the third term of corresponding potential function [in Eq. (23)] vanishes. For the extremal case, using Eq. (25), it is easy to see that $r_+ = r_- = (BM/2\Lambda)^{1/3} = 3B/2 = r_s$. The outer horizon, inner horizon, and ring singularity coincide and the metric is given by

$$ds^2 = -\frac{\Lambda}{r}(r+3B) \left(r - \frac{3}{2}B\right)^2 dt^2 + \frac{dr^2}{\Lambda r(r+3B)} + r^2 d\theta^2, \quad (26)$$

$$B = 2\sqrt{\frac{M}{27\Lambda}}.$$

(iv) We now consider the causal structure of the black hole solution. Since $-g_{tt}$ in Eq. (24) is cubic, we are unable to generally integrate g_{rr} to get the tortoise coordinate r_* . However, we can still deduce what the black hole causal structures look like. Note first that Eq. (24) asymptotically approaches the BTZ metric; therefore, the null infinity is timelike. If $r_+ > r_s > r_-$, then one may deduce that the ring singularity is spacelike. Thus, the causal structure of case $r_+ > r_s > r_-$ is like the static and uncharged BTZ causal structure. We will not repeat the drawing here. If $r_+ > r_- > r_s$, then the ring singularity would be timelike (every time we cross a horizon, the space and time coordinates interchange roles). As a result, the causal structure of case $r_+ > r_- > r_s$ looks like the spinning BTZ black hole. For the extremal case, $r_+ = r_- > r_s$, it looks like the extremal BTZ case.

C. Second trigonometric potential: $\cot\sqrt{2}\phi$

There is a second solution if we consider both $\mathcal{T} = \cot\sqrt{2}\phi$. The action and solutions are

$$\mathcal{L} = \mathcal{R} - 4(\nabla\phi)^2 + 2\Lambda + 2\Lambda \cot^2\sqrt{2}\phi + \left(\frac{M}{2L} - 2\Lambda\right) \cot^4\sqrt{2}\phi + \left(\frac{M}{2L} - 2\Lambda\right) \cot^6\sqrt{2}\phi, \quad (27)$$

$$ds^2 = -\left[-M\left(1 - \frac{L}{r^2}\right) + \Lambda r^2\right] dt^2 + \frac{\left(1 - \frac{2L}{r^2}\right)^2}{-M\left(1 - \frac{L}{r^2}\right) + \Lambda r^2} dr^2 + r^2 d\theta^2, \quad r^2 = 2L \sec^2(\sqrt{2}\phi). \quad (28)$$

$M = \mathcal{M}(\infty)$ is the mass parameter. As before, we only consider $M > 0$ and the dilaton shown in Eq. (28) implies $L > 0$. The properties of this solution are very similar to those

of the previous one. Again, there is a ring curvature singularity located at $r_s^2 = 2L$. The range of r^2 is, therefore, $\infty > r^2 > 2L$ if $\Lambda > 0$. When $L = 0$, Eq. (28) reduces to the BTZ case. One must have $\Lambda > 0$ for the existence of an event horizon(s). The location of the outer and inner horizons are given by

$$r_{\pm}^2 = \frac{M \pm \sqrt{M^2 - 4\Lambda LM}}{2\Lambda}. \quad (29)$$

It can be checked that r_+^2 (r_-^2) is always greater (less) than r_s^2 . Thus the inner horizon is physically irrelevant. For the extremal case, $r_+^2 = r_-^2 = r_s^2 = 2L = M/2\Lambda$; all horizons coincide together and the third and fourth terms of the corresponding potential function in Eq. (27) vanish. The causal structures of Eq. (28) can be deduced in a manner similar to Eq. (24). We will not repeat the discussion here.

V. EXACT SOLUTIONS IN ST GRAVITY

We now look for solutions in ST gravity described by field equations (9) and (10). Similar to the dilaton gravity cases studied in the previous section, we consider a potential function which was previously investigated in four-dimensional ST gravity. We note that a potential of the form $\Sigma_n c_n \phi^n$ (a polynomial of degree $n=4$) was discussed in [36] for a special class of ST gravity in four dimensions. Here, we again generalize V to a polynomial: $\Sigma_n c_n \phi^n$. We will adjust the coefficients c_n to get exact solutions. Given this form of $V(\phi)$, we find that there exist three exact solutions corresponding to three different choices of scalar.

