

Pair creation of black holes during inflation

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Black holes came into existence together with the universe through the quantum process of pair creation in the inflationary era. We present the instantons responsible for this process and calculate the pair creation rate from the no boundary proposal for the wave function of the universe. We find that this proposal leads to physically sensible results, which fit in with other descriptions of pair creation, while the tunneling proposal makes unphysical predictions. We then describe how the pair-created black holes evolve during inflation. In the classical solution, they grow with the horizon scale during the slow roll down of the inflaton field; this is shown to correspond to the flux of field energy across the horizon according to the first law of black hole mechanics. When quantum effects are taken into account, however, it is found that most black holes evaporate before the end of inflation. Finally, we consider the pair creation of magnetically charged black holes, which cannot evaporate. In standard Einstein-Maxwell theory we find that their number in the presently observable universe is exponentially small. We speculate how this conclusion may change if dilatonic theories are applied. [S0556-2821(96)00222-6]

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I. INTRODUCTION

It is generally assumed that the universe began with a period of exponential expansion called inflation. This era is characterized by the presence of an effective cosmological constant Λ_{eff} due to the vacuum energy of a scalar field ϕ . In generic models of chaotic inflation [1,2], the effective cosmological constant typically starts out large and then decreases slowly until inflation ends when $\Lambda_{\text{eff}} \approx 0$. Correspondingly, these models predict cosmic density perturbations to be proportional to the logarithm of the scale. On scales up to the current Hubble radius H_{now}^{-1} , this agrees well with observations of near scale invariance. However, on much larger length scales of order $H_{\text{now}}^{-1} \exp(10^5)$, perturbations are predicted to be on the order of 1. Of course, this means that the perturbational treatment breaks down; but it is an indication that black holes may be created, and thus warrants further investigation.

An attempt to interpret this behavior was made by Linde [3,4]. He noted that in the early stages of inflation, when the strong density perturbations originate, the quantum fluctuations of the inflaton field are much larger than its classical decrease per Hubble time. He concluded that, therefore, there would always be regions of the inflationary universe where the field would grow, and so inflation would never end globally ("eternal inflation"). However, this approach only allows for fluctuations of the field. One should also consider fluctuations which change the topology of space-time. This topology change corresponds to the formation of a pair of black holes. The pair creation rate can be calculated using instanton methods, which are well suited to this nonperturbative problem.

One usually thinks of black holes forming through gravitational collapse, and so the inflationary era may seem an unlikely place to look for black holes, since matter will be hurled apart by the rapid cosmological expansion. However, there are good reasons to expect black holes to form through the quantum process of pair creation. We have already pointed out the presence of large quantum fluctuations during inflation. They lead to strong density perturbations and thus potentially to spontaneous black hole formation. But second, and more fundamentally, it is clear that in order to pair create *any* object, there must be present a force which pulls the pair apart. In the case of a virtual electron-positron pair, for example, the particles can only become real if they are pulled apart by an external electric field. Otherwise, they would just fall back together and annihilate. The same holds for black holes: examples in the literature include their pair creation on a cosmic string [5], where they are pulled apart by the string tension; or the pair creation of magnetically charged black holes on the background of Melvin's Universe [6], where the magnetic field prevents them from recollapsing. In our case, the black holes will be separated by the rapid cosmological expansion due to the effective cosmological constant. So we see that this expansion, which we naively expected to prevent black holes from forming, actually provides just the background needed for their quantum pair creation.

Since inflation has ended, during the radiation and matter-dominated eras until the present time, the effective cosmological constant was nearly zero. Thus, the only time when black hole pair creation was possible in our universe was during the inflationary era, when Λ_{eff} was large. Moreover, these black holes are unique since they can be so small that quantum effects on their evolution are important. Such tiny black holes could not form from the gravitational collapse of normal baryonic matter, because degeneracy pressure will support white dwarfs or neutron stars below the Chandrasekhar limiting mass.

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In the standard semiclassical treatment of pair creation, one finds two instantons: one for the background, and the other for the objects to be created on the background. From the instanton actions I_{bg} and I_{obj} one calculates the pair creation rate Γ :

$$\Gamma = \exp[-(I_{\text{obj}} - I_{\text{bg}})], \quad (1.1)$$

where we neglect a prefactor. This prescription has been very successfully used by a number of authors recently [7–13] for the pair creation of black holes on various backgrounds.

In this paper, however, we will obtain the pair creation rate through a somewhat more fundamental, but equivalent procedure: since we have a cosmological background, we can use the Hartle-Hawking no boundary proposal [14] for the wave function of the universe. We will describe the creation of an inflationary universe by a de Sitter-type gravitational instanton, which has the topology of a four-sphere S^4 . In this picture, the universe starts out with the spatial size of one Hubble volume. After one Hubble time, its spatial volume will have increased by a factor of $e^3 \approx 20$. However, by the de Sitter no hair theorem, we can regard each of these 20 Hubble volumes as having been nucleated independently through gravitational instantons. With this interpretation, we are allowing for black hole pair creation, since some of the new Hubble volumes might have been created through a different type of instanton that has the topology $S^2 \times S^2$ and thus represents a pair of black holes in de Sitter space [15]. Using the framework of the no boundary proposal (reviewed in Sec. II), one can assign probability measures to both instanton types. One can then estimate the fraction of inflationary Hubble volumes containing a pair of black holes by the fraction Γ of the two probability measures. This is equivalent to saying that Γ is the pair creation rate of black holes on a de Sitter background. We will thus reproduce Eq. (1.1).

In Sec. III A we follow this procedure using a simplified model of inflation, with a fixed cosmological constant, before going to a more realistic model in Sec. III B. In Sec. III C we show that the usual description of pair creation arises naturally from the no boundary proposal, and Eq. (1.1) is recovered. We find that Planck size black holes can be created in abundance in the early stages of inflation. Larger black holes, which would form near the end of inflation, are exponentially suppressed. The tunneling proposal [16], on the other hand, predicts a catastrophic instability of de Sitter space and is unable to reproduce Eq. (1.1).

We then investigate the evolution of black holes in an inflationary universe. In Sec. IV their classical growth is shown to correspond to energy-momentum flux across the black hole horizon. Taking quantum effects into account, we find in Sec. V that the number of neutral black holes that survive into the radiation era is exponentially small. On the other hand, black holes with a magnetic charge can also be pair created during inflation. They cannot decay, because magnetic charge is topologically conserved. Thus, they survive and should still be around today. In Sec. VI, however, we show that such black holes would be too rare to be found in the observable universe. We summarize our results in Sec. VII. We use units in which $m_P = \hbar = c = k = 1$.

