Asymptotically free $\hat{\mathrm{U}}(1)$ Kac-Moody gauge fields in $3+1$ dimensions

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$\hat{\mathrm{U}}(1)$ Kac-Moody gauge fields have the infinite dimensional $\hat{\mathrm{U}}(1)$ Kac-Moody group as their gauge group. The pure gauge sector, unlike the usual U(1) Maxwell Lagrangian, is nonlinear and nonlocal; the Euclidean theory is defined on a $(d+1)$-dimensional manifold $\mathcal{R}_d \times \mathcal{S}^1$ and, hence, is also asymmetric. We quantize this theory using the background field method and examine its renormalizability at one loop by analyzing all the relevant diagrams. We find that, for a suitable choice of the gauge field propagators, this theory is one-loop renormalizable in $3+1$ dimensions. This pure $\hat{\mathrm{U}}(1)$ Kac-Moody gauge theory in $3+1$ dimensions has only one running coupling constant and the theory is asymptotically free. When fermions are added the number of independent couplings increases and a richer structure is obtained. Finally, we note some features of the theory which suggest its possible relevance to the study of anisotropic condensed matter systems, in particular, that of high-temperature superconductors. [S0556-2821(96)04520-1]

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I. INTRODUCTION

Kac-Moody groups embody the infinite dimensional symmetries that underlie string theory and conformal field theory [1]. It is natural to realize the Kac-Moody group $\hat{G}$ as a gauge symmetry, that is, as a group of gauge transformations of gauge fields, and to obtain a Lagrangian invariant under these gauge transformations.

A Lagrangian of vector gauge fields coupled to matter fields that is invariant under Kac-Moody gauge transformations for an arbitrary group $\hat{G}$ was obtained in [2]. A Kac-Moody group [3] $\hat{G}$ consists of all mappings of the circle $\mathcal{S}^1$ into the compact Lie group G. Let $\mathcal{R}_d$ be a Euclidean space consisting of one time dimension and $(d-1)$ space dimensions. The gauge transformation at a point $x \in \mathcal{R}_d$, namely, $\Phi(x)$, is an element of $\hat{G}$ and has the form ($\int \equiv \int_0^R d\sigma$; that is, $\mathcal{S}^1$ has radius $R/2\pi$)

$$\Phi(x)=e^{i\Lambda(x)}\exp\left(i\int \phi^{\alpha}(x,\sigma)Q^{\alpha}(\sigma)\right). \tag{1.1}$$

$Q^{\alpha}(\sigma)$ are the generators of $\hat{G}$, $\phi^{\alpha}(x,\sigma)$ and $\Lambda(x)$ are dimensionless gauge functions, and α is the Lie group index. We can consider $\sigma \in \mathcal{S}^1$ to be a new spacelike direction in the theory, which is then effectively defined on $\mathcal{R}_d \times \mathcal{S}^1$. The $\mathcal{S}^1$ direction has a special significance since each point in $\mathcal{S}^1$ carries its own noncommuting charge $Q(\sigma)$. Hence non-Abelian gauge fields $A_{\mu}^{\alpha}(x,\sigma)$ are defined on the manifold $\mathcal{R}_d \times \mathcal{S}^1$. Since $\Phi(x)$ has a central extension, there is also a U(1) gauge field $B_{\mu}(x)$ defined on only $\mathcal{R}_d$.

Just as for the case of general $\hat{G}$, the Lagrangian for $\hat{\mathrm{U}}(1)$ Kac-Moody gauge fields is nonlinear, nonlocal, and asymmetric. Quantizing the latter theory is of particular importance because (i) this is the starting point for the analysis of more complex Kac-Moody gauge fields [2] and (ii) the new nonlinearities arising from the Kac-Moody nature of the gauge group can be studied in their simplest and most essential form for the $\hat{\mathrm{U}}(1)$ case.

In Sec. II we review the $\hat{\mathrm{U}}(1)$ Kac-Moody gauge symmetry and derive the Lagrangian; in Sec. III we quantize the theory via the Feynman path integral using the background field method. In Sec. IV we compute the one-loop β function, and in Sec. V we analyze all the one-loop Feynman diagrams, identify the divergent diagrams, and show that the theory is one-loop renormalizable in 3+1 dimensions. In Sec. V we draw some conclusions and highlight a possible application of the theory as an effective low-energy description of anisotropic superconductors. The Appendixes contain some computational details and a discussion of Ward identities.

II. KAC-MOODY GAUGE-FIELD LAGRANGIAN

The $\hat{\mathrm{U}}(1)$ Kac-Moody generators satisfy the commutation relation

$$[Q(\sigma),Q(\sigma')]=i\kappa\delta'(\sigma-\sigma'), \tag{2.1}$$

where the prime symbol ($\prime$) on functions means $\partial/\partial\sigma$ and κ is a real number. For gauge transformations given by Eq. (1.1), we have

$$\Phi(x)Q(\sigma)\Phi^{\dagger}(x)=Q(\sigma)+\kappa\phi'(x,\sigma). \tag{2.2}$$

To determine the gauge transformation of the gauge fields, consider the link variable ($a=$ lattice spacing)

$$U_{\mu}(x)=\exp\left(iagB_{\mu}(x)+iae\int A_{\mu}(x,\sigma)Q(\sigma)\right), \tag{2.3}$$

where g and e are dimensionful coupling constants. We define Kac-Moody gauge transformations by

$$U_{\mu}(x)\to\Phi(x)U_{\mu}(x)\Phi^{\dagger}(x+a\hat{\mu}), \tag{2.4}$$

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and in the $a \to 0$ limit we obtain

$$\delta A_\mu(x,\sigma) = -\frac{1}{e}\partial_\mu \phi(x,\sigma), \tag{2.5}$$

$$\delta B_\mu(x) = -\frac{1}{g}\partial_\mu \Lambda(x) + \frac{\kappa e}{g}\int \phi'(x,\sigma)A_\mu(x,\sigma) - \frac{\kappa}{2g}\int \phi'(x,\sigma)\partial_\mu \phi(x,\sigma). \tag{2.6}$$

Consider next the open plaquette (untraced Wilson loop) starting and ending at x, that is,

$$W_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x). \tag{2.7}$$

To leading order in a we have

$$W_{\mu\nu}(x) = \exp(ia^2 g\chi_{\mu\nu})\exp\left(ia^2 e\int F_{\mu\nu}(x,\sigma)Q(\sigma)\right), \tag{2.8}$$

where the pure phase is

$$\chi_{\mu\nu} = f_{\mu\nu} + \frac{\kappa e^2}{g}\int A'_\mu A_\nu, \tag{2.9}$$

with

$$f_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Note that the term $\kappa\int A'_\mu A_\nu$ in the phase $\chi_{\mu\nu}$ arises due to the central extension in the Kac-Moody algebra and is absent for gauge theories based on compact Lie groups. Under gauge transformations (2.4) we have, upon using Eq. (2.2),

$$W_{\mu\nu}(x) \to \Phi(x)W_{\mu\nu}(x)\Phi^\dagger(x) \tag{2.10}$$

$$= \exp(ia^2 g\chi_{\mu\nu})\exp\left[ia^2 e\left(\int F_{\mu\nu}Q + \kappa\int F_{\mu\nu}\phi'\right)\right], \tag{2.11}$$

which yields

$$\delta F_{\mu\nu} = 0 \tag{2.12}$$

and

$$\delta\chi_{\mu\nu} = \frac{\kappa e}{g}\int F_{\mu\nu}\phi'. \tag{2.13}$$

Note that unlike the case of compact Lie groups, the Kac-Moody gauge transformation (2.10) induces an inhomogeneous transformation of $\int F_{\mu\nu}\phi'$ on $F_{\mu\nu}$ and hence induces a change in the phase $\chi_{\mu\nu}$. To obtain a gauge-invariant Lagrangian we need to introduce a new field $A_5(x,\sigma)$ to cancel the variation $\int F_{\mu\nu}\phi'$, and which transforms as

$$\delta A_5(x,\sigma) = -\frac{1}{e}\phi'(x,\sigma). \tag{2.14}$$

Defining the antisymmetric tensor

$$\Gamma_{\mu\nu}(x) = \chi_{\mu\nu}(x) + \frac{\kappa e^2}{g}\int F_{\mu\nu}A_5(x,\sigma), \tag{2.15}$$

we have, from Eqs. (2.12), (2.13), and (2.14),

$$\delta\Gamma_{\mu\nu} = 0. \tag{2.16}$$

It has been shown in [2] that $\Gamma_{\mu\nu}$ can be generalized to the Kac-Moody group $\hat{G}$ with G non-Abelian. Next define

$$F_{\mu 5}(x,\sigma) = \partial_\mu A_5 - A'_\mu, \tag{2.17}$$

which from Eqs. (2.5) and (2.14) is gauge invariant,

$$\delta F_{\mu 5} = 0. \tag{2.18}$$

Hence we have the following classical Lagrangian [4] in $\mathcal{R}_d \times \mathcal{S}^1$ which is invariant under the Kac-Moody gauge transformations (2.5), (2.6), and (2.14),

$$\mathcal{L}(x) = \frac{1}{4}\Gamma_{\mu\nu}^2(x) + \frac{1}{4}\int d\sigma d\sigma' F_{\mu\nu}(x,\sigma)h(\sigma-\sigma') \times F_{\mu\nu}(x,\sigma') + \frac{1}{2}\int d\sigma d\sigma' F_{\mu 5}(x,\sigma) \times f(\sigma-\sigma')F_{\mu 5}(x,\sigma'), \tag{2.19}$$

and where, as defined above ($\partial_5 = \partial/\partial\sigma$),

$$\Gamma_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{\kappa e^2}{g}\int A'_\mu A_\nu + \frac{\kappa e^2}{g}\int F_{\mu\nu}A_5,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.20}$$

$$F_{\mu 5} = \partial_\mu A_5 - \partial_5 A_\mu.$$

Recall that $A_\mu = A_\mu(x,\sigma)$, $A_5 = A_5(x,\sigma)$, and $B_\mu = B_\mu(x)$. Since $\Gamma_{\mu\nu}(x)$ has an asymmetric coupling in the σ direction, we anticipate a similar asymmetry in the pure A_μ, A_5 sector and this is reflected in the functions h and f which have been introduced in Eq. (2.19). These functions are required to be positive but are otherwise undetermined.

