

Duality between electric and magnetic black holes

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A number of attempts have recently been made to extend the conjectured S duality of Yang-Mills theory to gravity. Central to these speculations has been the belief that electrically and magnetically charged black holes, the solitons of quantum gravity, have identical quantum properties. This is not obvious, because, although duality is a symmetry of the classical equations of motion, it changes the sign of the Maxwell action. Nevertheless, we show that the chemical potential and charge projection that one has to introduce for electric but not magnetic black holes exactly compensate for the difference in action in the semiclassical approximation. In particular, we show that the pair production of electric black holes is not a runaway process, as one might think if one just went by the action of the relevant instanton. We also comment on the definition of the entropy in cosmological situations, and show that we need to be more careful when defining the entropy than we are in an asymptotically flat case.

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I. INTRODUCTION

The idea of duality has received considerable attention recently, particularly in the context of string theory. This is a subject with a long history, which may be traced back to Olive and Montonen's conjectured duality in Yang-Mills theory [1]. In the $N = 4$ Yang-Mills theory, one has two kinds of particles: small fluctuations in the scalar or Yang-Mills fields, and magnetic monopoles. The small fluctuations couple to the Yang-Mills field like electrically charged particles couple to the Maxwell field. They are therefore regarded as electrically charged elementary states. But the magnetic monopoles, which are the solitons of the theory, can also claim to be regarded as particles. Olive and Montonen conjectured [1] that there was a dual Yang-Mills theory, with coupling constant $g' = 1/g$. Monopoles in the dual theory would behave like the elementary electrically charged states of the original theory, and vice versa. This concept of duality was later extended to a lattice of theories related by the discrete group $SL(2, Z)$. There is some evidence that the low energy scattering of monopoles is consistent with what one would expect from this duality, which is called S duality, but no proof has been given that it goes beyond a symmetry of the equations of motion to a symmetry of the full quantum theory.

Despite this lack of proof, there has been extensive speculation on how S duality could extend to gravity and string-inspired supergravity theories [2]. The suggestion is that extreme, nonrotating black holes should be identified as the solitons of the theory. These states do

have some particlelike properties, as there are families of electric and magnetic black holes, which fall into multiplets under the action of the global supersymmetry group at infinity. The similarity with other solitons has been increased by our recent discovery [3] that all extreme black holes have zero entropy, as one would expect for elementary particles. However, the original Montonen-Olive idea of duality was supposed to relate electrically charged elementary states, or small fluctuations in the fields, with magnetically charged monopoles, or solitons. But in the gravitational case there are both magnetically and electrically charged solitons. This has led people to try to identify extreme black holes, the solitons, with electrically or magnetically charged elementary states in string theory [4,5]. The only evidence so far is that one can find black holes with the same masses and charges as a certain class of elementary states [5]. But this is not very surprising, because the masses are determined by the charges and Bogomol'nyi bounds in both cases.

Behind all these attempts to extend S duality to extreme black holes is the idea that electrically and magnetically charged black holes behave in a similar way. This is true in the classical theory, because duality between electric and magnetic fields is a symmetry of the equations. This does not, however, imply that it is a symmetry of the quantum theory, as the action is not invariant under duality. The Maxwell action is $F^2 = B^2 - E^2$, and it therefore changes sign when magnetic fields are replaced by electric. The purpose of this paper is to show that despite this difference in the action, the semiclassical approximations to the Euclidean path integral for dual electric and magnetic solutions are identical, at least where we have been able to evaluate them. In particular, we show that the rate at which black holes are pair created in cosmological and electromagnetic backgrounds is duality invariant.

We will now define our terms more precisely. It is well known that the Einstein-Maxwell equations exhibit

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duality. One can replace magnetic fields with electric fields and a solution remains a solution. More precisely, if (g, F) are a metric and field tensor that satisfy the field equations, then $(g, *F)$ also satisfy the equations, where $*F$ is the Lorentzian dual of F , that is,

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \quad (1)$$

with $\epsilon_{0123} = \sqrt{-g}$, and g the determinant of the metric. If F represents a magnetic field, $*F$ will represent an electric field, referred to as the dual electric field. In particular, for every magnetically charged black hole solution, there is a corresponding electrically charged black hole solution. This electric-magnetic duality extends to theories with a dilaton. The only difference is that one now takes

$$*F_{\mu\nu} = \frac{1}{2}e^{-2\alpha\phi}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \quad (2)$$

and

$$\phi \rightarrow -\phi, \quad (3)$$

where ϕ is the dilaton field. We will, however, restrict attention to the duality (1) in Einstein-Maxwell theory for the sake of simplicity.