A. Case (i): $\phi = r/(r - 3B/2)$

The first example corresponds to the choice

$$\phi = \frac{r}{r - 3B/2}. \quad (30)$$

Now Eq. (12) yields the coupling function

$$\omega = \frac{2}{1 - \phi}. \quad (31)$$

By following the similar procedures outlined above, the action and solution are

$$\mathcal{L} = \phi \mathcal{R} - \frac{2}{1 - \phi} (\nabla\phi)^2 + 2(3 - 3\phi + \phi^2)\Lambda\phi + \frac{8M}{27B^2} (1 - \phi)^3, \quad (32)$$

$$ds^2 = -\left[-M\left(1 - \frac{B}{r}\right) + \Lambda r^2\right] dt^2 + \frac{dr^2}{-M(1 - B/r) + \Lambda r^2} + r^2 d\theta^2, \quad \phi = \frac{r}{r - 3B/2}. \quad (33)$$

It can be checked that Eq. (10) is satisfied as well. M is the mass parameter calculated using Eq. (16) in the background $W_0 = W(M=0)$ and the limit $r \rightarrow \infty$. We require that $M > 0$.

Unlike the dilaton case, B is no longer required to be positive. The metric looks like the Schwarzschild-(anti-)de Sitter metric except for the position of the mass term. If $\Lambda = 0$, the metric is exactly the same form as the four-dimensional Schwarzschild case. There are several properties of metric (33).

(i) As $r \rightarrow \infty$, $\phi \rightarrow 1$, $V(\phi) \rightarrow 2\Lambda$, and $\omega(\phi) \rightarrow \infty$, the metric asymptotically approaches the BTZ metric. Obviously, if $B = 0$, then $\phi = 1$ and the metric is exactly the BTZ black hole. The BTZ black hole is, therefore, modified by the scalar in Eq. (30).

(ii) Although $\mathcal{R} = -6\Lambda$ (constant curvature scalar), the Krestmann scalar diverges as $r \rightarrow 0$. The only curvature singularity is located at $r = 0$. Note that unlike the dilaton case, $r = 3B/2 > 0$ does not correspond to a curvature singularity but rather to a diverging $\phi(r)$ and $V(\phi)$ in Eqs. (30) and (32). This situation is similar to the situation in the four-dimensional Bekenstein black hole [37], where the conformal coupling scalar in the action diverges at the event horizon, and to the situation in several classes of two-dimensional black holes [25], where the kinetic term of a scalar field in the action diverges at the outermost event horizon. In [25,37] the geometry is still well behaved at the event horizon. In the present case, the geometry is well behaved at $r = 3B/2$, although a nonfreely falling test particle coupling to $\phi(r)$ and traveling towards the event horizon from infinity might encounter a diverging barrier at this value of r . We will not consider this as a disaster since the existence of such a barrier solely depends on how the nonfreely falling test particle couples to ϕ , and there is no unique way of determining what this coupling should be (see discussions in [25]). The only restriction we impose is $r_+ \neq 3B/2$ in order to avoid a diverging entropy S , since $S \propto \phi(r_+)$ (see next section).

(iii) An interesting situation is given by the conditions $B > 0$, $\Lambda > 0$, and $4M \geq 27B^2\Lambda$ in which the metric (33) generally admits an outer and an inner horizon. This example shows that the existence of an inner horizon is not necessarily due to the presence of an electric charge or angular momentum. If the equality sign holds for the latter, the outer and inner horizons will coincide, $r_+ = r_- = 3B/2$. One may expect a diverging entropy for this extremal black hole. However, an extremal black hole needs a separate treatment in thermodynamics.