II. NO BOUNDARY PROPOSAL

We shall give a brief review; more comprehensive treatments can be found elsewhere [17]. According to the no boundary proposal, the quantum state of the universe is defined by path integrals over Euclidean metrics $g_{\mu\nu}$ on compact manifolds M . From this it follows that the probability of finding a three metric h_{ij} on a spacelike surface Σ is given by a path integral over all $g_{\mu\nu}$ on M that agree with h_{ij} on Σ . If the space-time is simply connected (which we shall assume), the surface Σ will divide M into two parts, M_+ and M_- . One can then factorize the probability of finding h_{ij} into a product of two wave functions, Ψ_+ and Ψ_- . Ψ_+ (Ψ_-) is given by a path integral over all metrics $g_{\mu\nu}$ on the half-manifold M_+ (M_-) which agree with h_{ij} on the boundary Σ . In most situations Ψ_+ equals Ψ_- . We shall, therefore, drop the suffixes and refer to Ψ as the wave function of the universe. Under inclusion of matter fields, one arrives at the prescription

$$\Psi[h_{ij}, \Phi_\Sigma] = \int D(g_{\mu\nu}, \Phi) \exp[-I(g_{\mu\nu}, \Phi)], \quad (2.1)$$

where (h_{ij}, Φ_Σ) are the three-metric and matter fields on a spacelike boundary Σ and the path integral is taken over all compact Euclidean four geometries $g_{\mu\nu}$ that have Σ as their only boundary and matter field configurations Φ that are regular on them; $I(g_{\mu\nu}, \Phi)$ is their action. The gravitational part of the action is given by

$$I_E = -\frac{1}{16\pi} \int_{M_+} d^4x g^{1/2} (R - 2\Lambda) - \frac{1}{8\pi} \int_\Sigma d^3x h^{1/2} K, \quad (2.2)$$

where R is the Ricci scalar, Λ is the cosmological constant, and K is the trace of K_{ij} , the second fundamental form of the boundary Σ in the metric g .

The wave function Ψ depends on the three metric h_{ij} and on the matter fields Φ on Σ . It does not, however, depend on time explicitly, because there is no invariant meaning to time in cosmology. Its independence of time is expressed by the fact that it obeys the Wheeler-DeWitt equation. We shall not try to solve the Wheeler-DeWitt equation directly, but we shall estimate Ψ from a saddle-point approximation to the path integral.

We give here only a brief summary of this semiclassical method; the procedure will become clear when we follow it through in the following section. We are interested in two types of inflationary universes: one with a pair of black holes, and the other without. They are characterized by spacelike sections of different topology. For each of these two universes, we have to find a classical Euclidean solution to the Einstein equations (an instanton), which can be analytically continued to match a boundary Σ of the appropriate topology. We then calculate the Euclidean actions I of the two types of solutions. Semiclassically, it follows from Eq. (2.1) that the wave function is given by

$$\Psi = \exp(-I), \quad (2.3)$$

where we neglect a prefactor. We can thus assign a probability measure to each type of universe:

$$P = |\Psi|^2 = \exp(-2I^{\text{Re}}), \quad (2.4)$$

where the superscript Re denotes the real part. As explained in the introduction, the ratio of the two probability measures gives the rate of black hole pair creation on an inflationary background Γ .

III. CREATION OF NEUTRAL BLACK HOLES

The solutions presented in this section are discussed much more rigorously in an earlier paper [18]. We shall assume spherical symmetry. Before we introduce a more realistic inflationary model, it is helpful to consider a simpler situation with a fixed positive cosmological constant Λ but no matter fields. We can then generalize quite easily to the case where an effective cosmological “constant” arises from a scalar field.

A. Fixed cosmological constant

1. The de Sitter solution

First, we consider the case without black holes: a homogeneous isotropic universe. Since $\Lambda > 0$, its spacelike sections will simply be round three-spheres. The wave function is given by a path integral over all metrics on a four manifold M_+ bounded by a round three-sphere Σ of radius a_Σ . The corresponding saddle-point solution is the de Sitter space-time. Its Euclidean metric is that of a round four-sphere of radius $\sqrt{3/\Lambda}$:

$$ds^2 = d\tau^2 + a(\tau)^2 d\Omega_3^2, \quad (3.1)$$

where τ is Euclidean time, $d\Omega_3^2$ is the metric on the round three-sphere of unit radius, and

$$a(\tau) = \sqrt{\frac{3}{\Lambda}} \sin \sqrt{\frac{\Lambda}{3}} \tau. \quad (3.2)$$

For $a_\Sigma = 0$, the saddle-point metric will only be a single point. For $0 < a_\Sigma < \sqrt{3/\Lambda}$ it will be part of the Euclidean four-sphere, and when $a_\Sigma = \sqrt{3/\Lambda}$, the saddle-point metric will be half the four-sphere. When $a_\Sigma > \sqrt{3/\Lambda}$ there will be no real Euclidean metric which is a solution of the field equations with the given boundary conditions. However, we can regard Eq. (3.2) as a function on the complex τ plane. On a line parallel to the imaginary τ axis defined by $\tau^{\text{Re}} = \sqrt{(3/\Lambda)}(\pi/2)$, we have

$$a(\tau)|_{\tau^{\text{Re}} = \sqrt{(3/\Lambda)}(\pi/2)} = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} \tau^{\text{Im}}. \quad (3.3)$$

This describes a Lorentzian de Sitter hyperboloid, with τ^{Im} serving as a Lorentzian time variable. One can thus construct a complex solution, which is the analytical continuation of the Euclidean four-sphere metric. It is obtained by choosing a contour in the complex τ plane from 0 to $\tau^{\text{Re}} = \sqrt{(3/\Lambda)}(\pi/2)$ and then parallel to the imaginary τ axis. One can regard this complex solution as half the Euclidean four-sphere joined to half of the Lorentzian de Sitter hyperboloid (Fig. 1).

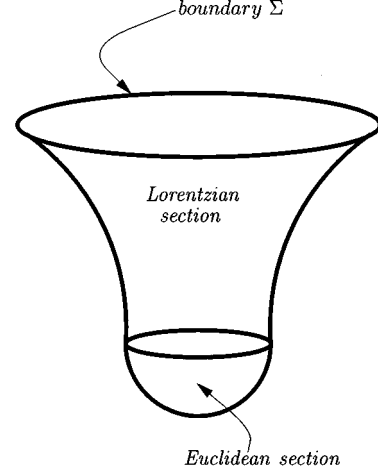


FIG. 1. The creation of a de Sitter universe. The lower region is half of a Euclidean four-sphere, embedded in five-dimensional Euclidean flat space. The upper region is a Lorentzian four-hyperboloid, embedded in five-dimensional Minkowski space.

The Lorentzian part of the metric will contribute a purely imaginary term to the action. This will affect the phase of the wave function but not its amplitude. The real part of the action of this complex saddle-point metric will be the action of the half Euclidean four-sphere:

$$I_{\text{ds}}^{\text{Re}} = -\frac{3\pi}{2\Lambda}. \quad (3.4)$$

Thus, the magnitude of the wave function will still be $e^{3\pi/2\Lambda}$, corresponding to the probability measure

$$P_{\text{ds}} = \exp\left(\frac{3\pi}{\Lambda}\right). \quad (3.5)$$

2. The Schwarzschild–de Sitter solution

We turn to the case of a universe containing a pair of black holes. Now, the cross sections Σ have topology $S^2 \times S^1$. Generally, the radius of the S^2 varies along the S^1 . This corresponds to the fact that the radius of a black hole immersed in de Sitter space can have any value between zero and the radius of the cosmological horizon. The minimal two-sphere corresponds to the black hole horizon, the maximal two-sphere to the cosmological horizon. The saddle-point solution corresponding to this topology is the Schwarzschild–de Sitter universe. However, the Euclidean section of this space-time typically has a conical singularity at one of its two horizons and thus does not represent a regular instanton. This is discussed in detail in the Appendix. There we show that the only regular Euclidean solution is the degenerate case where the black hole has the maximum possible size. It is also known as the Nariai solution and given by the topological product of two round two-spheres:

$$ds^2 = d\tau^2 + a(\tau)^2 dx^2 + b(\tau)^2 d\Omega_2^2, \quad (3.6)$$

where x is identified with period 2π , $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$, and

$$a(\tau) = \sqrt{\frac{1}{\Lambda}} \sin \sqrt{\Lambda} \tau, \quad b(\tau) = \sqrt{\frac{1}{\Lambda}} = \text{const.} \quad (3.7)$$