Now, if $g_{\mu\nu}$ is a Lorentzian metric, its determinant will be negative, so $\epsilon_{\mu\nu\rho\sigma}$ as defined above will be real. However, if $g_{\mu\nu}$ is a Euclidean metric, its determinant will be positive, and thus $\epsilon_{\mu\nu\rho\sigma}$ will be imaginary. That is, the Lorentzian duality (1) takes real magnetic fields to real electric fields in Lorentzian space, but real magnetic fields to imaginary electric fields in Euclidean space. This is consistent, as an electric field that is real in a Lorentzian space is imaginary in its Euclidean continuation. One might therefore think that, in using the Euclidean path integral, one should use Euclidean duality instead of Lorentzian duality, and replace magnetic fields with electric fields that were real in Euclidean space. That is, perhaps one should take

$$*F_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \quad (4)$$

instead of (1). This duality also has the advantage that it leaves the Maxwell action unchanged. However, it reverses the sign of the energy momentum tensor, so the solutions would have different geometry. That is, if $*F$ is given by (4), then it is no longer true that $(g, *F)$ satisfy the field equations whenever (g, F) do. In particular, there is no extreme black hole solution with real electric fields in Euclidean space. It seems therefore that if duality is to be a symmetry of black holes, it must be a duality between real electric and magnetic fields in Lorentzian space, rather than in Euclidean space.

There is then a difference in action between the dual electric and magnetic solutions. What effect will this have? One of the most interesting applications of the Euclidean path integral approach is the study of semiclassical instabilities, or tunneling processes. One uses instantons, Euclidean solutions of the field equations, to

estimate the rate at which such classically forbidden tunneling processes occur. The rate at which a process occurs is just given by the partition function Z , defined by

$$Z = \int d[g]d[A]e^{-I}, \quad (5)$$

where the integral is subject to some appropriate boundary conditions at infinity. When there is a Euclidean solution which satisfies the boundary conditions, we can approximate the integral by the saddle point, which gives $Z \approx e^{-I}$, where I is the action of the instanton, so it would seem that the difference in action between dual solutions must surely imply a difference in the rate for such processes. In particular, the Euclidean black hole solutions can be used as instantons for black hole nucleation or pair creation, and we might therefore think that electrically and magnetically charged black holes should be produced at different rates. However, a more careful analysis of the partition function shows that this is not the case.

The point is that magnetic and electric solutions differ not only in their actions, but in the nature of the boundary conditions we can impose on them. If we consider a single black hole, we can choose a particular charge sector in the magnetic case, but we have to introduce a chemical potential for the charge in the electric case. That is to say, we can impose the magnetic charge as a boundary condition at infinity, but we can only impose the chemical potential, and not the electric charge, as a boundary condition in the electric case. Thus the partition function in the magnetic case is a function of the temperature and charge, $Z(\beta, Q)$, while in the electric case the partition function is a function of the chemical potential ω , rather than Q , $Z(\beta, \omega)$. It is not surprising to find that these two quantities differ. What we need to do is obtain a partition function $Z(\beta, Q)$ in the electric case. To do this, we must introduce a charge projection operator [6].

The introduction of the charge projection operator is like performing a Fourier transform on the wave function, to trade ω for its canonically conjugate momentum Q . The effect of this transform is to make the partition function as a function of charge the same for the electrically and magnetically charged black holes. The difference in action precisely cancels the additional term introduced in the partition function by the Fourier transform.

We can also calculate $Z(\beta, Q)$ in the electric case directly, by using (5) with an action which is adapted to holding the electric charge fixed. To make the action give the classical equations of motion under a variation which holds the electric charge on the boundary fixed, we need to include an additional surface term in the action. This will make the action of dual electric and magnetic solutions identical.