From the kinetic term for the B_μ field, its canonical mass dimension is deduced to be $(d-2)/2$. We choose the A_μ and A_5 fields to have the mass dimension $(d-1)/2$, which is the appropriate canonical dimension for gauge fields in $d+1$ dimensions. Then the coupling constant e has mass dimension $(3-d)/2$ and

$$\lambda \equiv \frac{\kappa e^2}{g} \tag{2.21}$$

has mass dimension $(2-d)/2$, while the mass dimension of the functions $h(\sigma)$ and $f(\sigma)$ in Eq. (2.19) is 1. The Fourier transform of $h(\sigma)$ [and similarly for $f(\sigma)$] can, therefore, be written as ($\Sigma_n \equiv \Sigma_{n=-\infty}^{\infty}$)

$$h(\sigma) = \frac{1}{R}\sum_n h_n e^{ip_n\sigma}, \tag{2.22}$$

where $p_n=2\pi n/R$, $n\in\mathcal{Z}$, R is the length of the compact σ dimension, and the Fourier coefficients h_n are dimensionless. We will eventually be interested in the case $d=3$, and since R is the natural scale in the theory, we can define a dimensionless coupling constant

$$\overline{\lambda}=\lambda/\sqrt{R}. \tag{2.23}$$

We now discuss the coupling of fermions to the gauge theory. Dirac fermions $\psi(x,\sigma)$ and $\overline{\psi}(x,\sigma)$ defined on the $(d+1)$-dimensional manifold $\mathcal{R}_d\times\mathcal{S}^1$ may be minimally coupled to the gauge fields A_μ and A_5 to give the action [5]

$$S_1^F=\int d^dx d\sigma\overline{\psi}(x,\sigma)[(\partial_\mu-ieA_\mu)\gamma_\mu+(\partial_5-ieA_5)\gamma_5+m]\psi(x,\sigma), \tag{2.24}$$

which is invariant under the usual U(1) transformations (2.5) and (2.14) for the gauge fields and

$$\delta\psi=(e^{-i\phi}-1)\psi,\quad \delta\overline{\psi}=(e^{i\phi}-1)\overline{\psi} \tag{2.25}$$

for the Dirac fermions. The coupling of fermions to $B_\mu(x)$ is more subtle [2]. Consider the $\hat{\mathrm{U}}(1)$ super Kac-Moody algebra in which Majorana fermions $H(\sigma)$, satisfying $\{H(\sigma),H(\sigma')\}=\delta(\sigma-\sigma')$, together with $Q(\sigma)$ form the generators. The super Virasoro algebra formed from $Q(\sigma)$ and $H(\sigma)$ has $c=1+1/2$ with a (suitably regularized) supercharge operator $G(\sigma)=(1/\kappa)Q(\sigma)H(\sigma)$. Kac-Moody fermions $\Psi(x)$ are defined on $\mathcal{R}_d$, and form an arbitrary irreducible representation of the super Virasoro algebra. They have Kac-Moody invariant coupling to the gauge fields as

$$S_2^F=\int d^dx\overline{\Psi}(x)\left[\left(\partial_\mu-ie\int A_\mu(x,\sigma)Q(\sigma)-igB_\mu(x)\right)\times\gamma_\mu+M\right]\Psi(x)+\tau\int d^dx\Psi(x)\times\int[G(\sigma)-eA_5(x,\sigma)H(\sigma)]\Psi(x). \tag{2.26}$$

Under the Kac-Moody gauge transformation (1.1) the Kac-Moody fermions transform as

$$\Psi(x)\to\Phi(x)\Psi(x),\quad \overline{\Psi}(x)\to\overline{\Psi}(x)\Phi^\dagger(x), \tag{2.27}$$

and, as shown in Ref. [2], the action (2.26) is invariant. Note that the first term of Eq. (2.26) is the minimal coupling piece. The τ term in S_2^F has been introduced to attenuate the high mass states. The thermodynamical partition function of the free Lagrangian for $\Psi(x)$ given in Eq. (2.26) has been studied in [6] and exhibits a maximum limiting temperature. In this paper, for simplicity, we will discuss the quantum theory of the gauge fields only in the presence of the ordinary Dirac fermions (2.24).

III. PATH-INTEGRAL QUANTIZATION

We define the quantum theory through the Feynman path integral and the Faddeev-Popov *ansatz* for the gauge fixing. It is convenient to use the background field formalism [7] whereby the fields are split into background fields (A_μ,A_5,B_μ) and quantum fields (a_μ,a_5,b_μ). The classical action (2.19) is then invariant under the transformation

$$\delta(A_5+a_5)=-\frac{1}{e}\phi', \tag{3.1}$$

$$\delta(A_\mu+a_\mu)=-\frac{1}{e}\partial_\mu\phi, \tag{3.2}$$

$$\delta(B_\mu+b_\mu)=-\frac{1}{g}\partial_\mu\Lambda+\frac{\kappa e}{g}\int\phi'(A_\mu+a_\mu)-\frac{\kappa}{2g}\int\phi'(\partial_\mu\phi). \tag{3.3}$$

The action with given background fields is still invariant under a gauge transformation of the quantum fields alone. These quantum gauge transformations are deduced by deleting the classical fields on the left-hand side of (3.1)–(3.3):

$$\delta^Q a_5=-\frac{1}{e}\phi', \tag{3.4}$$

$$\delta^Q a_\mu=-\frac{1}{e}\partial_\mu\phi, \tag{3.5}$$

$$\delta^Q b_\mu=-\frac{1}{g}\partial_\mu\Lambda+\frac{\kappa e}{g}\int\phi'(A_\mu+a_\mu)-\frac{\kappa}{2g}\int\phi'(\partial_\mu\phi). \tag{3.6}$$

Hence to define the background field path integral $Z[A_\mu,A_5,B_\mu]$, the quantum gauge fields must be gauge fixed and an infinite group volume factored out. However, the gauge-fixed action is required to be invariant under background field transformations which are of the same form as the classical gauge invariance. That is,

$$\delta^B A_5=-\frac{1}{e}\phi', \tag{3.7}$$

$$\delta^B A_\mu=-\frac{1}{e}\partial_\mu\phi, \tag{3.8}$$

$$\delta^B B_\mu=-\frac{1}{g}\partial_\mu\Lambda+\frac{\kappa e}{g}\int\phi'(A_\mu)-\frac{\kappa}{2g}\int\phi'(\partial_\mu\phi), \tag{3.9}$$

$$\delta^B b_\mu=\frac{\kappa e}{g}\int a_\mu\phi', \tag{3.10}$$

$$\delta^B a_5=\delta^B a_\mu=0. \tag{3.11}$$

A gauge-fixing term for the a_μ and b_μ fields which is invariant under (3.7)–(3.11) is

$$\delta(\partial_\mu a_\mu)\,\delta\left(\partial_\mu b_\mu+\frac{\kappa e^2}{g}\int A'_\mu a_\mu-c_\mu\right), \tag{3.12}$$

with $c_\mu(x)$ an arbitrary function. To see the invariance of the Dirac-δ-function constraints in Eq. (3.12), consider the variation of the argument of the second term,

$$\delta^B\left(\partial_\mu b_\mu+\frac{\kappa e^2}{g}\int A'_\mu a_\mu-c_\mu\right)=\partial_\mu\delta^B b_\mu+\frac{\kappa e^2}{g}\int (a_\mu\delta^B A'_\mu+A'_\mu\delta^B a_\mu)=\partial_\mu\left(\frac{\kappa e}{g}\int a_\mu\phi'\right)+\frac{\kappa e^2}{g}\int\frac{(-\partial_\mu\phi')}{e}a_\mu$$

$$=\frac{\kappa e}{g}\int \phi'\partial_\mu a_\mu=0,$$

where in obtaining the last line we used that, from Eq. (3.12), $\partial_\mu a_\mu=0$. Also, the constraint on the a_μ field in Eq. (3.12) is trivially invariant under Eq. (3.11). When the gauge-fixing equation (3.12) is implemented in the path integral, the constraint on the b_μ field may be exponentiated as usual by averaging over the c_μ variable with a Gaussian weight. However, as noted above, it is crucial for the invariance of Eq. (3.12) that the a_μ field be in the strict Landau gauge. We now show that the Faddeev-Popov ghosts due to the above gauge fixing are noninteracting. Let $\Delta_{\rm FP}$ be the Faddeev-Popov determinant corresponding to the insertion of identity into the path integral:

$$1\equiv\Delta_{\rm FP}\int [D\Lambda D\phi]\,\delta(\partial_\mu a_\mu^Q)\,\delta\left(\partial_\mu b_\mu^Q+\frac{\kappa e^2}{g}\int A'_\mu a_\mu^Q-c_\mu\right). \tag{3.13}$$

The superscript Q on the quantum fields in the above equation denotes the gauge transform of the fields according to the variations given in (3.4)–(3.6). Thus,

$$\Delta_{\rm FP}^{-1}=\int [D\Lambda D\phi]\,\delta\left(-\frac{1}{e}\partial^2\phi+\partial_\mu a_\mu\right)\delta\left(-\frac{1}{g}\partial^2\Lambda+\partial_\mu b_\mu+\Omega(A_\mu,a_\mu,\phi)\right) \tag{3.14}$$

$$\begin{aligned}
&=\int [D\phi]\,\delta\left(-\frac{1}{e}\partial^2\phi+\partial_\mu a_\mu\right)\int [D\overline{\Lambda}]\,\delta(\overline{\Lambda})\det{}^{-1}\left[\partial\left(-\frac{1}{g}\partial^2\Lambda+\cdots\right)\Big/\partial\Lambda\right]\\
&=\det{}^{-1}\left(-\frac{1}{g}\partial^2\right)\int [D\phi]\,\delta\left(-\frac{1}{e}\partial^2\phi+\partial_\mu a_\mu\right)\\
&=\det{}^{-1}\left(-\frac{1}{g}\partial^2\right)\det{}^{-1}\left(-\frac{1}{e}\partial^2\right).
\end{aligned} \tag{3.15}$$

In Eq. (3.14) we have denoted by Ω some terms which are independent of the gauge parameter Λ. Then in the next line we carried out first the $\int[D\Lambda]$ integral which gave an inverse determinant of a Laplacian (free field propagator). Finally the $\int[D\phi]$ integral was done to give another free field determinant, showing that $\Delta_{\rm FP}$ represents free ghost fields. Notice that the above demonstration was simple because of the order in which the gauge-parameter integrals were done. If, however, one integrated in Eq. (3.14) first over the ϕ field, one would have obtained a nontrivial Jacobian which becomes trivial only after the Λ integral is done.