In this case the radius b of the S^2 is constant in the S^1 direction. The black hole and the cosmological horizon have equal radius $1/\sqrt{\Lambda}$ and no conical singularities are present. Thus, by requiring the smoothness of the Euclidean solution, the instanton approach not only tells us about probability measures, but also about the size of the black hole. There will be no saddle-point solution unless we specify $b_\Sigma = 1/\sqrt{\Lambda}$. We are then only free to choose the radius a_Σ of the one-sphere on Σ . For this variable, the situation is similar to the de Sitter case. There will be real Euclidean saddle-point metrics on M_+ for $a_\Sigma \leq 1/\sqrt{\Lambda}$. For larger a_Σ there will again be no Euclidean saddle point, but we find that

$$a(\tau)|_{\tau_{\text{Re}}} = \sqrt{(1/\Lambda)} (\pi/2) = \sqrt{\frac{1}{\Lambda}} \cosh \sqrt{\Lambda} \tau^{\text{Im}}. \quad (3.8)$$

This corresponds to the Lorentzian section of the degenerate Schwarzschild–de Sitter space-time, in which the S^1 expands rapidly, while the two-sphere (and, therefore, the black hole radius) remains constant. Again, we can construct a complex saddle point, which can be regarded as half a Euclidean $S^2 \times S^2$ joined to half of the Lorentzian solution. The real part of the action will be the action of the half of a Euclidean $S^2 \times S^2$:

$$I_{\text{Sds}}^{\text{Re}} = -\frac{\pi}{\Lambda}. \quad (3.9)$$

The corresponding probability measure is

$$P_{\text{Sds}} = \exp\left(\frac{2\pi}{\Lambda}\right). \quad (3.10)$$

We divide this by the probability measure (3.5) for a universe without black holes to obtain the pair creation rate of black holes in de Sitter space:

$$\Gamma = \frac{P_{\text{Sds}}}{P_{\text{ds}}} = \exp\left(-\frac{\pi}{\Lambda}\right). \quad (3.11)$$

Thus, the probability for pair creation is very low, unless Λ is close to the Planck value, $\Lambda = 1$.

B. Effective cosmological constant

Of course, the real universe does not have a cosmological constant of order the Planck value. However, in inflationary cosmology it is assumed that the universe starts out with a very large effective cosmological constant, which arises from the potential V of a scalar field ϕ . The exact form of the potential is not critical. So for simplicity, we chose V to be the potential of a field with mass m , but the results would be similar for a $\lambda\phi^4$ potential. To account for the observed fluctuations in the microwave background [19], m has to be on the order of 10^{-5} to 10^{-6} [20]. The wave function Ψ will now depend on the three metric h_{ij} and the value of ϕ on Σ . By a gauge choice one can take ϕ to be constant on Σ ,

and we shall do so for simplicity. For $\phi > 1$ the potential acts like an effective cosmological constant

$$\Lambda_{\text{eff}}(\phi) = 8\pi V(\phi). \quad (3.12)$$

One proceeds in complete analogy to the fixed cosmological constant case. For small three-geometries and $\phi > 1$, there will be real Euclidean metrics on M_+ , with ϕ almost constant. If the three-geometries are rather larger, there will, again, not be any real Euclidean saddle-point metrics. There will, however, be complex saddle points. These can, again, be regarded as a Euclidean solution joined to a Lorentzian solution, although neither the Euclidean nor Lorentzian metrics will be exactly real. Apart from this subtlety, which is dealt with in Ref. [18], the saddle-point solutions are similar to those for a fixed cosmological constant, with the time-dependent Λ_{eff} replacing Λ . The radius of the pair-created black holes will now be given by $1/\sqrt{\Lambda_{\text{eff}}}$. As before, the magnitude of the wave function comes from the real part of the action, which is determined by the Euclidean part of the metric. This real part will be

$$I_{S^3}^{\text{Re}} = -\frac{3\pi}{2\Lambda_{\text{eff}}(\phi_0)} \quad (3.13)$$

in the case without black holes, and

$$I_{S^2 \times S^1}^{\text{Re}} = -\frac{\pi}{\Lambda_{\text{eff}}(\phi_0)} \quad (3.14)$$

in the case with a black hole pair. Here, ϕ_0 is the value of ϕ in the initial Euclidean region. Thus, the pair creation rate is given by

$$\Gamma = \frac{P_{S^2 \times S^1}}{P_{S^3}} = \exp\left[-\frac{\pi}{\Lambda_{\text{eff}}(\phi_0)}\right]. \quad (3.15)$$

C. Discussion

Let us interpret this result. Since $0 < \Lambda_{\text{eff}} \leq 1$, we get $\Gamma < 1$ and so black hole pair creation is suppressed. In the early stages of inflation, when $\Lambda_{\text{eff}} \approx 1$, the suppression is weak, and black holes will be plentifully produced. However, those black holes will be very small, with a mass on the order of the Planck mass. Larger black holes, corresponding to lower values of Λ_{eff} at later stages of inflation, are exponentially suppressed. We shall see in the following two sections that the small black holes typically evaporate immediately, while sufficiently large ones grow with the horizon and survive long after inflation ends (that is, long in early universe terms).

We now understand how the standard prescription for pair creation, Eq. (1.1), arises from this proposal: by Eq. (2.4),

$$\Gamma = \frac{P_{S^2 \times S^1}}{P_{S^3}} = \exp[-(2I_{S^2 \times S^1}^{\text{Re}} - 2I_{S^3}^{\text{Re}})], \quad (3.16)$$

where I^{Re} denotes the real part of the Euclidean action of a complex saddle-point solution. But we have seen that this is equal to half of the action of the complete Euclidean solution. Thus, $I_{\text{obj}} = 2I_{S^2 \times S^1}^{\text{Re}}$ and $I_{\text{bg}} = 2I_{S^3}^{\text{Re}}$, and we recover Eq. (1.1).

The prescription for the wave function of the universe has long been one of the central, and arguably one of the most disputed, issues in quantum cosmology [14,21]. According to Vilenkin's tunneling proposal [16], Ψ is given by e^{+I} rather than by e^{-I} . This choice of sign is inconsistent with Eq. (1.1), as it leads to the inverse result for the pair creation rate: $\Gamma_{\text{TP}} = 1/\Gamma_{\text{NBP}}$. In our case, we would get $\Gamma_{\text{TP}} = \exp(+\pi/\Lambda_{\text{eff}})$. Thus, black hole pair creation would be enhanced, rather than suppressed. de Sitter space would be catastrophically unstable to the formation of black holes. Since the radius of the black holes is given by $1/\sqrt{\Lambda_{\text{eff}}}$, large black holes would be even more likely than small ones. Clearly, the tunneling proposal cannot be maintained. On the other hand, Eq. (3.15), which was obtained from the no boundary proposal, is physically very reasonable. It allows topological fluctuations near the Planckian regime, but suppresses the formation of large black holes at low energies. Thus, the consideration of the cosmological pair production of black holes lends strong support to the no boundary proposal.