(iv) A special case of interest is given by the condition $B = -2\sqrt{M/27\Lambda}$. This condition yields $V = 2\Lambda$ in Eq. (32). If we carry out a scalar redefinition $\phi = 1 - \frac{1}{8}\Psi^2$, the corresponding theory and solution are

$$S = \int d^3x \sqrt{-g} [(1 - \frac{1}{8}\Psi^2)\mathcal{R} - (\nabla\Psi)^2 + 2\Lambda], \quad (34)$$

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\theta^2,$$

$$F = \Lambda \left[r^2 - \frac{M}{\Lambda} - \frac{2}{r} \left(\frac{M}{3\Lambda} \right)^{3/2} \right],$$

$$\Psi = \sqrt{8} \sqrt{\frac{\sqrt{M/3\Lambda}}{r + \sqrt{M/3\Lambda}}}. \quad (35)$$

The event horizon is located at

$$r_+ = 2 \sqrt{\frac{M}{3\Lambda}}. \quad (36)$$

The above action and solution are exactly the conformal black hole obtained in [13] by Martinez and Zanelli.¹ Starting from action (34), the authors solved the field equations directly to get Eqs. (35).

(v) Note that the coupling function (31) is a function of r . It has different signs for different ranges of r . For example, $\omega(\phi) > 0$ for $B > 0$ and $r < 3B/2$, and $\omega(\phi) < 0$ for $r > 3B/2$. In some sense, the kinetic term flips sign. This situation is very similar to the cases in [25]. The physics of the transition $\omega > 0 \leftrightarrow \omega < 0$ and the stability of the above solution are open questions to be investigated.

B. Case (ii): $\phi = 2/(2 + kr)$

By assuming a different form of scalar, we get

$$\mathcal{L} = \phi \mathcal{R} + \frac{2}{\phi} (\nabla \phi)^2 + 2 \left(\Lambda - \frac{Ak^2}{4} \right) \phi^3, \quad (37)$$

$$ds^2 = -[A(1 + kr) + \Lambda r^2] dt^2 + \frac{dr^2}{A(1 + kr) + \Lambda r^2} + r^2 d\theta^2,$$

$$\phi = \frac{2}{2 + kr}. \quad (38)$$

This metric is exactly the same form as the one obtained in a special class of two-dimensional ST gravity [2]. It is easy to show that the mass $M \rightarrow 0$ as $r \rightarrow \infty$ in Eq. (38) if $k \neq 0$. Thus, the analogous ADM mass vanishes. This “massless” black hole is in fact a Brans-Dicke-type black hole with a non-vanishing potential. If $k = 0$, A is identified as the negative mass parameter, that is to say, $A = -M$, and the usual BTZ metric is recovered. Note that a similar form of metric (38) was previously obtained in [6] in three-dimensional Brans-Dicke theory with a different choice of $\phi(r)$, and with $V(\phi) = 0$ and $\omega = -2/\phi$.

The potential function in Eq. (37) and scalar in Eq. (38) both diverge at $r = -2/k$. The existence of the event horizon(s) can easily be investigated by graphical analysis. It is interesting to note that for the extremal case, $Ak^2 = 4\Lambda$, the potential function vanishes.

¹Note that the mass parameter $\tilde{M}$ in [13] is one-eighth of M shown in Eq. (35). That is, $8\tilde{M} = M$. It is because the authors defined the entropy in general relativity as one-fourth of the area of the event horizon, while we use the definition [3,15] that in three-dimensional general relativity, the entropy is twice the area.

C. Case (iii): $\phi = r^2/(r^2 - 2L)$

Finally, a third choice of an asymptotically constant scalar yields

$$\mathcal{L} = \phi \mathcal{R} - \frac{4\phi - 1}{2\phi(1 - \phi)} (\nabla \phi)^2 + \frac{M}{2L} + 6 \left(2\Lambda - \frac{M}{2L} \right) \phi + 18 \left(-\Lambda + \frac{M}{4L} \right) \phi^2 + 2 \left(4\Lambda - \frac{M}{L} \right) \phi^3, \quad (39)$$

$$ds^2 = - \left[-M \left(1 - \frac{L}{r^2} \right) + \Lambda r^2 \right] dt^2 + \frac{dr^2}{-M \left(1 - \frac{L}{r^2} \right) + \Lambda r^2} + r^2 d\theta^2, \quad \phi = \frac{r^2}{r^2 - 2L}. \quad (40)$$