We are particularly interested in instantons describing black hole pair creation. To obtain pair creation of black holes, one has to have some force that is pulling the holes apart. The case that has been extensively studied is the formation of charged black holes in a background elec-

tric or magnetic field [3,7–10]. Here the negative electromagnetic potential energy of the holes in the background electric or magnetic field can compensate for the positive rest mass energy of the black holes. The pair creation of magnetically charged black holes in a background magnetic field has been the subject of most work in this area, and the action and pair creation rate for this case have been calculated in [7,8]. It was assumed in earlier work that the treatment of the electric case was a trivial extension of the magnetic; we now realize that this is not in fact the case. We consider the pair creation of electric black holes in a background electric field, and show by calculating $Z(\beta, Q)$ directly that the pair creation rate in this case is the same as in the magnetic case.

The effective cosmological constant in the inflationary period of the Universe can also accelerate objects away from each other, and so it should be possible to find instantons describing the pair production of black holes in a cosmological background. In the case without gauge fields, the relevant solution is the Schwarzschild–de Sitter metric. This has been interpreted in the past as a single black hole in a de Sitter universe, but it really represents a pair of black holes at antipodal points on the three sphere space section of the de Sitter universe, accelerating away from each other. If one takes $t = i\tau$, one obtains a Euclidean metric. One can remove the conical singularities in this metric if the black hole and cosmological horizons have the same temperature. For the Schwarzschild–de Sitter metric, this occurs in the limiting case known as the Nariai metric, which is just the analytical continuation of $S^2 \times S^2$, with both spheres having the same radius [11].

If you cut this solution in half, you get the amplitude to propagate from nothing to a three-surface Σ with topology $S^2 \times S^1$ according to the no boundary proposal. One can regard $S^2 \times S^1$ as corresponding to the space section of the Nariai universe, which will settle down to two black holes in de Sitter space (see [11] for more details). The action of $S^2 \times S^2$ is $I = -2\pi/\Lambda$. This is greater than the action $I = -3\pi/\Lambda$ of S^4 , which corresponds to de Sitter space. Thus the amplitude to pair create neutral black holes in de Sitter space is suppressed, as one would hope.¹

One can also consider the pair creation of electrically or magnetically charged black holes in de Sitter space. Here the relevant solutions are the Reissner–Nordström–de Sitter metrics, which can again be extended to Euclidean metrics. More than one instanton can be constructed in this case; these instantons are discussed in more detail in [14–16]. We will consider only the simplest case, where the instanton is again $S^2 \times S^2$, but

where the spheres now have different radii. The action for the magnetic instanton is less negative than the neutral case. Thus the pair creation of magnetic black holes is suppressed relative to neutral black holes, which is itself suppressed relative to the background de Sitter space. All this is what one might expect on physical grounds. But in the electric case, the action is less than the action of the neutral case, and can be less than the action of the background de Sitter space if the electric charge is large enough. This at first seemed to suggest that de Sitter space would be unstable to decay by pair production of electrically charged black holes.

Presumably, we have to apply a charge projection operator to obtain comparable partition functions here, as in the single black hole case. However, the $S^2 \times S^2$ instanton has no boundary, so we at first thought that it was not possible to have a chemical potential in this case. However, as we said above, what we actually want to consider is the amplitude to propagate from nothing to a three-surface Σ with topology $S^2 \times S^1$, and we can impose the potential on the boundary Σ . The instanton giving the semiclassical approximation to this amplitude is just half of $S^2 \times S^2$. In the magnetic case, the magnetic charge can be given as a boundary condition on this surface, but in the electric case, the boundary gives only the potential ω . If we again make the Fourier transform to trade ω for Q , the semiclassical approximation to the wave function as a function of charge is the same for the electrically and magnetically charged black holes. Thus the pair creation of both magnetic and electric black holes is suppressed in the early Universe.

We will also discuss the entropy of black hole solutions. For the asymptotically flat black holes, the partition function $Z(\beta, Q)$ can be interpreted as the canonical partition function, while $Z(\beta, \omega)$ can be interpreted as the grand canonical partition function. Using the instantons to approximate the partition function, we can show that the entropy of the asymptotically flat black holes is $S = \mathcal{A}_{\text{BH}}/4$ for both electrically and magnetically charged black holes.