IV. CLASSICAL EVOLUTION

We shall now consider neutral black holes created at any value $\phi_0 > 1$ of the scalar field and analyze the different effects on their evolution. Before we take quantum effects into account, we shall display the classical solution for a Universe containing a pair of black holes. We shall demonstrate explicitly that it behaves according to the first law of black hole mechanics.

With a rescaled inflaton potential

$$V(\phi) = \frac{1}{8\pi} m^2 \phi^2, \quad (4.1)$$

the effective cosmological constant will be

$$\Lambda_{\text{eff}} = m^2 \phi^2. \quad (4.2)$$

In the previous section we learned that the black hole radius remains constant, at $1/\sqrt{\Lambda}$, in the Lorentzian regime. But this was for the simple model with fixed Λ . The effective cosmological constant in Eq. (4.2) is slightly time dependent. Thus, we might expect the black hole size to change during inflation.

Indeed, for $\pi/(2m\phi_0) < t < (\phi_0)/m$, approximate Lorentzian solutions are given by [18]

$$\phi(t) = \phi_0 - mt, \quad (4.3)$$

$$a(t) = \frac{1}{m\phi_0} \cosh \left[m \int_0^t \phi(t') dt' \right], \quad (4.4)$$

$$b(t) = \frac{1}{m\phi(t)}, \quad (4.5)$$

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 d\Omega_2^2. \quad (4.6)$$

Since we are dealing with a degenerate solution, the radii r_b and r_c of the black hole and cosmological horizons are equal:

$$r_b = r_c = b(t). \quad (4.7)$$

According to Eq. (4.5) they will expand slowly together during inflation as the scalar field rolls down to the minimum of the potential V and the effective cosmological constant decreases. At the end of inflation they will be approximately equal to m^{-1} .

One can think of this increase of the horizons as a classical effect, caused by a flow of energy-momentum across them. If the scalar field were constant, its energy-momentum tensor would act exactly such as a cosmological constant. The flow of energy-momentum across the horizon would be zero. However, the scalar field is not constant but is rolling downhill in the potential to the minimum at $\phi = 0$. This means that there is an energy-momentum flow across the horizon equal to

$$\dot{M} = AT_{ab} l^a l^b = \frac{1}{\phi^2}, \quad (4.8)$$

where $A = 4\pi b^2$ is the horizon area, T_{ab} is the energy-momentum tensor for the massive scalar field, given by

$$T_{ab} = \frac{1}{4\pi} \partial_a \phi \partial_b \phi - \frac{1}{8\pi} g_{ab} (\partial_c \phi \partial^c \phi + m^2 \phi^2), \quad (4.9)$$

and l^a is a null vector tangent to the horizon:

$$l^a = \frac{\partial}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x}. \quad (4.10)$$

One would expect the horizons to respond to this flow of energy across them by an increase in area according to the first law of black hole mechanics [22]:

$$\dot{M} = \frac{\kappa}{8\pi} \dot{A}, \quad (4.11)$$

where κ is the surface gravity of the horizon. We will show that this equation is indeed satisfied if the horizon growth is given by Eq. (4.5).

The values of κ for general Schwarzschild-de Sitter solutions are derived in the Appendix. Because of the slow change of the effective cosmological constant we can approximate the surface gravity at any time t in our model by the surface gravity in the model with a fixed cosmological constant $\Lambda = \Lambda_{\text{eff}}(t)$. In the degenerate case which we are considering now, κ will thus be given by

$$\kappa = \sqrt{\Lambda_{\text{eff}}} = m\phi. \quad (4.12)$$

Equation (4.11) becomes

$$\dot{M} = \frac{m\phi}{8\pi} \frac{d}{dt} \left(\frac{4\pi}{m^2 \phi^2} \right) = \frac{1}{\phi^2}, \quad (4.13)$$

which agrees with Eq. (4.8).

It should be pointed out that this calculation holds not only for the black hole horizon, but also for the cosmological horizon. Moreover, an analogous calculation is possible for the cosmological horizon in an ordinary inflationary Universe without black holes. Thus, in hindsight we understand

the slow growth of the cosmological horizon during inflation as a manifestation of the first law of black hole mechanics.

V. QUANTUM EVOLUTION

So far we have been neglecting the quantum properties of the inflationary space-time presented above. It is well known that in a Schwarzschild–de Sitter universe, radiation is emitted both by the black hole and by the cosmological horizon [23]. To treat this properly, one should include the one-loop effective action of all the low mass fields in the metric $g_{\mu\nu}$. By using a supersymmetric theory one might avoid divergences in the one-loop term, but it would still be impossibly difficult to calculate in any but very simple metrics. Instead, we shall use an approximation in which the black hole and cosmological horizons radiate thermally with temperatures

$$T_b = \frac{\kappa_b}{2\pi}, \quad T_c = \frac{\kappa_c}{2\pi}. \quad (5.1)$$

This quantum effect must also be included in the calculation of the energy flow across the horizons. For the saddle-point metric, Eq. (4.6), it has no consequence: in the Nariai solution the black hole and cosmological horizons have the same radius and surface gravity. Thus, they radiate at the same rate. That means they will be in thermal equilibrium. The black holes will not evaporate, because they will be absorbing as much as they radiate. Instead, their evolution will be governed by the classical growth described above.

However, the Nariai metric is an idealization. (Strictly speaking, it does not even contain a black hole, but rather two acceleration horizons.) Because of quantum fluctuations there will be small deviations from the saddle-point solution, corresponding to a Schwarzschild–de Sitter space-time which is not quite degenerate. The radius b of the two-sphere will not be exactly constant along the S^1 , but will have a maximum r_c and a minimum r_b , which we identify with the cosmological and the black hole horizons, respectively. (The other black hole horizon of the pair will lie beyond the cosmological horizon and will not be visible in our universe.) Since the black hole horizon is slightly smaller, it will have a higher temperature than that of the cosmological horizon. Therefore, the black hole will radiate more than what it receives. There will thus be a net transfer of energy from the smaller horizon to the larger one. This will cause the larger horizon to grow faster, and the smaller one to shrink until the black hole vanishes completely. We show below that black holes can still grow with the cosmological horizon until the end of inflation, if they are either created sufficiently large, or start out very nearly degenerate. However, we shall see that none of these conditions is easily satisfied.

Black holes created during the final stages of inflation will survive until the end of inflation simply because they will be relatively large and cold. One can estimate the minimum size they must have by treating them as evaporating Schwarzschild black holes, which have a lifetime on the order of M^3 , where M is the mass at which the black hole is created. In terms of the value of the scalar field at creation, $M \approx b_0 = (m\phi_0)^{-1}$. Inflation ends after a time of ϕ_0/m . Therefore, black holes created at $\phi_0 \leq m^{-1/2}$ will certainly

survive until the end of inflation. They would continue to grow slowly during the radiation era, until the temperature of the radiation falls below that of the black holes. They will then start to evaporate. By Eq. (3.15), however, such black holes will be suppressed by a factor of

$$\Gamma = \exp(-\pi m^{-1}). \quad (5.2)$$

One must, therefore, investigate the possibility that the small black holes, which can be created in abundance, start out so nearly degenerate that they will grow with the cosmological horizon until they have reached the “safe” size of $m^{-1/2}$. We need to determine how nearly equal the horizon sizes have to be initially so that the black hole survives until the end of inflation.