M is the mass parameter. The metric has a curvature singularity at $r=0$. The scalar and its potential both diverge at $r^2=2L$ with $L>0$. The extremal limit is $4\Lambda L=M$. In this limit, the third term in the Lagrangian becomes 2Λ , while fourth, fifth, and sixth terms all vanish. The properties of Eq. (40) are very similar to those of the previous cases and we do not repeat the discussions here. Note that one can generalize the three scalars in Eqs. (30), (38), and (40) to $\phi = r^i/(C_1 r^j + C_2)$, where i and j are integers. However, we are unable to get exact solutions for arbitrary i and j .

Before ending this section, we note two things. First, a similar *ad hoc* procedure was adopted by Bechmann and Lechtenfeld [26] to study the modifications of the four-dimensional Schwarzschild black hole in general relativity. They reversed the role of the dilaton potential $V(\phi)$, keeping the choice of $V(\phi)$ initially unspecified and then solving it by using a desired $\phi(r)$ and an asymptotic form of the metric. As a result, the Schwarzschild mass parameter appears in the potential explicitly. In this and the previous section, we have done a similar thing: We studied how a dilaton or scalar modifies the three-dimensional BTZ black hole in general relativity. We use the freedom in the coefficients of the potential and coupling functions to construct exact solutions. Those potential forms are not completely arbitrary since they were previously studied in other dimensions. Second, note that the above dilaton and ST solutions are obviously not unique to the field equations followed from Eq. (1): changing the choices of those dilatons, scalars, and coefficients and/or the forms of those potential and coupling functions can lead to other solutions. They may or may not be the modification of BTZ solutions. Here, we just illustrate some possible solutions which are modifications of the BTZ case. In the next section, we will briefly discuss how the mass parameter in the actions can be absorbed by a three-form and some of the thermodynamic properties of the solutions.

VI. TEMPERATURE

In all of the above black hole solutions, the analogous ADM mass parameter $M = \mathcal{M}(\infty)$ identified at spatial infinity through Eqs. (15) and (16) appears as a universal constant in the actions. In this sense, black holes exist only with specific masses determined by some of the parameters in the actions. Black holes for these actions are thus rather like

discrete bound state solutions. This is not of much interest since one can only consider one solution of the field equations at a time. It is much more satisfactory to regard M as an integration constant rather than as a universal constant in the action. In order to treat M as a dynamical variable, one must be able to vary it in the action. This is impossible for the above black hole solutions since M appears explicitly in the potential functions. A way is needed to restore M to its status as an integration constant in such a way as to yield the same effective field equations and solutions.

Similar to the method used in four dimensions [16,38], where the cosmological constant is treated as a dynamical variable by coupling gravity to a four-form field, we will couple gravity to a gauge-invariant three-form field $H_{\mu\nu\rho}$, where $H_{\mu\nu\rho} = \nabla_{[\mu} A_{\nu\rho]}$ and $A_{\mu\nu}$ is a completely antisymmetric two-index gauge field. Note that three-form fields arise naturally in low energy string gravity in any dimension, except in two dimensions where any three-form vanishes. However, one can also regard the $H_{\mu\nu\rho}$ as a possible matter field to any nonstring theory of gravity. Here, our emphasis is on how a three-form matter field can introduce a dynamical variable to three-dimensional dilaton and ST gravitational theories. We only discuss $H_{\mu\nu\rho}$ coupling in the dilaton gravity cases, since the generalization of it to ST gravity is straightforward.