For the cosmological solutions, the square of the wave function $\Psi(Q, \pi^{ij} = 0)$ can be regarded as the density of states or microcanonical partition function. Thus the entropy is just given by the $\ln$ of the wave function. Using the instantons to approximate this density of states, we find that the entropy is $S = \mathcal{A}/4$, where $\mathcal{A}$ is the total area of all the horizons in the instanton.

In Sec. II, we review the calculation of the action for the Reissner–Nordström black holes, and the introduction of the charge projection operator. In Sec. III, we describe the Reissner–Nordström–de Sitter solution, and derive an instanton which can be interpreted as describing black hole pair production in a background de Sitter space. We then calculate its action. We go on to argue, in Sec. IV, that a charge projection can be performed in this case as well, and that the partition function as a function of the charge is the same in the electric and magnetic cases. In Sec. V, we review the electric Ernst solution, and obtain the instanton which describes pair creation of electrically charged black holes in an electric field. In Sec. VI, we calculate the action for this instanton, and

¹If one were to use the tunneling proposal [12,13] instead of the no boundary proposal, one would find that the probability of the pair creation of neutral black holes was enhanced rather than suppressed relative to the probability for the spontaneous formation of a de Sitter universe. This is further evidence against the tunneling proposal.

thus obtain the pair creation rate. In Sec. VII, we review the derivation of the entropy for the Reissner-Nordström black holes, and discuss its definition for the Reissner-Nordström-de Sitter solutions.

II. ACTION AND CHARGE PROJECTION IN REISSNER-NORDSTRÖM SPACETIME

Let us first consider asymptotically flat black hole solutions. To simplify the later calculation of the entropy, we will evaluate the action of these black holes by a Hamiltonian decomposition, following the treatment given in [17]. If there is a Maxwell or Yang-Mills field, one takes spatial components of the vector potential A_i as the canonical coordinates on three-surfaces of constant time. The conjugate momenta are the electric field components E^i . The time component A_t of the potential is regarded as a Lagrange multiplier for the Gauss law constraint $\text{div} E = 0$. Let us first assume that the manifold has topology $\Sigma \times S^1$. Then the action is

$$I = - \int dt \left[\int_{\Sigma_t} (p^{\mu\nu} \dot{g}_{\mu\nu} + E^i \dot{A}_i) - H \right]. \quad (6)$$

There is a well-known ambiguity in the gravitational action for manifolds with boundary, as one can add any function of the boundary data to the action, and its variation will still give the same equations of motion [18]. We will adopt the approach of [17], and require that the action of some suitable background vanish. We define a suitable background to be one which agrees with the solution asymptotically, that is, which induces the same metric and gauge fields on S_∞^2 . If we assume that the background is a solution of the equations of motion, the Hamiltonian H is [19]

$$\begin{aligned} H = & \frac{1}{8\pi} \int_{\Sigma_t} (N\mathcal{H} + N^i \mathcal{H}_i + N A_t \text{div} E) \\ & - \frac{1}{8\pi} \int_{S_\infty^2} [N(^2K - ^2K_0) + N^i p_{ij} \\ & + 2N A_t (E - E_0)], \end{aligned} \quad (7)$$

where 2K is the extrinsic curvature of the boundary S_∞^2 of the surface Σ_t , E is the electric field, and 2K_0 and E_0 represent these quantities evaluated in the background.

In order to get the action in this canonical form, we have had to integrate by parts the terms in the action involving spatial gradients of A_t . This produces the A_t surface term in the Hamiltonian. This surface term is zero for magnetic monopoles and magnetic black holes. It is also zero for any solution with electric fields, but no horizons, because one can choose a gauge in which A_t vanishes at infinity. Thus, the existence of this surface term in the Hamiltonian does not seem to have been generally noticed. However, it is nonzero for electrically charged black holes, because the gauge transformation required to make $A_t = 0$ at infinity is not regular on the horizon.