Therefore we may ignore (except for gravity or nonzero temperature) the ghosts and write the gauge-fixed action, including external sources J_μ, K_μ, and L, as

$$\begin{aligned}
S=\lim_{\alpha\to 0}\int d^dx\Big\{&\mathcal{L}(A_5+a_5,A_\mu+a_\mu,B_\mu+b_\mu)\\
&+\frac{1}{2\alpha}\int(\partial_\mu a_\mu)^2\Big\}+\int d^dx\left\{\frac{1}{2\eta}\left(\partial_\mu b_\mu+\frac{\kappa e^2}{g}\int A'_\mu a_\mu\right)^2\right.\\
&\left.+\int J_\mu a_\mu+\int La_5+K_\mu b_\mu\right\},
\end{aligned} \tag{3.16}$$

with the Lagrangian $\mathcal{L}$ given by Eqs. (2.19) and (2.20) and the limit $\alpha\to 0$ enforcing the Landau gauge condition for a_μ. This action is invariant under the background transformation (3.7)–(3.11) plus the following transformation for the sources,

$$\delta^B J_\mu=-\frac{\kappa e}{g}K_\mu\phi', \tag{3.17}$$

$$\delta^B K_\mu=\delta^B L=0, \tag{3.18}$$

and therefore the quantum field theory, defined by the background path integral

$$Z[A_\mu,A_5,B_\mu,J_\mu,K,L_\mu]=\int [Da_\mu Da_5 Db_\mu]e^{-S_{\rm eff}}, \tag{3.19}$$

is invariant under the transformations (3.7)–(3.11) and (3.17) and (3.18). The one-particle-irreducible (1PI) effective action of the ''vacuum'' theory (that is without background fields) can be obtained by computing, using Eq. (3.19), 1PI graphs with no external quantum fields [7], and by identifying the background fields in Eq. (3.19) with the fields appearing in the 1PI action. This 1PI action is invariant under the Kac-Moody gauge transformations (3.7)–(3.9).

For a one-loop calculation we only need to expand Eq. (3.16) up to second order in the quantum fields. Terms linear in the quantum fields may be ignored as they do not contribute to the 1PI effective action [7]. Therefore the part of Eq. (3.16) relevant for this paper is given by

$$S=S_0+S_2, \tag{3.20}$$

where S_0 is the classical action corresponding to the Lagrangian (2.19) and

$$S_2=\int d^dx\frac{1}{4}\left\{\tilde{f}_{\mu\nu}+\lambda\int d\sigma(a'_\mu A_\nu+A'_\mu a_\nu+F_{\mu\nu}a_5+\tilde{F}_{\mu\nu}A_5)\right\}^2+\int d^dx\frac{\lambda}{2}\Gamma_{\mu\nu}\int d\sigma(a'_\mu a_\nu+\tilde{F}_{\mu\nu}a_5)+\int d^dx\frac{1}{2\eta}\left(\partial_\mu b_\mu\right.$$
$$\left.+\lambda\int d\sigma A'_\mu a_\mu\right)^2+\int d^dx\int d\sigma d\sigma'\left(\frac{1}{4}\tilde{F}_{\mu\nu}(\sigma)h(\sigma-\sigma')\tilde{F}_{\mu\nu}(\sigma')+\frac{1}{2}\tilde{F}_{\mu5}(\sigma)f(\sigma-\sigma')\tilde{F}_{\mu5}(\sigma')\right). \tag{3.21}$$

Where there is no ambiguity we have suppressed the x and σ dependence of some of the fields in the above equations. The tensors $F_{\mu\nu}$, $f_{\mu\nu}$, and $\Gamma_{\mu\nu}$ were defined in Sec. II and correspond to the background fields while the tilde denotes the same tensors but now formed from the quantum fields, that is, $\tilde{f}_{\mu\nu}=\partial_\mu b_\nu-\partial_\nu b_\mu$, and so forth.

The free propagators (see Fig. 1) for the quantum gauge fields are given in momentum space by (for simplicity we choose the Feynman gauge $\eta=1$ for the b_μ field)

$$(b_\mu b_\nu)\equiv\Delta_{\mu\nu}(p)=\frac{\delta_{\mu\nu}}{p^2}, \tag{3.22}$$

$$(a_\mu a_\nu)\equiv D_{\mu\nu}(P)=\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{f_n p_n^2+h_n p^2}, \tag{3.23}$$

$$(a_5a_5)\equiv S(P)=\frac{1}{f_n p^2}, \tag{3.24}$$

$$(a_\mu a_5)=0, \tag{3.25}$$

where $P\equiv(p_\mu,p_n)$, $p_n=2\pi n/R$, $p=(p_\mu^2)^{1/2}$, and f_n and h_n are the Fourier coefficients of $h(\sigma)$ and $f(\sigma)$ [see Eq. (2.22)]. Note that because of the strict Landau gauge condition on the a_μ field, there is no mixed $(a_\mu a_5)$ propagator at the tree level.

From S_2 one obtains by inspection the interaction vertices. Then using Wick's theorem together with the propagators given above the various contributions to the one-loop 1PI action can be written down. In Appendix B we use equivalent functional methods to obtain specific one-loop contributions necessary for the β-function calculation in the next section. We remark that as both the β-function calculation and the renormalizability of the theory can be discussed economically by studying the 1PI action, it is unnecessary to first determine explicit Feynman rules for the vertices and then reconstruct diagrams from those rules. Nevertheless, in Sec. V when we discuss the renormalizability of the theory we draw Feynman diagrams as a convenient shorthand to illustrate the various N-point Green's functions contained in the one-loop 1PI action. Finally note that the fermionic action (2.24), being conventional, can be incorporated in the above analysis in the usual way [7].

IV. β FUNCTION

A. General

The tree-level action (3.16) has been written in terms of ''bare'' quantum and background fields, and bare coupling constants λ and e. When the quantum fields are integrated out in the path integral (3.19) to generate the full 1PI effective action, UV divergences will be present in the new terms which have been added to the classical action. As the full 1PI action is guaranteed to be gauge invariant in the background gauge (when a gauge-invariant regulator is used), the divergences will take a gauge-invariant form [7]. If the theory is renormalizable, which we show below to be the case at one loop, the divergences can be removed by rescaling the fields and couplings in the bare classical Lagrangian so that the new Lagrangian, and hence the physical partition function, is finite when written in terms of finite ''renormalized'' quantities. Consider the one-loop UV-divergent correction $\Delta\Gamma^2_{\mu\nu}$ to $\Gamma^2_{\mu\nu}$. Then we want to write

$$\Gamma^2_{\mu\nu}+\Delta\Gamma^2_{\mu\nu}\equiv\Gamma^{(r)2}_{\mu\nu}, \tag{4.1}$$

by renormalizing the bare tensor (we employ dimensional regularization with $d=3-2\epsilon$)

$$\Gamma_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+\mu_0^\epsilon\sqrt{R\lambda}\int d\sigma(A'_\mu A_\nu+F_{\mu\nu}A_5), \tag{4.2}$$

where μ_0 is some fixed mass scale (such as R), into the renormalized tensor

$$\Gamma^r_{\mu\nu}=\partial_\mu B^r_\nu-\partial_\nu B^r_\mu+\mu^\epsilon\sqrt{R\lambda_r}\int d\sigma(A^{r\prime}_\mu A^r_\nu+F^r_{\mu\nu}A^r_5), \tag{4.3}$$

defined at the new mass scale μ. The bare quantities in Eq. (4.2) are related to the renormalized ones in Eq. (4.3) by the Z factors:

$$B_\mu=\sqrt{Z_B}B^r_\mu, \tag{4.4}$$

$$A_\mu=\sqrt{Z_A}A^r_\mu, \tag{4.5}$$

$$A_5=\sqrt{Z_5}A^r_5, \tag{4.6}$$

and

$$\overline{\lambda}=Z_{\lambda}\overline{\lambda_r}\left(\frac{\mu}{\mu_0}\right)^{\epsilon}. \tag{4.7}$$