The generalized action for such a three-form coupled to dilaton gravity can be written as [16]

$$S = \int d^3x \sqrt{-g} [\mathcal{R} - 4(\nabla \phi)^2 + V(\phi) + l(\phi) H_{\mu\nu\rho} H^{\mu\nu\rho}], \quad (41)$$

where $l(\phi)$ is another coupling function. The $H_{\mu\nu\rho}$ equation of motion is given by

$$\nabla^\mu (l(\phi) H_{\mu\nu\rho}) = 0, \quad (42)$$

and its solution is

$$H_{\mu\nu\rho} = \frac{q}{l(\phi)} \epsilon_{\mu\nu\rho}, \quad (43)$$

where ϵ is the volume three-form, and q is an integration constant (dynamical variable) which is proportional to the conserved charge (analogous to the electric charge for the two-form Maxwell field) associated with the gauge field $A_{\mu\nu}$. Mathematically, $V(\phi)$ is a zero-form, $\partial_\mu \phi$ is a one-form, and $H_{\mu\nu\rho}$ is a three-form. Thus, action (41) accommodates n -form ($n=0,1,3$) fields in three-dimensional dilaton gravity.

It is not difficult to check that the last term in action (41) produces an effective potential term: $V_{\text{eff}} = 6q^2/l(\phi)$ in the right-hand side of Eq. (2) and $dV_{\text{eff}}/d\phi$ in the left-hand side of Eq. (3). For the Lagrangian (23) of the black hole solution (24), we write the effective potential term as

$$V_{\text{eff}} = \frac{4M}{27B^2} \cot^6 \phi, \quad (44)$$

by setting $q^2 = M$ and $l^{-1}(\phi) = \frac{2}{81} B^2 \cot^6 \phi$. Thus, for a fixed B , M can now be treated as an integration constant (dynamical variable). For the second dilaton black hole solution (28), we can write

$$V_{\text{eff}} = \frac{M}{2L} (\cot^4 \sqrt{2} \phi + \cot^6 \sqrt{2} \phi), \quad (45)$$

by setting $q^2 = M$ and $l^{-1}(\phi) = L/12 (\cot^4 \sqrt{2} \phi + \cot^6 \sqrt{2} \phi)$. To sum up, by using the generalized n -form action (41), the mass parameter of the dilaton black hole solutions to the effective field equations of Eq. (41) is now an integration constant instead of universal constant. One can also add a similar three-form matter field to the ST gravity to represent the part(s) of the potential functions that has mass dependence.

We first compute the entropy of the dilaton black hole solutions. It is now well recognized that nonextremal black hole entropy S depends only on the geometry of the horizon. General arguments such as those in four dimensions [16,39] show that

$$S = 4\pi r_+ \quad (46)$$

in three-dimensional stationary black holes in Einstein gravity coupled to “dirty matter” (e.g., dilaton and generalized n -form-gauged fields), where r_+ is the outermost event horizon given by Eqs. (25) and (29). The temperature is generally given by

$$T = \frac{\kappa_H}{2\pi} \frac{1}{N}, \quad (47)$$

where $N(r)$ is the lapse function for a stationary black hole spacetime, and κ_H is the surface gravity [16,40]. For an asymptotically nonflat spacetime, $N(r)$ does not approach 1 as $r \rightarrow \infty$. In fact, $\kappa_H/2\pi$ is just the temperature at the spatial location where $N(r) = 1$, not the Hawking temperature. For an asymptotically flat black hole, one recovers the usual Hawking temperature $T_H = \kappa_H/2\pi$ since $N(\infty) \rightarrow 1$. We only show the surface gravity in the following. For the black hole metric (24), the temperature is

$$T = \frac{-MB + 2\Lambda r_+^3}{4\pi r_+ (r_+ - 3B/2)}. \quad (48)$$

It is easy to check that when $B=0$, the above expression reduces to the BTZ case studied by Brown, Creighton, and Mann in [15]. Recall that for the extremal black hole, one has $r_+^3 = MB/2\Lambda = (3B/2)^3$ and, therefore, $T_{\text{ext}} = 1/4\pi r_+$ which is nonvanishing. For the second dilaton black hole (28), its temperature is

$$T = \frac{(-ML + \Lambda r_+^4)}{2\pi r_+ (r_+^2 - 2L)}. \quad (49)$$

When $L=0$ the temperature reduces to the BTZ case and $T = 1/2\pi r_+$ for the extremal case.