If we take the thermal radiation into account, the flow across the horizons now consists of two parts: the classical term due to the energy flow of the scalar field, as well as the net radiation energy transfer, given by Stefan’s law. Applying the first law of black hole mechanics to each horizon, we get

$$\frac{\kappa_b}{8\pi} \dot{A}_b = m^2 r_b^2 - (\sigma A_b T_b^4 - \sigma A_c T_c^4), \quad (5.3)$$

$$\frac{\kappa_c}{8\pi} \dot{A}_c = m^2 r_c^2 + (\sigma A_b T_b^4 - \sigma A_c T_c^4), \quad (5.4)$$

where $\sigma = \pi^2/60$ is the Stefan-Boltzmann constant. Using Eq. (5.1), we obtain two coupled differential equations for the horizon radii:

$$\dot{r}_b = \frac{1}{\kappa_b r_b} \left[m^2 r_b^2 - \frac{1}{240\pi} (r_b^2 \kappa_b^4 - r_c^2 \kappa_c^4) \right], \quad (5.5)$$

$$\dot{r}_c = \frac{1}{\kappa_c r_c} \left[m^2 r_c^2 + \frac{1}{240\pi} (r_b^2 \kappa_b^4 - r_c^2 \kappa_c^4) \right]. \quad (5.6)$$

The exact functional relation between the surface gravities and the horizon radii is generally nontrivial. However, the above evolution equations can be simplified if one takes into account that a nucleated black hole pair must be very nearly degenerate if it is to survive until the end of inflation. We can, therefore, write

$$r_b(t) = b(t)[1 - \epsilon(t)], \quad r_c(t) = b(t)[1 + \epsilon(t)], \quad (5.7)$$

with $\epsilon_0 \ll 1$. The surface gravities can also be expressed in terms of ϵ (see Appendix); to first order they are given by

$$\kappa_b = \frac{1}{b(t)} \left[1 + \frac{2}{3} \epsilon(t) \right], \quad \kappa_c = \frac{1}{b(t)} \left[1 - \frac{2}{3} \epsilon(t) \right]. \quad (5.8)$$

We assume that $b(t)$ behaves as in Eq. (4.5) for the Nariai solution as long as $\epsilon(t) \ll 1$. Equations (5.5) and (5.6) are then identically satisfied to zeroth order in ϵ . In the first order, they give an evolution equation for ϵ :

$$\dot{\epsilon} = \left(\frac{2}{3} m^2 b + \frac{1}{180\pi} \frac{1}{b^3} \right) \epsilon. \quad (5.9)$$

This equation can be integrated to give

$$\epsilon(t) = \epsilon_0 \left(\frac{\phi}{\phi_0} \right)^{2/3} \exp \left\{ \frac{1}{720\pi} m^2 \phi_0^4 \left[1 - \left(\frac{\phi}{\phi_0} \right)^4 \right] \right\}. \quad (5.10)$$

For the unsuppressed, Planck size black holes we have $\phi_0 = m^{-1}$. If they grow with the horizon, they will reach the safe size, which corresponds to $\phi = m^{-1/2}$, after a time

$$t_{\text{safe}} = m^{-2} (1 - m^{1/2}). \quad (5.11)$$

If a Planck size black hole is to have survived until this time, i.e., if $\epsilon(t_{\text{safe}}) \leq 1$, then the initial difference in horizon sizes may not have been larger than

$$\epsilon_0^{\text{max}} = m^{1/3} \exp \left[- \frac{m^{-2}}{720\pi} (1 - m^2) \right]. \quad (5.12)$$

The probability $P(\epsilon_0 \leq \epsilon_0^{\text{max}})$ that the two horizons start out so nearly equal obviously depends on the distribution of the initial sizes of the two horizons. The semiclassical treatment of the quantum fluctuations which cause the geometry to differ from the degenerate case, for general values of the effective cosmological constant, is an interesting problem by itself, and beyond the scope of this paper. We hope to return to it in a forthcoming paper on complex solutions in quantum cosmology. However, here we are working at the Planck scale, so that the semiclassical approximation will break down anyway. It, therefore, seems reasonable to assume that the initial sizes of the horizons are distributed roughly uniformly between zero and a few Planck lengths. This means that

$$P(\epsilon_0 \leq \epsilon_0^{\text{max}}) \approx \epsilon_0^{\text{max}} \approx \exp \left(- \frac{m^{-2}}{720\pi} \right). \quad (5.13)$$

Since $m \approx 10^{-6}$, we conclude by comparison with the suppression factor (5.2) that it is considerably less efficient to create Planck size black holes which would grow to the safe size than just to create the large black holes. Both processes, however, are exponentially suppressed.

VI. CHARGED BLACK HOLES

Although, nearly all neutral black holes will evaporate during inflation, those that have a magnetic charge will not be able to because there are no magnetically charged particles for them to radiate. They can only evaporate down to the minimum mass necessary to support their magnetic charge. Let us, therefore, introduce a Maxwell term in the action and reexamine the pair creation of primordial black holes:

$$I = - \frac{1}{16\pi} \int_{M_+} d^4x g^{1/2} (R - 2\Lambda - F_{\mu\nu} F^{\mu\nu}) - \frac{1}{8\pi} \int_{\Sigma} d^3x h^{1/2} K. \quad (6.1)$$

The $S^2 \times S^2$ bubbles in space-time foam can now carry magnetic flux. The action of the Maxwell field will reduce their probability with respect to neutral bubbles. Thus, magnetically charged black holes will also be pair created during

inflation, though in smaller numbers than neutral black holes. However, once created, they can disappear only if they meet a black hole with the opposite charge, which is unlikely. So they should still be around today.

We shall estimate the number of primordial charged black holes present in the observable universe. For the purpose of calculating the pair creation rate during inflation, we can use the solutions for a fixed cosmological constant, since Λ_{eff} changes slowly. There exists a three-parameter family of Lorentzian charged Schwarzschild–de Sitter solutions. They are usually called Reissner–Nordström–de Sitter solutions and labeled by the charge q and the “mass” μ of the black hole, and by the cosmological constant Λ :

$$ds^2 = -U(r)dt^2 + U(r)^{-1}dr^2 + r^2 d\Omega_2^2, \quad (6.2)$$

where

$$U(r) = 1 - \frac{2\mu}{r} + \frac{q^2}{r^2} - \frac{1}{3}\Lambda r^2. \quad (6.3)$$