[We are implicitly assuming that the theory is multiplicatively renormalizable, a fact which we will verify to one loop by explicit calculations. Therefore the discussion of renormalization here is sufficient. We defer to future investigations details (see Abbott [7]) which manifest themselves at higher orders.] As the Kac-Moody gauge invariance (3.7)–(3.9) of the 1PI action is maintained in the background gauge, the renormalizations (4.4)–(4.7) imply certain relations among the Z factors. Substituting Eqs. (4.4)–(4.7) into Eq. (4.2) we obtain

$$\Gamma_{\mu\nu}=\sqrt{Z_B}\left(\partial_{\mu}B^r_{\nu}-\partial_{\nu}B^r_{\mu}+\mu^{\epsilon}\sqrt{R}\,\overline{\lambda_r}z_1\int d\sigma(A^{r\prime}_{\mu}A^r_{\nu}+z_2F^r_{\mu\nu}A^r_5)\right), \tag{4.8}$$

where

$$z_1\equiv\frac{Z_AZ_{\lambda}}{\sqrt{Z_B}}, \tag{4.9}$$

$$z_2\equiv\left(\frac{Z_5}{Z_A}\right)^{1/2}. \tag{4.10}$$

By definition, the object in large parentheses in Eq. (4.8) is the *finite* renormalized tensor $\Gamma^r_{\mu\nu}$, Eq. (4.3), and so one must have

$$z_1=z_2=1. \tag{4.11}$$

Therefore one obtains the background gauge Ward identities (compare the case of Yang-Mills theory in [7])

$$Z_5=Z_A \tag{4.12}$$

and

$$Z_{\lambda}=\frac{\sqrt{Z_B}}{Z_5}. \tag{4.13}$$

Hence

$$\overline{\lambda}=\frac{\sqrt{Z_B}}{Z_5}\left(\frac{\mu}{\mu_0}\right)^{\epsilon}\overline{\lambda_r}, \tag{4.14}$$

from which the beta function $\beta_{\lambda}=\mu\partial\overline{\lambda_r}/\partial\mu$ can be determined by differentiating both sides of Eq. (4.14) with respect to μ and taking $\epsilon\to 0$. It should be noted that the relation (4.12) also follows consistently from the renormalization of the gauge-invaraint tensor $F_{\mu 5}$, and that Eq. (4.13) is the analogue of the relation $Z_g^{\mathrm{YM}}=1/\sqrt{Z_A}$ for Yang-Mills theory in the background gauge [7]. In Appendix A, we reobtain the identities (4.12) and (4.13) by considering the renormalization of the gauge-transformation rules (2.5), (2.6), and (2.14), and also derive some identities for two-point functions of the theory.

FIG. 1. The a_{μ} propagator (wavy line), the a_5 propagator (curly line), the b_{μ} propagator (dashed line), and the Dirac fermion propagator (solid line).

In Secs. IV B and IV C, we determine the wave function renormalization factors required in Eq. (4.14) to one-loop order in the pure gauge theory. Note that though Z_5 and Z_A are equal from Eq. (4.12), we have used the former in Eq. (4.14) as it is easier to determine directly (only two diagrams need be analyzed). In Sec. IV D we will discuss how including the Dirac fermions (2.24) changes the number of independent couplings, and also compute the new one-loop β functions.

B. Z_B

Two terms from S_2, Eq. (3.21), contribute to the one-loop correction $\Delta\Gamma^2_{\mu\nu}$ and these determine Z_B. (In terms of two-point functions these contributions correspond to the two Feynman diagrams in Fig. 2.) We obtain (see Appendix B)

$$\Delta\Gamma^2_{\mu\nu}(x)=-4\mathcal{T}\Gamma^2_{\mu\nu}(x), \tag{4.15}$$

where

$$\mathcal{T}=\frac{-\overline{\lambda}^2R}{8}(4-d)\mathcal{S}(d)\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q^2+1)}+\text{convergent terms as } d\to 3 \tag{4.16}$$

and

$$\mathcal{S}(d)\equiv\mu_0^{2\epsilon}\left(\frac{2\pi}{R}\right)^{d-2}\sum_n\frac{|n|^{d-2}}{h_n^{d/2}}f_n^{(d-4)/2}. \tag{4.17}$$

Since the d-dimensional loop momentum integral is ultraviolet (UV) finite for $d\leqslant 3$, the only possible ultraviolet (UV) divergence as $d\to 3$ comes from the sum over n. To evaluate Eq. (4.17) we have to specify the functions f_n and h_n. We choose (see below)

$$h_n=c_1|n|+c_1', \tag{4.18}$$

$$f_n=c_2|n|+c_2'. \tag{4.19}$$

For the boundedness of the path integral we require the constants c_i and c_i' to be non-negative. If either of c_1 or c_2 were zero, then, since the Riemann zeta function $\zeta(s)=\Sigma_1^{\infty}1/n^s$ has a pole only for $s=1$ [specifically $\zeta(s\to 1)\to 1/(s-1)$], the sum (4.17) would be finite as $d\to 3$. For the choice (4.18) and (4.19) with c_1 and c_2 nonzero, and as $\epsilon\to 0$,

$$S(d=3-2\epsilon)\to\frac{2\pi}{R}\frac{1}{\sqrt{c_1^3c_2}}\frac{1}{\epsilon}+O(\epsilon^0). \tag{4.20}$$

Hence, using the minimal subtraction scheme

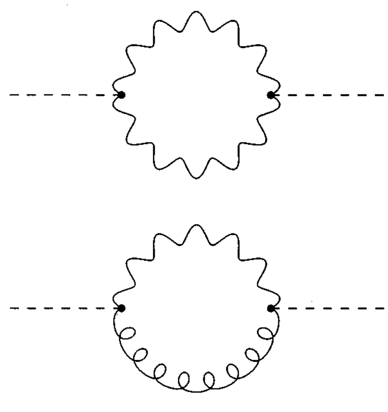

FIG. 2. Diagrams contributing to Z_B.

$$\Delta\Gamma^2_{\mu\nu}(x)=\frac{\overline{\lambda^2}}{4\epsilon}\frac{1}{\sqrt{c_1^3c_2}}\Gamma^2_{\mu\nu}(x) \tag{4.21}$$

and since $\Gamma^2_{\mu\nu}+\Delta\Gamma^2_{\mu\nu}\equiv\Gamma^{(r)2}_{\mu\nu}=Z_B^{-1}\Gamma^2_{\mu\nu}$ this implies

$$Z_B^{-1}=1+\frac{\overline{\lambda^2}}{4\epsilon}\frac{1}{\sqrt{c_1^3c_2}}. \tag{4.22}$$

As $\lambda=\lambda_r+O(\lambda_r^3)$, therefore

$$Z_B=1-\frac{\overline{\lambda_r^2}}{4\epsilon}\frac{1}{\sqrt{c_1^3c_2}}. \tag{4.23}$$

To motivate the choice (4.18) and (4.19), let us assume that h_n and f_n, for large n, have the powerlike asymptotic forms

$$h_n\to c_s|n|^s, \tag{4.24}$$

$$f_n\to c_t|n|^t. \tag{4.25}$$

For integral $s,t>0$, the sum (4.17) for large n is then

$$\sim\sum_n |n|^{d-2}|n|^{t[(d-4)/2]}|n|^{-sd/2}. \tag{4.26}$$

From the properties of the Riemann ζ function, as $d\to 3$, this has a divergence only if

$$3s+t=4. \tag{4.27}$$

The only positive integral solution to Eq. (4.27) is $s=t=1$. If the condition (4.27) is not satisfied, then the n sum is finite as $d\to 3$ and we would obtain $Z_B=1$. Furthermore, as $Z_5=1$ (see next subsection) independent of f_n and h_n, this implies that the one-loop β function for λ is zero for the *ansatz* (4.24) and (4.25) unless $s=t=1$. Thus our choice (4.18) and (4.19), and any others equivalent to it at large n, would give a running coupling while other choices of h_n and f_n with asymptotic forms (4.24) and (4.25) would give a UV finite theory at one loop.

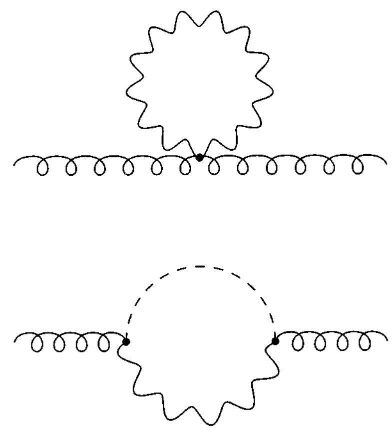

FIG. 3. Gauge field diagrams contributing to the A_5 self-energy.

C. Z_5

The one-loop contribution to the 1PI effective action which is quadratic in A_5 determines Z_5 (see Appendix B for details). Equivalently, Z_5 is determined by two diagrams contributing to the self-energy of the A_5 field: The tadpole and nontadpole diagrams shown in Fig. 3. Neither of these involves an internal n summation. Since the a_μ propagator is massive, the tadpole digram has a potential UV divergence but this divergence is linear and vanishes in dimensional regularization: Recall that in dimensional regularization (DR) the scaleless integral $(d>2)$

$$\int\frac{d^dq}{q^2}=0, \tag{4.28}$$

so that

$$\int\frac{d^dq}{q^2+m^2}=\int d^dq\left(\frac{1}{q^2+m^2}-\frac{1}{q^2}\right)=\int\frac{d^dq(-m^2)}{q^2(q^2+m^2)} \tag{4.29}$$

is finite as $d\to 3$. Similarly a potential linear UV divergence in the other diagram is removed in DR. Hence $Z_5=1$ at one-loop order. As a cross-check, we have verified, by looking at the $(A_\mu A_\nu)$ and $(A_\mu A_5)$ self-energies, that they are also finite near $d=3$ so that $Z_5=Z_{A_\mu}=1$ as required by gauge invariance in the background gauge. Furthermore, the fact that the divergence in the B_μ field two-point function could be renormalized by a wave function renormalization and that the A_μ and A_5 two-point functions are finite justifies *a posteriori* our assumption in the last subsection of multiplicative renormalizability of the two-point functions at one loop. Thus it follows from Eq. (4.14), Eq. (4.23), and $Z_5=1$ that

$$\beta_\lambda=\frac{-\overline{\lambda_r^3}}{4\sqrt{c_1^3c_2}}. \tag{4.30}$$

The four-dimensional pure $\hat{\mathrm{U}}(1)$ Kac-Moody gauge theory is asymptotically free.