We now turn to the ST black hole solutions. Because of the presence of ϕ in the Ricci scalar in action (1), the entropy is modified to be [16,40]

$$S = 4\pi r_+ \phi(r_+). \quad (50)$$

Thus, for black holes (33) and (40), the entropies are

$$S = \frac{4\pi r_+^2}{r_+ - 3B/2} \quad (51)$$

and

$$S = \frac{4\pi r_+^3}{r_+^2 - 2L}, \quad (52)$$

respectively. Note that in both solutions the entropy is ill defined in the extremal cases where $r_+ = 3B/2$ and $r_+^2 = 2L$, respectively. The temperature is still given by the surface gravity divided by the lapse [40]. For Eq. (33), it is given by

$$T = \frac{1}{4\pi} \left(-\frac{MB}{r_+^2} + 2\Lambda r_+ \right) \quad (53)$$

and for Eq. (40), it becomes

$$T = \frac{1}{2\pi} \left(-\frac{ML}{r_+^3} + \Lambda r_+ \right). \quad (54)$$

In both cases, T vanishes in the extremal limits. In addition, they reduce to the BTZ temperature when $B=0$ and $L=0$, respectively. In particular, for the MZ solution (35), recall that one has $B = -2\sqrt{M/27\Lambda}$. Using Eqs. (51) and (53), we get $S = (8\pi/3)r_+$ and $T = (9\Lambda/16\pi)r_+$. Equation (36) implies that $M = (3\Lambda/4)r_+^2$. Now, it is easy to check that $dM = TdS$ as shown in [13].

Note that using the formalisms discussed in [16,40], one may compute various thermodynamic quantities for all the black hole solutions in this paper and study in detail their thermodynamics.²

VII. QUANTUM PROBE OF SINGULARITY

In general relativity, a spacetime is said to be singular if it is (timelike, null, spacelike) geodesically incomplete. In particular, if a spacetime admits a timelike singularity, then the evolution of a test particle is not defined after a finite proper time. Using the dilaton solutions of this paper as examples, we will investigate whether they are examples of static spacetimes which admit timelike singularities in which the evolution of a quantum test particle is completely well defined near the singular regions of the classical spacetimes.

Note that if the spacetime is free of any timelike singularity and globally hyperbolic, then the evolution of a quantum test particle must be essentially well defined. We will adopt the point of view [17] that a quantum particle is non-singular when the evolution of any state is uniquely defined

²For example, one may study the first law, $dE = TdS + \dots$ (work terms), where E is the quasilocal energy. Note that one should let the asymptotic value of dilaton and scalar to be arbitrary, $\phi(\infty) \rightarrow \phi_o$, since ϕ_o is considered as one of the thermodynamic variables [21].

for all time. If this is not the case, then there is some loss of predictability and we will say that the particle is singular. The existence of a nonsingular quantum particle can be translated into the existence of the self-adjointness of the spatial part of the wave operator [17,41]. A relativistic spinless particle with mass $m \geq 0$ is described quantum mechanically by a positive frequency solution to the wave equation of mass m , that is the Klein-Gordon equation in curved spacetime,

$$\nabla^2 \psi - m^2 \psi = 0, \quad (55)$$

where ψ is the wave function of the particle. For a static spacetime with a timelike Killing vector field ξ^μ with Killing parameter t (the time coordinate x^0), the wave equation (55) can be rewritten in the form

$$\frac{\partial^2 \psi}{\partial t^2} = \Xi D^i (\Xi D_i \psi) - \Xi^2 m^2 \psi, \quad (56)$$

where $\Xi^2 = -\xi^\mu \xi_\mu = -g_{tt}$, and D_i ($i=1,2$) is the spatial covariant derivative on Σ , a static slice. Let $\mathcal{A}$ denote (minus) the operator on the right-hand side:

$$\mathcal{A} = -\Xi D^i (\Xi D_i) + \Xi^2 m^2. \quad (57)$$

The question is then whether or not $\mathcal{A}$ is essentially self-adjoint for a spacetime which admits a timelike singularity. It is our adopted criterion for defining the quantum regularity or singularity. The meaning of self-adjointness (see [17]) can be summarized as follows: Consider the equation $\mathcal{A}\psi \pm i\psi = 0$ ($i = \sqrt{-1}$). Using the separation of variables $\psi = Y(r)\Phi(\theta)$ and metric (4), one gets the following radial equation for Y :