We are interested in the cases where the black holes are magnetically, rather than electrically, charged. Then, the Maxwell field is given by

$$F = q \sin\theta d\theta \wedge d\phi. \quad (6.4)$$

In an appropriate region of the parameter space, U has three positive roots, which we denote, in ascending order, by r_i , r_o , and r_c and interpret as the inner and outer black hole horizons and the cosmological horizon. They can serve as an alternative parametrization of the solutions. For general values of q , μ , and Λ the metric (6.2) has no regular Euclidean section. The black holes which can be pair created through regular instantons lie on three intersecting hypersurfaces in the parameter space [24], as seen in Fig. 2. They are called the *cold*, the *lukewarm*, and the *charged Nariai* solutions. In the cold case, the instanton is made regular by setting $r_i = r_o$, which corresponds to an extremal black hole. The lukewarm hypersurface is characterized by the condition $q = \mu$. It corresponds to a nonextremal black hole which has the same surface gravity and temperature as the cosmological horizon. This property is shared by the charged Nariai solution, which has $r_o = r_c$. For $q = 0$, its mass is given by $\mu = 1/(3\sqrt{\Lambda})$ and it coincides with the neutral Nariai Universe we discussed earlier. For larger charge and mass, it is still the direct product of two round two-spheres; however, with different radii. The cold, lukewarm, and de Sitter solutions all coincide for $q = \mu = 0$. The largest possible mass and charge for the lukewarm solution is $q = \mu = 3/(4\sqrt{3\Lambda})$, where it coincides with the charged Nariai solution. The largest possible mass and charge for any regular instanton is attained at the point where the charged Nariai and the cold hypersurfaces meet. This *ultracold* case has $q = 1/(2\sqrt{\Lambda})$ and $\mu = 2/(3\sqrt{2\Lambda})$. It admits two distinct solutions of different actions. All of these solutions are presented and discussed in detail in the comprehensive paper by Mann and Ross [11], where the actions are calculated as well. We shall now apply these results in the context of inflation.

Let us consider an inflationary scenario in which the effective cosmological constant $\Lambda_{\text{eff}} = m\phi$ starts out near the Planck value and then decreases slowly. As in the neutral

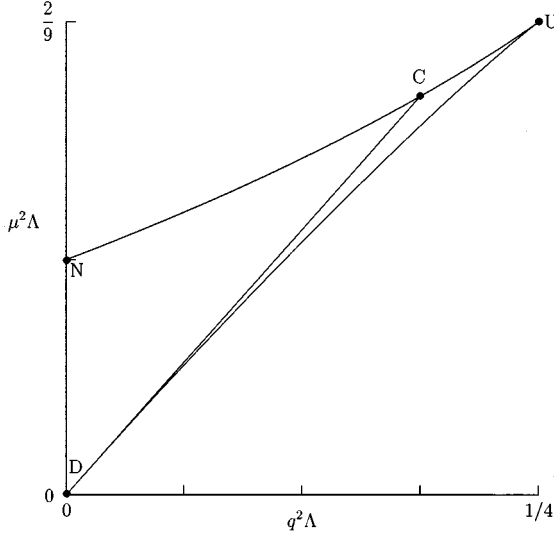


FIG. 2. The hypersurfaces in the parameter space of the Reissner–Nordström–de Sitter solution which admit regular instantons. The plot is of the dimensionless quantities $\mu^2\Lambda$ vs $q^2\Lambda$. The points N and D represent the uncharged Nariai and de Sitter solutions. The curve DC corresponds to the lukewarm instantons. At C it meets the curve NU of charged Nariai instantons. The cold instantons lie on the curve CU; the point U represents the two ultracold solutions.

case, one would expect the creation of charged black holes to be least suppressed for large Λ , i.e., at the earliest stage of inflation. However, unlike the neutral case, a magnetically charged black hole cannot be arbitrarily small since it must carry at least one unit of magnetic charge:

$$q_0 = \frac{1}{2e_0}, \quad (6.5)$$

where $e_0 = \sqrt{\alpha}$ is the unit of electric charge, and $\alpha \approx 1/137$ is the fine structure constant. In the following we shall only consider black holes with $q = q_0$, since they are the first to be created, and since more highly charged black holes are exponentially suppressed relative to them. We see from Fig. 3 that pair creation first becomes possible through the ultracold instanton U_1 , when Λ_{eff} has decreased to the value of $\Lambda^U = 1/(4q_0^2) = \alpha$. Since this solution has relatively high action, however, black hole production becomes more efficient at a slightly later time. It will then occur mainly through the charged Nariai instanton, which has lower action than the cold black hole. When

$$\Lambda_{\text{eff}} = \Lambda^E = (4\sqrt{3} - 6)\alpha, \quad (6.6)$$

the pair creation rate reaches its peak and is given by

$$\begin{aligned} \Gamma &= \exp[-(I^E - I_{\text{ds}})] \\ &= \exp\left[-\frac{(2 + \sqrt{3})\pi}{2\alpha}\right], \end{aligned} \quad (6.7)$$

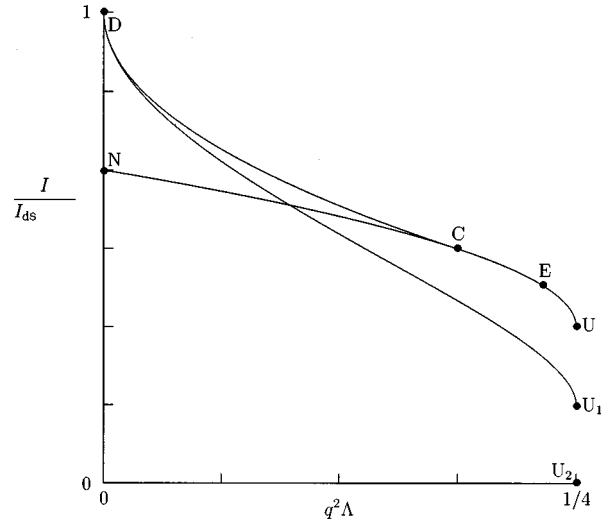


FIG. 3. The action of the different types of Reissner–Nordström–de Sitter instantons, as a fraction of the action of the de Sitter instanton, $I_{\text{ds}} = -3\pi/\Lambda$. Note that the point U does not correspond to an instanton solution in this diagram; the two ultracold instantons are labeled by U_1 and U_2 . E is the instanton with the highest pair creation rate during inflation.

by Eq. (1.1). As Λ_{eff} decreases further, the pair creation rate starts to decrease and soon becomes vastly suppressed. For

$$0 < \Lambda_{\text{eff}} < \Lambda^C = \frac{3}{4}\alpha, \quad (6.8)$$

the lukewarm instanton has the lowest action,

$$I_{\text{luk}} = -\frac{3\pi}{\Lambda_{\text{eff}}} + \pi\sqrt{\frac{3}{\alpha\Lambda_{\text{eff}}}}. \quad (6.9)$$

It will, therefore, dominate the black hole production until the end of inflation.

If we ask how many charged black holes were produced during the entire inflationary era, we have to take into account that the density of the black holes pair created at the early stages of inflation will be reduced by the subsequent inflationary expansion. As a consequence, most of the charged primordial black holes in our universe were produced near the end of inflation, as we shall show here. The number of such black holes per Hubble volume pair created during one Hubble time when $\phi = \phi_{\text{pc}}$ is given by

$$\begin{aligned} \Gamma(\phi_{\text{pc}}) &= \exp[-(I_{\text{luk}} - I_{\text{ds}})] \\ &= \exp\left[-\sqrt{\frac{3}{\alpha}} \frac{\pi}{m\phi_{\text{pc}}}\right]. \end{aligned} \quad (6.10)$$

At the end of inflation, these black holes have a number density per Hubble volume of

$$d_{\text{end}}(\phi_{\text{pc}}) = \Gamma(\phi_{\text{pc}}) \phi_{\text{pc}}^3 \exp\left[-\frac{3}{2}\phi_{\text{pc}}^2\right], \quad (6.11)$$

where the cubic factor is due to the growth of the Hubble radius and the exponential factor reflects the inflationary expansion of space.