To date, the known renormalizable and asymptotically free theories in four dimensions are of the Yang-Mills [9]

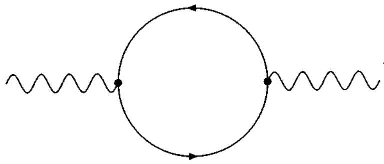

FIG. 4. Fermion loop contributing to Z_5.

type: These are based on a non-Abelian Lie group [10,11]. The result of this paper shows that the four-dimensional $\hat{\mathrm{U}}(1)$ gauge theory, which is nonlocal in one spatial coordinate, can also be asymptotically free. Furthermore, as we show in Sec. V, this theory is at least one-loop renormalizable. Whether the theory is renormalizable to all orders is an interesting open question.

Physically, it has been understood [12] that the crucial ingredient responsible for asymptotic freedom in Yang-Mills theory is the self-interaction of spin one (vector) fields which makes the vacuum of Yang-Mills theory behave like a paramagnetic medium [13]. As the $\hat{\mathrm{U}}(1)$ Kac-Moody gauge theory also has self-interacting vector fields, asymptotic freedom in this case probably has a similar physical interpretation. However, we remind the reader that unlike Yang-Mills theory, the $(3+1)$-dimensional Kac-Moody gauge theory is nonlocal or, equivalently, it may be interpreted as a local theory but with an infinite set of noncommuting charges.

D. Including Dirac fermions

The Dirac fermions $\psi(x,\sigma)$ and $\overline{\psi}(x,\sigma)$ couple only to the gauge fields A_μ and A_5 as indicated in Eq. (2.24), and the resulting theory with gauge fields and Dirac fermions has two independent coupling constants λ and e. The wave function renormalization constant Z_A now obtains a quantum-electrodynamic-like contribution [8] from the fermion-loop diagram of Fig. 4,

$$Z_5=Z_A=1-\frac{e^2N}{12\pi^2\epsilon}, \tag{4.31}$$

where N is the number of fermion flavors. The divergence which Eq. (4.31) absorbs (through a rescaling of the bare fields A_μ and A_5) is that in the loop correction to the usual kinetic terms $\frac{1}{4}F^2_{\mu\nu}$ and $\frac{1}{2}F^2_{\mu 5}$, that is, Eq. (2.19) with $h(\sigma-\sigma')=f(\sigma-\sigma')=\delta(\sigma-\sigma')$. Therefore with Dirac fermions present, the renormalizability of the theory seems to restrict the choice (4.18) and (4.19) further to $h_n=f_n=1$. However, as we argue below, it is still possible to choose the more general form

$$h_n=f_n=1+c|n|. \tag{4.32}$$

With Eq. (4.32), the kinetic terms for the A_μ and A_5 fields in Eq. (2.19) are now split into two pieces each: The 1 in Eq. (4.32) gives rise to the conventional kinetic term which is renormalized by Eq. (4.31) while the $c|n|$ part gives rise to another kinetic term. Since at one loop there is no ultraviolet divergence corresponding to the latter kinetic term, consistency demands that c may be nonzero only if it renormalizes inversely to Eq. (4.31), so that the unconventional kinetic term is unrenormalized. That is, we require

$$Z_cZ_A=1, \tag{4.33}$$

where the bare parameter c is related to the renormalized parameter c_r by

$$c=Z_cc_r. \tag{4.34}$$

(Actually what is being renormalized is a mass parameter c/R.)

Now the dimensionless bare charge e is related to the renormalized dimensionless charge e_r by

$$e=\left(\frac{\mu}{\mu_0}\right)^{\epsilon}Z_ee_r. \tag{4.35}$$

Since from Eq. (2.24) and gauge invariance in the background gauge (compare with the discussion in the last section)

$$Z_e=\frac{1}{\sqrt{Z_A}}=\frac{1}{\sqrt{Z_5}}, \tag{4.36}$$

therefore [8]

$$\beta_e=\mu\frac{\partial e_r}{\partial\mu}=-e_r\epsilon+\frac{e_r^3N}{12\pi^2}, \tag{4.37}$$

as in quantum electrodynamics (QED). (We have kept the term of order ϵ in β_e as it is needed below.) Then one obtains

$$\gamma_c\equiv\mu\frac{\partial c_r}{\partial\mu}=\frac{c_re_r^2N}{6\pi^2}. \tag{4.38}$$

Note that there is no term of order ϵ on the right-hand side of the last equation because, as mentioned before, c/R is a mass parameter of fixed mass dimension 1, so that no compensating mass parameter μ^ϵ enters in its dimensional continuation.

Using the values of β_e, γ_c, and Z_5 above, and Z_b from Eq. (4.23) with $c_1=c_2=c$, we obtain, from Eq. (4.14),

$$\beta_\lambda=-\frac{\overline{\lambda_r^3}}{4c_r^2}+\frac{\overline{\lambda_r}e_r^2N}{6\pi^2} \tag{4.39}$$

to lowest order. The contribution of the fermions is of opposite sign to that of the gauge fields just as in quantum chromodynamics (QCD). However, unlike QCD, we have two independent coupling constants appearing in Eq. (4.39) so that β_λ can be arranged to have a nontrivial zero in weak coupling: For e_r small, β_λ vanishes at

$$\overline{\lambda_r^\star}=e_r\frac{2c_r}{\pi}\left(\frac{N}{6}\right)^{1/2}. \tag{4.40}$$

Since $\overline{\lambda_r^\star}\sim O(e_r)$ even for c_r of order 1 and N not too large, the result (4.40) is self-consistent as it is within the perturbative regime for *both* $\overline{\lambda_r}$ and e_r. Thus, starting from short distances (but e_r still small), $\overline{\lambda_r}$ tends to increase at longer distances, but before it can become too large the fermion coupling drives it to $\lambda_r^\star$. This point is infrared stable since $\beta_\lambda'(\lambda_r^\star)>0$. However, $\lambda_r^\star$ is not a fixed point of the whole system since e_r continues to run.

V. RENORMALIZABILITY

We want to show that the theory as defined by Eq. (3.16) is at least one-loop renormalizable. That is, the UV divergences in all terms in the one-loop 1PI effective action can be absorbed by rescaling the terms in the tree Lagrangian (2.19). Note that since the tensors $\Gamma_{\mu\nu}$, $F_{\mu\nu}$, $F_{\mu 5}$ are individually gauge invariant, radiative corrections could in principle generate more gauge-invariant terms than are in the Lagrangian (2.19), such as

$$\Delta\mathcal{L}=\int \{\alpha_1(\partial_\mu F_{\mu\nu})F_{\nu 5}+\alpha_2\Gamma_{\mu\nu}F_{\mu\nu}+\alpha_3 F_{\mu\nu}\partial^2 F_{\mu\nu}+\cdots\}. \tag{5.1}$$

In this section we show that, remarkably, all the one-loop UV divergences have the same form as the minimal Lagrangian (2.19) so that no extra terms such as Eq. (5.1) are needed for renormalization. This is also fortunate because some of the terms in Eq. (5.1) are not positive definite and so unsuitable for inclusion in the tree-level Lagrangian (see Fig. 4). More specifically, we will sketch renormalizability of the theory for the case

$$h_n=c_1|n|+c_1',\qquad f_n=c_2|n|+c_2', \tag{5.2}$$

discussed in the last section [for Eqs. (4.24) and (4.25) with $s,t>1$ the theory is even more convergent]. The possible terms such as Eq. (5.1) in the 1PI action correspond to contributions from various N-point 1PI diagrams formed from the vertices in Eq. (3.21) with the quantum fields (a_μ,b_μ,a_5) along the internal lines and the background fields (A_μ,B_μ,A_5) on the external legs [7]. In the next subsection we analyze subsets of such diagrams in the pure gauge theory and in Sec. V C we consider fermions (2.24). For the purpose of determining the UV degree of divergence of the one-loop diagrams, it is sufficient to consider the large-n limit in which case we can treat the discrete sum as a continuous integral by making the replacements

$$\frac{n}{R}\to k_z \quad \text{and} \quad \frac{1}{R}\sum_n \int d^3k \to \int dk_z d^3k. \tag{5.3}$$

Thus according to Eq. (5.2), in a power-counting analysis the propagators (3.23) and (3.24) for the a_μ and a_5 fields have a cubic decay $(\sim 1/|p|^3)$ at large momenta in contrast to the usual quadratic form of propagators such as Eq. (3.22).

A. Case A: diagrams containing at least one external B_μ line

(i) Exactly two B_μ lines. We have already shown (in Sec. IV B) that diagrams contributing to the one-loop correction of $\Gamma_{\mu\nu}^2$ have a UV divergence which is removed by wave function renormalization, and that no other divergences remain in those diagrams.

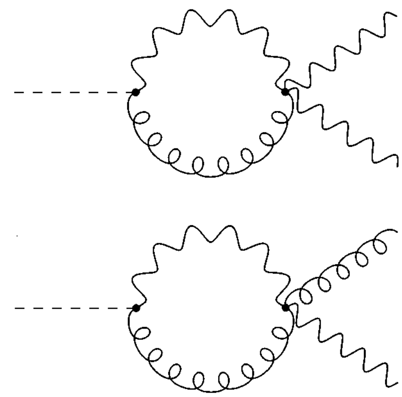

FIG. 5. UV-divergent diagrams contained in the one-loop correction to $\Gamma_{\mu\nu}^2$.

(ii) At least two B_μ and some other external lines. Since the diagrams in case A(i) were logarithmically divergent, adding any more external vertices makes the diagrams UV convergent.