$$Y'' + \left(\frac{f'}{f} - \frac{h'}{2h} + \frac{1}{r} \right) Y' - \left(\frac{c^2 h}{f r^2} + \frac{m^2 h}{f} \pm i \frac{h}{f^2} \right) Y = 0. \quad (58)$$

c^2 is an eigenvalue of (minus) the Laplacian on the one-sphere. The operator $\mathcal{A}$ will be essentially self-adjoint in the neighborhood of a radial coordinate if one of the two solutions (for each constant c^2 and each sign of the complex term) to Eq. (58) fails to be square integrable with respect to the measure $r\sqrt{h}/\Xi^2 = r\sqrt{h}/f$, that is if

$$I = \int Y Y^* \frac{r\sqrt{h}}{f} dr \quad (59)$$

diverges in the neighborhood of the radial point of question [17]. Here, we will take the radial point near the timelike singularity. Define a new radial coordinate $dr_* = g_{rr} dr$ so that the radial null geodesics follow curves of constant $t \pm r_*$. If the singularity is at a finite r_* , then it is timelike.

Since there are many solutions with naked timelike singularities in three spacetime dimensions, we plan not to investigate them all. Our intention in this section is to just illustrate the idea by using several examples in dilaton cases. We first investigate the naked solution (22). The curvature singularity is located at $r=0$ and the finiteness of r_* at $r=0$ indicates that it is timelike. As $r \rightarrow 0$, the m^2 and complex terms in Eq. (58) may be ignored. In addition, $f'/f \rightarrow 1/r$ and

$h'/2h \rightarrow 1/r$. Thus, in the neighborhood of $r=0$, Eq. (58) becomes $rY'' + Y' - (c^2/\Lambda B^3)Y = 0$ which has the solution $Y = C_1 I_0(\alpha\sqrt{r}) + C_2 K_0(\alpha\sqrt{r})$, where C_1 and C_2 are arbitrary constants, $\alpha = 2c/\sqrt{\Lambda B^3}$, and I_0 and K_0 are the modified Bessel functions of first and second kind of zero order, respectively. $K_0 \rightarrow \ln(r)$ and $I_0 \rightarrow 1$ as $r \rightarrow 0$. Therefore, it is easy to see that Eq. (59) is finite near $r=0$. Thus, $\mathcal{A}$ is not self-adjoint near $r=0$. Previous examples of spacetimes with timelike singularities which lead to singular quantum particles can be found in four-dimensional negative mass Schwarzschild solution and Reissner-Nordström solution [17].

Next, we consider solution (24). We set $M=0$ and choose $\Lambda > 0$. Now, the ring curvature singularity at $r_s = 3B/2$ is naked and timelike. Note that $f(r_s) \neq 0$ but $h(r_s) = 0$. As $r \rightarrow r_s$, the c^2 , m^2 , and complex terms in Eq. (58) can be ignored. Only the $h'/2h$ term is dominating. As a result, one has $Y' \propto \sqrt{h}$. If we define $\rho = 2r - 3B$, then $\rho = 0 \Leftrightarrow r = r_s$ and near $\rho = 0$, the only form of solution is $Y = C_1 \rho + C_2 \ln(\rho + 3B)$. It can be checked that I in Eq. (59) is finite and $\mathcal{A}$ is not self-adjoint. The final example is the solution (28). Similar to the previous case, we set $M=0$. We further set $\rho^2 = r^2 - 2L$ and $\rho = 0$ is the location of the ring curvature singularity. The repetition of the previous analysis indicates that $\mathcal{A}$ is not self-adjoint near $\rho = 0$. Thus, all three examples above lead to singular quantum test particles. It is worthwhile mentioning the static and chargeless BTZ case with $M < 0$. It has no curvature timelike singularity at $r=0$. It is easy to check that near $r=0$, one solution has the form $Y \propto r^{-c/\sqrt{-M}}$ which can lead to a diverging I near $r=0$. Thus, the operator $\mathcal{A}$ is self-adjoint as expected.