Since we are dealing with exponential suppressions, practically all primordial magnetically charged black holes in the Universe were created at the value of ϕ_{pc} which makes d_{end} maximal. This occurs for

$$\phi_{\text{pc}}^{\text{max}} \approx \left(\frac{\pi^2}{3\alpha m^2} \right)^{1/6}, \quad (6.12)$$

so that the approximate total number per Hubble volume at the end of inflation is given by

$$D_{\text{end}} \approx d_{\text{end}}(\phi_{\text{pc}}^{\text{max}}) \approx \left(\frac{\pi^2}{3\alpha m^2} \right)^{1/2} \exp \left[- \left(\frac{9\pi^2}{16\alpha m^2} \right)^{1/3} \right]. \quad (6.13)$$

We take the values of the Hubble radius to be

$$H_{\text{end}}^{-1} = m^{-1}, \quad H_{\text{eq}}^{-1} = 10^{54}, \quad H_{\text{now}}^{-1} = 10^{59}, \quad (6.14)$$

respectively at the end of inflation, at the time of equal radiation and matter density, and at the present time. Therefore, since the end of inflation the density has been reduced by a factor of $10^{-91} m^{-3/2}$ and the Hubble volume has increased by $10^{177} m^3$. We multiply D_{end} by these factors to obtain the number of primordial black holes in the presently observed universe:

$$D_{\text{now}} \approx 10^{86} \left(\frac{\pi^2}{3\alpha} \right)^{1/2} \exp \left[- \left(\frac{9\pi^2}{16\alpha m^2} \right)^{1/3} \right]. \quad (6.15)$$

With $\alpha = 1/137$ and $m = 10^{-6}$, the exponent will be on the order of -10^5 . Thus, in ordinary Einstein-Maxwell theory, it is very unlikely that the observed universe contains even one magnetic black hole.

However, in theories with a dilaton both the value of the electric charge and the effective Newton's constant can vary with time. If both were much higher in the past, the effective value of αm^2 would not be so small and the present number of magnetic black holes could be much higher. We are currently working on this question.

VII. SUMMARY AND CONCLUSIONS

Since the quantum pair creation of black holes can be investigated semiclassically using instanton methods, it has been widely used as a theoretical laboratory to obtain glimpses at quantum gravity. The inflationary era is the only time when we can reasonably expect the effect to have taken place in our own universe. We chose to work in a very simple model of chaotic inflation, which allowed us to expose quite clearly all the important qualitative features of black hole pair creation. The inflationary universe was approximated as a de Sitter solution with a slowly varying cosmological constant. Similarly, neutral black holes produced during inflation were described by a degenerate Schwarzschild-de Sitter solution. Their pair creation rate was estimated from the no boundary proposal, by comparing the probability measures assigned to the two solutions. We

found that Planck size black holes are plentifully produced, but at later stages in inflation, when the black holes would be larger, the pair creation rate is exponentially suppressed. This fits in with the usual instanton prescriptions for pair creation. The tunneling proposal, on the other hand, fails to make physically reasonable predictions. The consideration of black hole pair creation thus lends support to the no boundary proposal.

We analyzed the classical and quantum evolution of neutral primordial black holes. Classically, the black hole horizon and the cosmological horizon have the same area and temperature. The two horizons grow slowly during inflation. We showed that this is due to the classical flow of scalar field energy across them, according to the first law of black hole mechanics. Quantum effects, however, prevent the geometry from being perfectly degenerate, causing the black hole to be hotter than the cosmological horizon. As a consequence, practically all neutral black holes evaporate before the end of inflation.

Finally, we turned to magnetically charged black holes, which can also be pair created during inflation. Even if they have only one unit of charge, they cannot evaporate completely and would still exist today. The pair creation rate is highest during the early stages of inflation, when the effective cosmological constant is still relatively large. Black holes created at that time, however, will be diluted by the inflationary expansion. Most of the charged primordial black holes were, therefore, created near the end of the inflationary era, where they would not be diluted as strongly. However, in Einstein-Maxwell theory they are so heavily suppressed that we must conclude that there are no primordial black holes in the observable universe. It will, therefore, be interesting to examine inflationary models that include a dilaton field. One would expect to obtain a much higher number of primordial black holes, which may even allow us to constrain some of these models.

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APPENDIX

In this Appendix we show how to calculate the surface gravities of the two horizons in the Schwarzschild-de Sitter solution. This space-time possesses a regular Euclidean section only in the degenerate (Nariai) case, where the two horizons have the same radius. Neutral black holes pair created during inflation will, therefore, start out nearly degenerate. We present a suitable coordinate transformation for the nearly degenerate metric, introducing a small parameter ϵ , which parametrizes the deviation from degeneracy. The surface gravities and Euclidean action are calculated to second order in ϵ , yielding a negative mode in the action. We explain why our results differ from those obtained in Ref. [15].

The Lorentzian Schwarzschild-de Sitter solution has the metric

$$ds^2 = -U(r)dt^2 + U(r)^{-1}dr^2 + r^2 d\Omega_2^2, \quad (A1)$$

where

$$U(r) = 1 - \frac{2\mu}{r} - \frac{1}{3}\Lambda r^2. \quad (A2)$$

For $0 < \mu < (1/3)\Lambda^{-1/2}$, U has two positive roots r_b and r_c , corresponding to the black hole and the cosmological horizon. The space-time admits a timelike Killing vector field

$$K = \gamma_t \frac{\partial}{\partial t}, \quad (A3)$$

where γ_t is a normalization constant. The surface gravities κ_b and κ_c , given by

$$\kappa_{b,c} = \lim_{r \rightarrow r_{b,c}} \left[\frac{(K^a \nabla_a K_b)(K^c \nabla_c K^b)}{-K^2} \right]^{1/2}, \quad (A4)$$

depend on the choice of γ_t . To obtain the correct value for the surface gravity, one must normalize the Killing vector in the right way. In the Schwarzschild case ($\Lambda = 0$) the natural choice is to have $K^2 = -1$ at infinity; this corresponds to $\gamma_t = 1$ for the standard Schwarzschild metric. However, in our case there is no infinity, and it would be a mistake to set $\gamma_t = 1$. Instead, one needs to find the radius r_g for which the orbit of the Killing vector coincides with the geodesic going through r_g at constant angular variables. This is the two-sphere at which the effects of the cosmological expansion and the black hole attraction balance out exactly. An observer at r_g will need no acceleration to stay there, just like an observer at infinity in the Schwarzschild case. One must normalize the Killing vector on this “geodesic orbit.” Note that this is a general prescription which will also give the correct result in the Schwarzschild limit. It is straightforward to show that

$$r_g = \left(\frac{3\mu}{\Lambda} \right)^{1/3}, \quad (A5)$$

so that

$$\gamma_t = U(r_g)^{-1/2} = [1 - (9\Lambda\mu^2)^{1/3}]^{-1/2}. \quad (A6)$$

Equation (A4) then yields

$$\kappa_{b,c} = \frac{1}{2\sqrt{U(r_g)}} \left| \frac{\partial U}{\partial r} \right|_{r=r_{b,c}}. \quad (A7)$$