(iii) One B_μ and some other external lines. From the vertices in Eq. (3.21), one sees that at one-loop no gauge-invariant term in the 1PI action can be formed from exactly one Γ and one $F_{\mu\nu}$ (or $F_{\mu 5}$). So consider possible terms of the form $\Gamma(F)^n$, $n\geqslant 2$ (we have used a symbolic notation: indices and possible derivatives have been suppresed, and F here denotes $F_{\mu\nu}$ or $F_{\mu 5}$). Consider first $n=2$; some of the diagrams with one external B line and two external A lines (see Fig. 5) actually contribute to $\Delta\Gamma_{\mu\nu}^2$ and by gauge invariance their UV divergence is related to the wave function renormalization of the B_μ field already discussed in case A(i) above. There are also diagrams which are not part of corrections to $\Gamma_{\mu\nu}^2$ and so must be part of corrections to $\Gamma(F)^2$ (see, e.g., Fig. 6). We wish to show that the net contribution of diagrams such as in Fig. 6 to $\Gamma(F)^2$ is only a finite renormalization. Note that superficially Fig. 6 is logarithmically UV divergent but gauge invariance demands that the external A_5 vertex must eventually (either after doing the detailed calculation or after adding other similar diagrams) become a $\partial_\mu A_5$. This then makes the diagram UV convergent.

With the above effective power counting [that is, Eq. (5.3) plus gauge invariance], one can show that all diagrams con-

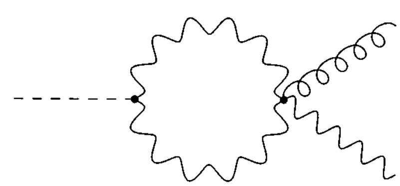

FIG. 6. A diagram not contained in $\Delta\Gamma_{\mu\nu}^2$. The curly external line is A_5, and the wavy external line is A_μ', while the dashed external line is $\partial_\mu B_\tau$.

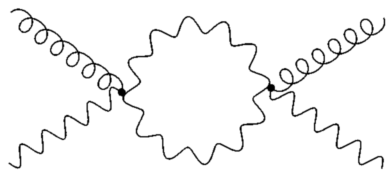

FIG. 7. A diagram contained in the one-loop correction to $\Gamma^2_{\mu\nu}$.

tributing to $\Gamma(F)^2$ are convergent. Then it follows that $\Gamma(F)^n (n \geqslant 2)$, radiative contributions to the 1PI action at one-loop order are UV finite.

B. Case B: diagrams *not* containing an external B_μ line

(i) Terms of the form $(F)^{2n+1}$, $n \geqslant 1$, in the 1PI action. It is impossible to form such terms at one-loop order as it requires an odd number of $A-b-a$ vertices which cannot be contracted since the b_μ field must occur in pairs.

(ii) Terms of the form $(F)^4$ in the 1PI action. Some of the diagrams containing four external A lines (see, e.g., Fig. 7) contribute to $\Delta\Gamma^2_{\mu\nu}$ and therefore their UV divergence is as before related to the wave function renormalization of the B_μ field. Here we need to consider those four A diagrams which contribute to F^4. Consider for example the diagram of Fig. 8. There is no internal n sum because of the b_μ propagator, and so superficially the UV degree of divergence is that of the integral $\int d^3 q q^3/q^6$ which is logarithmic. More explicitly integrals of the form

$$\int \frac{d^d q q_\mu q_\sigma q_\alpha}{q^4[m^2+(p-q)^2]} \tag{5.4}$$

appear but these are finite as $d \to 3$ because in an expansion around $p=0$, the first term is zero by symmetry and the rest are finite. Thus the superficially divergent diagram is actually UV finite.

(iii) Terms of the form F^{2n}, $n \geqslant 3$, in the 1PI action. Since the $n=2$ diagrams in case B(ii) were only (superficially) logarithmically divergent, the $n \geqslant 3$ diagrams are therefore convergent by the effective power counting rules discussed above.

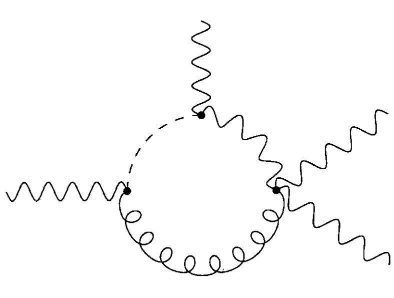

FIG. 8. A superficially UV-divergent diagram not contained in the one-loop correction to $\Gamma^2_{\mu\nu}$. It is UV finite.

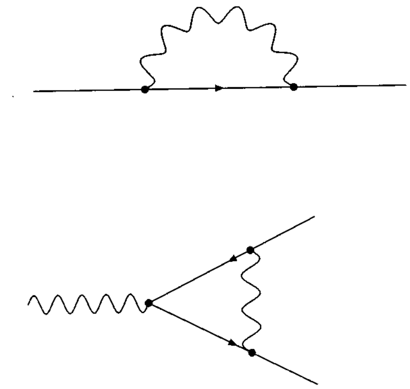

FIG. 9. Fermion self-energy and vertex-correction diagrams.

C. Case C: diagrams with Dirac fermion lines

Since the fermionic part of the action (2.24) looks like QED, all the *one-loop* diagrams are the same as there. As in QED, there is only one UV-divergent diagram with a closed fermion loop, Fig. 4. However, unlike QED, the fermion self-energy diagram and the vertex diagram in Fig. 9 are UV finite for $d=3$ because the a_μ propagator is more damped [see Eq. (3.23)] than the photon in normal QED. Thus the vertex renormalization factor Z_v and the fermion wave function renormalization factor Z_ψ corresponding to Fig. 9 equal identity.

VI. CONCLUSION

We have studied in this paper the quantization of Kac-Moody gauge-field theory introduced in Ref. [2]. Although such a theory had been written in [2] for the case of the Kac-Moody group $\hat{G}$ with G non-Abelian, we considered here the Abelian limit $G=\mathrm{U}(1)$ which still retained the interesting features of nonlinerality, nonlocality, and spatial asymmetry. The last two features are a result of interpreting the internal Kac-Moody space as a compact spatial coordinate.

We quantized the pure $\hat{\mathrm{U}}(1)$ Kac-Moody gauge theory (2.19) using the path-integral formulation in a background field and showed that in $3+1$ dimensions it was one loop renormalizable. Indeed, at one-loop, we found that because of the unusual propagators for the a_μ and a_5 fields, the theory was UV finite within dimensional regularization for a wide choice of the functions h_n and f_n appearing in the classical theory. However, for h_n and f_n behaving like $|n|$ for large n, we found that the dimensionless coupling λ_r was asymptotically free. When Dirac fermions (2.24) were added, the theory gained another coupling constant, e. At first sight, renormalizability and gauge invariance now seemed to demand that the functions h_n and f_n took the form (4.32) with $c=0$. However, by keeping c nonzero, we obtained a more interesting structure for the β functions: This could be done by promoting c from a fixed quantity to a masslike parameter which was allowed to be renormalized according to Eq. (4.33). In this way the constraints of gauge invariance and renormalizability could be satisfied and the beta function β_λ obtained two terms of opposing sign (4.39). The couplings e_r and $\overline{\lambda_r}$ could be chosen in the perturbative regime to make β_λ vanish.

If the theory is to be regarded as a fundamental theory, then its renormalizability should in the future be studied to all orders. It would also be interesting to quantize the theory by including the Kac-Moody fermions (2.26). We only note here from the actions (2.19), (2.24), and (2.26) that the partition function for the full theory comprising gauge fields and Kac-Moody fermions has the intriguing property of apparently being sensitive to the sign of the parameter κ appearing in the Kac-Moody algebra (2.1).

One might wonder about the theory in other dimensions, $d+1\neq 4$. We found that already at one loop the theory (3.16) was not renormalizable for $d=4$: For example, a divergence of the form $p^4R^2/(d-4)$ appeared in the self-energy of the A_5 field and such a term could not be absorbed by wave function renormalization. For $d=1$ and $d=2$ the pure gauge theory is free.

Instead of treating the $(3+1)$-dimensional theory as fundamental, it might be profitable to treat it as an effective theory in the sense of Wilson. Since the Lagrangian (2.19) is naturally asymmetric, one is tempted then to associate it with a description of high-temperature superconductors (HTS's), which are anisotropic systems. (At nonzero temperature the Kac-Moody gauge theory would be defined on $\mathcal{S}^1\times\mathcal{R}_{d-1}\times\mathcal{S}^1$.) It has been argued [14] that the non-Fermi-liquid behavior of the normal state of HTS's requires for its explanation a planar dynamical gauge field in the effective low-energy theory. This role in the Lagrangian (2.19) could be played by $B_\mu(x)$, while $A_\mu(x,\sigma)$, $A_5(x,\sigma)$, and the fermions would describe some other fields (perhaps the effective electromagnetic fields and fermionic quasiparticles, respectively). Of course detailed calculations are required to see if our proposal of Eq. (2.19) has indeed the required properties to describe the properties of HTS in the normal state, and if it has a phase transition at nonzero temperature to a superconducting state. In any case, regardless of a particular application, it would be interesting to study how the infrared dynamics of Eq. (2.19) depends on the anisotropy scale R, and what physical consequences issue from the Kac-Moody symmetry of the system.

ACKNOWLEDGMENTS

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APPENDIX A: WARD IDENTITIES

1. Renormalization constants

The main advantage of the background gauge is that the quantum 1PI effective action has the same gauge symmetry as the classical action [7]. One consequence of this is the simple relation among the renormalization constants derived in Sec. IV by considering the renormalization of gauge-invariant tensors. Here we show how those relations also follow consistently from the renormalization of the gauge-transformation rules (2.5), (2.6), and (2.14). Recall that by definition, under the rescalings (4.4)–(4.6), the bare tensor $\Gamma_{\mu\nu}$ becomes proportional to the renormalized tensor $\Gamma^r_{\mu\nu}$ which is of the same form as the bare tensor but now written in terms of renormalized fields and couplings. Therefore gauge invariance will be maintained only if the gauge variations (2.5), (2.6) and (2.14) for the bare fields also transform, under the same rescalings, into identical relations for the renormalized fields.