One can proceed similarly in the ST gravity cases, but with a different form of $\mathcal{A}$, which is modified to include the coupling of the scalar ϕ to quantum test particles (see [17]). We intend not to discuss them further. It is worthwhile to mention that there are no generic theorems to show that a quantum test particle must be nonsingular (in the sense of self-adjointness of $\mathcal{A}$) in a static spacetime with a timelike singularity. Thus, the examples shown above do not lead to any problem (it is indeed interesting to understand under what conditions a quantum particle will behave nonsingularly in a singular spacetime). However, when approaching the strong field regime, the action (1) might need to be modified (by properly introducing several extra dilaton and scalar fields) due to quantum gravitational effects. Then, one may use the modified solutions to investigate the singularity or regularity of a quantum test particle. In the above solutions, we just assume that they are all still valid in the neighborhood of the singularities.

VIII. CONCLUSION

We have found five black hole solutions to three-dimensional dilaton and ST gravity. When there are no cosmological horizons ($\Lambda > 0$), four of them asymptotically approach the BTZ metric at the spatial infinity. The forms of potential functions, which are polynomials, are motivated from their two or four-dimensional counterparts. We leave the coefficients of the polynomials, and the forms of coupling functions, unspecified. Here, we took the advantage of this arbitrariness to search for exact solutions. In the cases of

dilaton gravity, the standard vacuum theory is general relativity and we search for certain dilaton matter actions in which their solutions contain the BTZ spacetime as a special case. In the cases of ST gravity, we construct vacuum theories which yield similar solutions. Once particular coupling and potential functions have been determined, one can regard the action containing these functions as a separate matter or theory in its own right and further explore the solution space of its field equations. Obviously, the action is a primary quantity. It is more interesting to know in advance what actions one can use and then extracting the corresponding solutions, than to construct the actions from solutions.

In the dilaton gravity, three different forms of potential yield three solutions, respectively. Two of them are black holes with a ring curvature singularity at a finite proper circumference which is hidden by the outer horizon. In the ST gravity, three different choices of coefficient for a given form of potential yield three different black hole solutions. One solution contains the Martinez and Zanelli conformal black hole as a special case. In contrast with the dilaton cases, none of them has any ring singularity. The only curvature singularity is located at $r=0$ in each case.

In all of the solutions, we use the quasilocal formalism to identify the analogous ADM mass parameter M . We find that black holes exist only for their mass M determined by parameters in the potentials, and so M appeared as a universal constant in the action. However, by including a coupling of dilaton or ST gravity to a three-form field yields an effective potential term in the field equations. By setting M to be proportional to the charge of the three-form field, its status as a constant of integration is restored. We calculate the temperature and entropy of the black hole solutions. They also

provide examples to illustrate how one can investigate whether or not a quantum particle behaves nonsingularly in the sense of self-adjointness of the operator $\mathcal{A}$.

It is interesting to see whether or not one can also add angular momentum to the present dilaton and ST solutions, although the work is nontrivial. Our present solutions depend on the choice of an asymptotically constant $\phi(r)$, the coefficients of the potentials, and the coupling functions. By considering other choices of $\phi(r)$ and potential or coupling functions, one may extract more solutions which may or may not be containing the BTZ as special cases. Also, $\phi(r)$ is not necessarily required to be asymptotically constant, and examples of such black hole solutions can be found in three [4,42] and four dimensions [43]. We hope that our paper suggests a way for the derivation of more generic and interesting solutions. Recently, Cai and Zhao [44] used the inner (Cauchy) horizon of a class of $(2+1)$ -dimensional static charged dilaton black holes obtained in [4] to test the conjecture of Cauchy horizon stability suggested by Helliwell and Konkowski. The conjecture is found to be valid. Similar investigations were carried out in [45] for the spinning BTZ black hole. Similarly, one can check the conjecture using the inner (Cauchy) horizon of the ST black holes obtained in this paper. We intend to relate further details elsewhere.

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