In order to consider the pair production of black holes, we need to find a Euclidean instanton which can be analytically continued to the metric (A1). The obvious ansatz is

$$ds^2 = U(r)d\tau^2 + U(r)^{-1}dr^2 + r^2 d\Omega_2^2, \quad (A8)$$

where τ is Euclidean time. Again, one can define a constant γ_τ which will normalize the timelike Killing vector on the geodesic orbit. In order to avoid a conical singularity at a horizon, one needs to identify τ with an appropriate period τ^{id} , which is related to the surface gravity on the horizon by

$$\tau^{\text{id}} = 2\pi\gamma_\tau/\kappa. \quad (A9)$$

Usually, only one of the two horizons can be made regular in this way, since their surface gravities will be different. They will be equal only for $\mu = (1/3)\Lambda^{-1/2}$, when the two roots of U coincide. In this degenerate case one can remove both conical singularities simultaneously and obtain a regular instanton. As was first pointed out in [15], the fact that $r_b = r_c$ does not mean that the Euclidean region shrinks to zero. The coordinate system (A8) clearly becomes inappropriate when μ approaches its upper limit: the range of r becomes arbitrarily narrow while the metric coefficient $U(r)^{-1}$ grows without bound. One must, therefore, perform an appropriate coordinate transformation. If we write

$$9\mu^2\Lambda = 1 - 3\epsilon^2, \quad 0 \leq \epsilon \leq 1, \quad (A10)$$

the degenerate case corresponds to $\epsilon \rightarrow 0$. We then define new time and radial coordinates ψ and χ by

$$\begin{aligned} \tau &= \frac{1}{\epsilon\sqrt{\Lambda}} \left(1 - \frac{1}{2}\epsilon^2 \right) \psi, \\ r &= \frac{1}{\sqrt{\Lambda}} \left[1 + \epsilon \cos\chi - \frac{1}{6}\epsilon^2 + \frac{4}{9}\epsilon^3 \cos\chi \right]. \end{aligned} \quad (A11)$$

With this choice of ψ , we have $\gamma_\psi = \sqrt{\Lambda}$ to second order in ϵ , so that the Killing vector

$$K^a = \sqrt{\Lambda} \frac{\partial}{\partial \psi} \quad (A12)$$

has unit length on the geodesic orbit. We have chosen the new radial coordinate χ so that U vanishes to fourth order in ϵ for $\cos\chi = \pm 1$. This is necessary since U contains no zero and first order terms and we intend to calculate all quantities to second nontrivial order in ϵ :

$$U(\chi) = \sin^2\chi \epsilon^2 \left[1 - \frac{2}{3}\epsilon \cos\chi + \frac{2}{3}\epsilon^2 \cos^2\chi + \frac{8}{9}\epsilon^2 \right]. \quad (A13)$$

Thus, the black hole horizon corresponds to $\chi = \pi$ and the cosmological horizon to $\chi = 0$.

The new metric obtained from the coordinate transformations (A11) is

$$\begin{aligned} ds^2 &= \frac{1}{\Lambda} \left(1 - \frac{2}{3}\epsilon \cos\chi + \frac{2}{3}\epsilon^2 \cos^2\chi - \frac{1}{9}\epsilon^2 \right) \sin^2\chi d\psi^2 \\ &\quad + \frac{1}{\Lambda} \left(1 + \frac{2}{3}\epsilon \cos\chi - \frac{2}{9}\epsilon^2 \cos^2\chi \right) d\chi^2 \\ &\quad + \frac{1}{\Lambda} \left(1 + 2\epsilon \cos\chi + \epsilon^2 \cos^2\chi - \frac{1}{3}\epsilon^2 \right) d\Omega_2^2. \end{aligned} \quad (A14)$$

In the degenerate case, $\epsilon = 0$, this is the Nariai metric: the topological product of two round two-spheres, each of radius $1/\sqrt{\Lambda}$. There are no conical singularities if the Euclidean time ψ is identified with a period 2π . For general ϵ , the two

horizons cannot be made regular simultaneously; it is clear from a geometrical standpoint that ψ must be identified with period

$$\begin{aligned}\psi_{c,b}^{\text{id}} &= 2\pi \sqrt{g_{\chi\chi}} \Big|_{\chi=0,\pi} \left(\frac{\partial}{\partial\chi} \sqrt{g_{\psi\psi}} \Big|_{\chi=0,\pi} \right)^{-1} \\ &= 2\pi \left(1 \pm \frac{2}{3}\epsilon - \frac{1}{6}\epsilon^2 \right)\end{aligned}\quad (\text{A15})$$

to prevent a conical singularity at the cosmological or black hole horizon, respectively.

One can calculate the surface gravities κ_c and κ_b using the Euclidean version of Eq. (A4) and the Killing vector (A12); equivalently, one could use the relation $\kappa = 2\pi\gamma_\psi/\psi^{\text{id}}$ to obtain the same result:

$$\kappa_{c,b} = \sqrt{\Lambda} \left(1 \mp \frac{2}{3}\epsilon + \frac{11}{18}\epsilon^2 \right). \quad (\text{A16})$$

This equation is useful for the analysis of the radiation energy flux in a nearly degenerate Lorentzian Schwarzschild–de Sitter universe, since each horizon radiates approximately thermally with the temperature $T = \kappa/2\pi$.

We will now calculate the Euclidean action of the metric (A14) and show that it possesses a negative mode in the direction of decreasing black hole mass. The total instanton action is given by

$$I = -\frac{\Lambda\mathcal{V}}{8\pi} - \frac{A_c\delta_c}{8\pi} - \frac{A_b\delta_b}{8\pi}, \quad (\text{A17})$$

where $\mathcal{V}$ is the four-volume of the geometry, $A_{c,b}$ are the horizon areas, and $\delta_{c,b}$ are the conical deficit angles at the

horizons. Of course, all of these quantities depend on the value we choose for ψ^{id} . Obvious options are: to leave it at 2π even in the nondegenerate case, thus introducing a deficit angle on the cosmological horizon and an excess angle at the black hole horizon, or as an alternative, to make one of the two horizons regular, thereby causing a larger excess or deficit at the other horizon. The most interesting of these cases is the one in which we choose a regular cosmological horizon. In this case, the metric (A14) will lie on the interpolation between the Euclidean Nariai and de Sitter universes, since the latter has only a cosmological horizon, which ought to be regular. The Euclidean actions for these universes are $-2\pi/\Lambda$ for the Nariai, and $-3\pi/\Lambda$ for the de Sitter. Since no intermediate solution is known, one would expect the action to decrease monotonically as one moves away from the Nariai solution. In other words, this particular perturbation of the Nariai metric should correspond to a negative mode in the action. Indeed, if we identify ψ with the period ψ_c^{id} the action in Eq. (A17) turns out as

$$I = -\frac{2\pi}{\Lambda} - \frac{17\pi}{9\Lambda}\epsilon^2 + O(\epsilon^4). \quad (\text{A18})$$

(The same result is obtained for the period ψ_b^{id} while for $\psi^{\text{id}} = 2\pi$ the negative mode is given by $-20\pi\epsilon^2/9\Lambda$.)

The coordinate transformations, the perturbed Nariai metric, and the negative mode given here differ from those of Ref. [15] for various reasons. The authors of [15] did not ensure that $U=0$ on the horizons, and the Killing vector was not renormalized properly. Also, they identified Euclidean time with period 2π even in the nondegenerate case, which is not appropriate to the physical situation we are trying to analyze.

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