This alternative approach to the Ward identities (4.12) and (4.13) to be discussed here is instructive because it highlights an interesting feature of the theory. Notice that the transformations (2.5), (2.6), and (2.14) depend on various combinations of two couplings (e and g) and the parameter κ, but in the Lagrangian (2.19) these combine into a single effective coupling

$$\lambda=\kappa e^2/g. \tag{A1}$$

For the pure gauge theory we could of course scale our fields so that only one parameter appears in both the gauge transformations and the Lagrangian. However this would not be possible when matter is coupled so let us examine Eqs. (2.5), (2.6), and (2.14) as they are under the renormalizations (4.4)–(4.6), and also define the separate renormalizations

$$e=Z_e e_r, \tag{A2}$$

$$g=Z_g g_r. \tag{A3}$$

[In Eqs. (A2) and (A3) we have not extracted the factors of μ^ϵ and μ_0^ϵ so the couplings are dimensionful.] Then from Eq. (4.7) we have

$$Z_\lambda\equiv\frac{Z_e^2}{Z_g}. \tag{A4}$$

Now Eq. (2.14) remains form invariant; that is, it goes into

$$\delta A_5^r(x,\sigma)=-\frac{1}{e_r}\phi_r'(x,\sigma), \tag{A5}$$

under renormalization, only if

$$Z_\phi=Z_e\sqrt{Z_5}, \tag{A6}$$

where $\phi=Z_\phi\phi_r$. Then from Eq. (2.5) we obtain

$$\delta A_\mu^r(x,\sigma)=-\frac{1}{e_r}\sqrt{\frac{Z_5}{Z_A}}\partial_\mu\phi_r(x,\sigma), \tag{A7}$$

which implies

$$Z_5=Z_A \tag{A8}$$

for form invaraince to hold. Thus we have rederived the first identity (4.12) of Sec. IV. Next, defining $\Lambda=Z_\Lambda\Lambda_r$, the gauge transformation in Eq. (2.6) renormalizes to

$$Z_g\sqrt{Z_B}\delta B^r_\mu(x) = -\frac{Z_\Lambda}{g_r}\partial_\mu \Lambda_r(x) + Z_e Z_\phi \sqrt{Z_A}\frac{\kappa e_r}{g_r}\int \phi'_r(x,\sigma) A^r_\mu(x,\sigma) - Z^2_\phi \frac{\kappa}{2g_r}\int \phi'_r(x,\sigma)\partial_\mu \phi_r(x,\sigma). \quad \text{(A9)}$$

To maintain form (gauge) invariance we must choose $Z_\Lambda = Z_g\sqrt{Z_B}$. Then Eq. (A9) goes into the renormalized version of the bare transformation only if in addition two equalities are satisfied:

$$Z_g\sqrt{Z_B} = Z_e Z_\phi \sqrt{Z_A} = Z^2_\phi. \quad \text{(A10)}$$

Using Eqs. (A6) and (A8) these last constraints yield, for Z_λ in Eq. (A4),

$$Z_\lambda \equiv \frac{Z_e^2}{Z_g} \quad \text{(A11)}$$

$$= \sqrt{\frac{Z_B}{Z_5}}, \quad \text{(A12)}$$

which is precisely the second Ward identity (4.13) of Sec. IV.

When Dirac fermions are coupled to the theory we have the Ward identity (4.36),

$$Z_e = \frac{1}{\sqrt{Z_A}} = \frac{1}{\sqrt{Z_5}}, \quad \text{(A13)}$$

which imposes a constraint on the above analysis carried out for the pure gauge theory. Indeed Eqs. (A6) and (A13) imply

$$Z_\phi = 1. \quad \text{(A14)}$$

This is exactly what is required for the bare gauge transformation law for the fermions

$$\delta\psi = (e^{-i\phi} - 1)\psi \quad \text{(A15)}$$

to go into a renormalized transformation of the same form:

$$\delta\psi_r = (e^{-i\phi_r} - 1)\psi_r, \quad \text{(A16)}$$

with $\psi = Z_\psi \psi_r$. Alternatively, continuing in the spirit of the discussion prior to this paragraph, demanding that Eq. (A16) hold gives Eq. (A14) which in turn implies the Ward identity (A13) when combined with Eqs. (A6) and (A8).

2. Self-energies

Let $\mathcal{A}$ denote the 1PI effective action of the bare but regularized theory in the background gauge. The manifest symmetry of this action means $\delta^B \mathcal{A} = 0$ where the gauge variation is given in Eqs. (3.7)–(3.9). For infinitesimal gauge transformations we obtain

$$\int d^d x\, d\sigma \left[-\frac{1}{e}[\partial_\mu \phi(x,\sigma)]\frac{\delta \mathcal{A}}{\delta A_\mu(x,\sigma)} - \frac{1}{e}\phi'(x,\sigma)\frac{\delta \mathcal{A}}{\delta A_5(x,\sigma)}\right] + \int d^d x \left(-\frac{1}{g}\partial_\mu \Lambda(x) + \lambda \int \phi'(x,\sigma) A_\mu(x,\sigma)\right)\frac{\delta \mathcal{A}}{\delta B_\mu(x)} = 0. \quad \text{(A17)}$$

Taking appropriate derivatives of this equation generates various Ward identities. We will derive those for the two-point functions of the theory. Put $\phi = 0$, apply $\delta/\delta B_\nu$ to Eq. (A17) and then set the fields to zero to obtain

$$\int d^d x\, \partial_\mu \Lambda(x) \frac{\delta^2 \mathcal{A}}{\delta B_\mu(x)\,\delta B_\nu(y)}\bigg|_{A_5 = A_\mu = B_\mu = 0} = 0. \quad \text{(A18)}$$

Fourier transforming to momentum space and introducing the 1PI self-energy for the B field $\Pi^{BB}_{\mu\nu}(p)$, we obtain

$$p_\mu[(p^2\delta_{\mu\nu} - p_\mu p_\nu) + \Pi^{BB}_{\mu\nu}(p)] = 0, \quad \text{(A19)}$$

which gives the QED-like transversality condition

$$p_\mu \Pi^{BB}_{\mu\nu}(p) = 0. \quad \text{(A20)}$$

Note that in Eq. (A19) the tree-level tensor is manifestly transverse because the background field is *not* gauge fixed. The identity (A20) is verified at one loop in Appendix B.

Similarly setting $\Lambda = 0$ in Eq. (A17) and applying either $\delta/\delta A_5$ or $\delta/\delta A\nu$ to Eq. (A17) gives, respectively,

$$p_\mu[f_n p_\mu p_n + \Pi^A_{\mu 5}(P)] + p_n[(p^2 f_n + \Pi_{55}(P)] = 0 \quad \text{(A21)}$$

and

$$p_\mu[f_n p_n^2 \delta_{\mu\nu} + \Pi^{AA}_{\mu\nu}(P)] + p_n[f_n p_n p_\nu + \Pi^A_{5\nu}(P)] = 0, \quad \text{(A22)}$$

where the notation is fairly obvious. These last two Ward identities for the self-energies of the A_μ and A_5 fields are more complicated than that for the B field because of the anisotropic appearance of the former fields in the action. By taking either the $p_\mu = 0$ or $p_n = 0$ limit of the relations (A21) and (A22) we obtain the more useful identities

$$\Pi_{55}(p = 0, p_n \neq 0) = 0, \quad \text{(A23)}$$

$$\Pi^A_{5\nu}(p = 0, p_n \neq 0) = 0, \quad \text{(A24)}$$

$$p_\mu \Pi^{AA}_{\mu\nu}(p, p_n = 0) = 0, \quad \text{(A25)}$$

$$p_\mu \Pi^A_{\mu 5}(p, p_n = 0) = 0. \quad \text{(A26)}$$

The identity (A23) is tested to one-loop order in Appendix B. The last condition (A26) can be simplified by using the d-dimensional Lorentz invariance to give

$$\Pi^A_{\mu 5}(p \neq 0, p_n = 0) = 0, \quad \text{(A27)}$$

which is complementary to Eq. (A24).

APPENDIX B: THE EFFECTIVE ACTION

1. General

Functionally, the 1PI effective action $\mathcal{A}(A_\mu,B_\mu,A_5)$ at one loop is determined from the generating functional (3.19) evaluated at zero external sources [7,8] and using Eqs. (3.20) and (3.21),

$$e^{-\mathcal{A}(C)}=Z_2[C]\equiv e^{-S_c}\int[DQ]e^{-S_2(C,Q)}, \tag{B1}$$

where we have symbolically denoted all the quantum fields by Q and the classical fields by C. Since S_2 is quadratic in the quantum fields, the path integral may be evaluated formally to give a determinant and the correction to S_c is then given by the well-known expression $\ln\mathrm{Det}[G(C)]=\mathrm{Tr}\ln[G(C)]$ with $G(C)$ the propagator for the Q field in the C background. Particular one-loop 1PI N-point Green's functions can be obtained by expanding the Trln perturbatively.

We have chosen another standard way to obtain specific contributions to $\mathcal{A}$. Introduce sources j (one for each quantum field) in $Z_2[C]$,

$$Z_2[C,j]=e^{-S_c}\int[DQ]\exp\left(-S_2(C,Q)+\int jQ\right) \tag{B2}$$

$$=e^{-S_c}\exp\left[-\int\mathcal{L}_I\left(C,\frac{\delta}{\delta j}\right)\right]Z_0[j] \tag{B3}$$

$$=e^{-S_c}\left\{1-\int\mathcal{L}_I\left(C,\frac{\delta}{\delta j}\right)+\cdots\right\}Z_0[j], \tag{B4}$$

where $\mathcal{L}_I(C,Q)$ is the interaction part of the Lagrangian in Eq. (4.25), and, explicitly,

$$\begin{aligned}\ln Z_0[j]=\frac{1}{2}\int d^dx\,d^dy\Big\{\int d\sigma d\sigma'[J_\mu(x,\sigma)D_{\mu\nu}(x,\sigma;y,\sigma')\\ \times J_\nu(y,\sigma')+L(x,\sigma)S(x,\sigma;y,\sigma')L(y,\sigma')]\\ +K_\mu(x)\Delta_{\mu\nu}(x;y)K_\nu(y)\Big\}\end{aligned} \tag{B5}$$

or, in momentum space,

$$\begin{aligned}\ln Z_0[j]=\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\Big\{\frac{1}{R}\sum_n[J_\mu(-P)D_{\mu\nu}(P)J_\nu(P)\\ +L(-P)S(P)L(P)]+K_\mu(-p)\Delta_{\mu\nu}(p)K_\nu(p)\Big\},\end{aligned} \tag{B6}$$

where the propagators are those defined in Eq. (3.22)–(3.24).

From Eqs. (B4) and (B6) one calculates straightforwardly perturbative contributions to $Z_2[C]=Z_2[C,j=0]$ and hence to S_c. (Those readers conversant with Wick's theorem may obtain the same results as below almost by inspection.)

2. Z_B

Consider now the the one-loop correction W_1 to the piece $-\frac{1}{4}\int d^dx\Gamma^2_{\mu\nu}(x)$ in the classical part $(-S_c)$ of Eq. (B4). Noting the appropriate vertices in Eq. (4.25), one obtains, after some algebra,

$$W_1=\left(-\frac{1}{4}\right)^2\frac{4\lambda^2\mu_0^{2\epsilon}}{2}\int\frac{d^dp}{(2\pi)^d}\Gamma_{\mu\nu}(p)\Gamma_{\sigma\tau}(-p)\{\star\}, \tag{B7}$$

where we have defined

$$\begin{aligned}\{\star\}\equiv\int\frac{d^dk}{(2\pi)^d}\sum_n[-2k_n^2D_{\sigma\nu}(P-K)D_{\tau\mu}(K)\\ +4k_\mu k_\sigma S(P-K)D_{\tau\nu}(K)],\end{aligned} \tag{B8}$$

and where in the last line $k_n=2\pi n/R$ and $p_n=0$. The two diagrams for the B_μ-field self-energy shown in Fig. 2 correspond to the two terms in Eq. (B8). We are interested in the UV divergent part of W_1 as it will determine the renormalization constant Z_B discussed in Sec. IV. Thus one can perform a Taylor expansion about $p=0$ of the expression (B8). By power counting one sees that only the leading $(p=0)$ term gives a potentially divergent contribution. Dropping also some other terms which vanish when Lorentz indices are contracted in Eq. (B7), we obtain

$$\begin{aligned}\{\star\}_{p=0}=\int\frac{d^dk}{(2\pi)^d}\sum_n\Bigg\{\frac{4\delta_{\tau\nu}k_\mu k_\sigma}{f_nk^2(f_nk_n^2+h_nk^2)}\\ +\frac{-2k_n^2(\delta_{\tau\mu}\delta_{\sigma\nu}-2\delta_{\tau\mu}k_\sigma k_\nu/k^2)}{(f_nk_n^2+h_nk^2)^2}\Bigg\}.\end{aligned} \tag{B9}$$

Simplifying the tensor structure above and performing a change of variables gives

$$W_1=\mu_0^{2\epsilon}\frac{\lambda^2}{8}\int\frac{d^dp}{(2\pi)^d}\Gamma_{\mu\nu}(p)\Gamma_{\mu\nu}(-p)\{\star\star\}+\text{finite terms}, \tag{B10}$$

where

$$\{\star\star\}=\frac{4}{d}\sum_n\frac{1}{f_nh_n}\left(\frac{f_n}{h_n}\right)^{(d-2)/2}(k_n)^{d-2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{1+q^2} \tag{B11}$$

$$+2\left(1-\frac{2}{d}\right)\sum_n\frac{1}{h_n^2}\left(\frac{f_n}{h_n}\right)^{(d-4)/2}(k_n)^{d-2}\times\int\frac{d^dq}{(2\pi)^d}\frac{1}{(1+q^2)^2}. \tag{B12}$$

Remarkably, although the functions f_n and h_n appear in very different ways in the propagators in Eq. (B8), and so contribute differently in general to the two diagrams in Fig. 2, their contribution to the two terms in the UV-divergent part of W_1 is identical. Furthermore, in dimensional regularization one establishes through an integration by parts the identity

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(1+q^2)^2} = \left(1 - \frac{d}{2}\right) \int \frac{d^d q}{(2\pi)^d} \frac{1}{1+q^2}, \tag{B13}$$

and so one has the compact expression

$$\{\star\star\} = (4-d)\left(\frac{2\pi}{R}\right)^{d-2} \sum_n \frac{|n|^{d-2}}{f_n^2}\left(\frac{f_n}{h_n}\right)^{d/2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{1+q^2} \tag{B14}$$

$$= -(4-d) \times \left(\frac{2\pi}{R}\right)^{d-2} \sum_n \frac{|n|^{d-2}}{f_n^2}\left(\frac{f_n}{h_n}\right)^{d/2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(1+q^2)}. \tag{B15}$$

We have used Eq. (4.29) in the last step. Hence

$$W_1 = -4\mathcal{T}\left(-\frac{1}{4}\int d^d x \Gamma_{\mu\nu}^2\right), \tag{B16}$$

with $\mathcal{T}$ defined in Eq. (4.16), and finally

$$\Delta\Gamma_{\mu\nu}^2(x) = -4\mathcal{T}\Gamma_{\mu\nu}^2(x), \tag{B17}$$

as stated in Sec. IV B.

We now illustrate how the power-counting rules discussed in Sec. V indicate a UV logarithmic divergence in the wave function renormalization of the B_μ field for the *ansatz* (4.18) and (4.19). Consider the first diagram of Fig. 2: From Eq. (4.25) one sees that each vertex in that diagram comes from $\lambda \partial_\mu B_\nu \int a'_\mu a_\nu$. Therefore the UV degree of divergence of that diagram (with external ∂B legs) is

$$\sum_n \int \frac{d^3 q n^2}{(nq^2)(nq^2)} \sim \int^\Lambda \frac{d^4 q q^2}{q^3 q^3} \sim \ln\Lambda. \tag{B18}$$

In the second diagram of Fig. 2 each vertex comes from the term $\lambda \partial_\mu B_\nu \int \widetilde{F}_{\mu\nu} a_5$ in Eq. (4.25). Thus the UV degree of divergence is now

$$\sum_n \int \frac{d^3 q q^2}{(nq^2)(nq^2)} \sim \int^\Lambda \frac{d^4 q q^2}{q^3 q^3} \sim \ln\Lambda. \tag{B19}$$

These results agree with the explicit analysis in Sec. IV B.

By taking functional derivatives with respect to B_α and B_β, one can obtain from W_1 the one-loop self-energy correction $\Pi_{\alpha\beta}^{BB}(p)$ to the B field. This self-energy is automatically transverse because of the manifest gauge symmetry of the expression for W_1. Indeed one obtains, from Eq. (B7),

$$\Pi_{\alpha\beta}^{BB} \sim (p_\mu \delta_{\nu\alpha} - p_\nu \delta_{\mu\alpha})(p_\sigma \delta_{\tau\beta} - p_\tau \delta_{\sigma\beta}) \times (\text{rest}), \tag{B20}$$

so that

$$p_\alpha \Pi_{\alpha\beta}^{BB} = p_\beta \Pi_{\alpha\beta}^{BB} = 0, \tag{B21}$$

as required by the Ward identity (A20).

3. Z_5

Here we summarize the one-loop contribution to the 1PI effective action from terms quadratic in A_5. There are two pieces and these correspond to the two diagrams shown in Fig. 3. Using the functional method described above, the tadpole contribution to $(-S_c)$ is obtained as

$$\frac{-2\lambda^2 \mu_0^{2\epsilon}}{4} \int \frac{d^d p}{(2\pi)^d} \frac{1}{R} \sum_{p_n} A_5(P) A_5(-P) \{@\}, \tag{B22}$$

where

$$\{@\} \equiv \int \frac{d^d q}{(2\pi)^d} q^2 D_{\nu\nu}(Q)\big|_{q_n = p_n} \tag{B23}$$

$$= \frac{(d-1)}{4} \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{h_n q^2 + f_n p_n^2} \tag{B24}$$

$$= \frac{(d-1)}{4h_n} \int \frac{d^d q}{(2\pi)^d} \frac{-f_n p_n^2}{h_n q^2 + f_n p_n^2} \tag{B25}$$

is finite as $d \to 3$ because of Eq. (4.29). Hence the tadpole diagram does not contribute to Z_5. Similarly the other contribution to $(-S_c)$ is determined to be

$$\left(\frac{1}{4}\right)^2 \frac{\lambda^2 \mu_0^{2\epsilon}}{2} 4^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{R} \sum_{p_n} A_5(P) A_5(-P) \{@@\}, \tag{B26}$$

where

$$\{@@\} \equiv \int \frac{d^d q}{(2\pi)^d} \{q_\mu q_\tau \Delta_{\nu\sigma}(q)(p+q)_\sigma (p+q)_\nu D_{\tau\mu} \times (P+Q) + q_\mu q_\sigma \Delta_{\nu\tau}(q)(p+q)_\sigma (p+q)_\mu \times D_{\tau\nu}(P+Q)\} \cdot \delta_{q_n,0} \tag{B27}$$

is easily shown by algebraic manipulations such as Eq. (4.29) to be finite near $d=3$. So neither does this piece contribute to Z_5. Note also that the tadpole contribution to the A_5 self-energy is independent of p_μ and actually cancels the $p_\mu = 0$ part of the other self-energy diagram. This vanishing of the A_5 self-energy at zero momentum p_μ is in accordance with the Ward identity (A23) (note that in this example the *individual* contributions do not satisfy the Ward identity but their sum does as required).

In passing we note that for $d \to 4$, the nontadpole diagram contribution to the self-energy of the A_5 field contains a divergence proportional to p^2 which requires wave function renormalization, but there is also a divergence proportional to p^4 from Eq. (B27) which implies the nonrenormalizability of the theory in $4+1$ dimensions.

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