One can pass from a Lorentzian black hole solution to

a Euclidean one by introducing an imaginary time coordinate $\tau = -it$. One then has to identify τ with period $\beta = 2\pi/\kappa$ to make the metric regular on the horizon, where κ is the surface gravity of the horizon. One can then use the relation between the action and the Hamiltonian to calculate the action of the Euclidean black hole solution. As the solution is static, the Euclidean action (6) is β times the Hamiltonian. However, the Euclidean section for a nonextreme black hole does not have topology $\Sigma \times S^1$, and so (6) only gives the action of the region swept out by the surfaces of constant τ . This is the whole of the Euclidean solution, except for the fixed point locus of the time translation killing vector on the horizon. The contribution to the action from the corner between two surfaces τ_1 and τ_2 is

$$\frac{\kappa}{8\pi} (\tau_2 - \tau_1) \mathcal{A}_{\text{BH}}, \quad (8)$$

where $\mathcal{A}_{\text{BH}}$ is the area of the horizon. Thus the action is $I = \beta H - \mathcal{A}_{\text{BH}}/4$ [20]. For solutions of the field equations, the three-surface integral vanishes, because of the gravitational and electromagnetic constraint equations. Thus, the value of the Hamiltonian comes entirely from the surface terms.

Now we will calculate the action in this way for the nonextreme electric and magnetic Reissner-Nordström solutions. Recall that the Reissner-Nordström metric is given by

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 \\ & + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \end{aligned} \quad (9)$$

where M is the mass and Q is the charge of the black hole. The gauge potential for this solution is

$$F = Q \sin \theta d\theta \wedge d\phi \quad (10)$$

for a magnetically charged solution, and

$$F = -\frac{Q}{r^2} dt \wedge dr \quad (11)$$

for an electrically charged solution. We will not consider dyonic solutions. The metric has two horizons, at $r = r_\pm = M \pm \sqrt{M^2 - Q^2}$. We analytically continue $t \rightarrow i\tau$, and identify τ with period $\beta = 2\pi/\kappa$, where $\kappa = (r_+ - r_-)/2r_+^2$ is the surface gravity of the horizon at $r = r_+$. The surfaces of constant τ meet at the event horizon $r = r_+$, whose area is

$$\mathcal{A}_{\text{BH}} = 4\pi r_+^2 = \frac{4\pi}{\kappa} (M - QU), \quad (12)$$

where $U = Q/r_+$. The second equality is obtained by exploiting the definitions of $r_\pm$ and κ .

If we consider the magnetically charged black hole solution, the gauge potential will be

$$A = Q(1 - \cos \theta) d\phi, \quad (13)$$

where we have chosen a gauge which is regular on the

axis $\theta = 0$. For a magnetic black hole, the electromagnetic surface term in the Hamiltonian vanishes, and the Hamiltonian is just given by the gravitational surface term. However, as the background spacetime usually used to calculate the Hamiltonian for the Reissner-Nordström black holes is just periodically identified flat space, this surface term is equal to the usual Arnowitt-Deser-Misner (ADM) mass [17]. Thus the Hamiltonian is simply

$$H = M, \quad (14)$$

and if τ is identified with period $\beta = 2\pi/\kappa$, the action is

$$I = \beta M - \mathcal{A}_{\text{BH}}/4 = \frac{\pi}{\kappa}(M + QU). \quad (15)$$

For the electrically charged black hole solution, the gauge potential is

$$A = -i(Q/r - \Phi)d\tau, \quad (16)$$

where $\Phi = U$ is the potential at infinity and we have chosen a gauge which is regular on the black hole horizon. Note that this gauge potential is pure imaginary, as we have analytically continued $t \rightarrow i\tau$. We take the point of view that one should simply accept that the gauge potential in Euclidean space is imaginary; if one analytically continued the charge to obtain a real gauge potential, the metric would be changed, and one could no longer sensibly compare the electric and magnetic solutions, as they would no longer be dual solutions. In this case, the Hamiltonian is still just equal to the surface term, but now the electromagnetic surface term survives as well. The Hamiltonian can now be calculated to be

$$H = M - Q\Phi, \quad (17)$$

and we see that Φ may be interpreted as the electrostatic potential in this case. Thus, if τ is identified with period $\beta = 2\pi/\kappa$, the action is

$$I = \beta(M - Q\Phi) - \mathcal{A}_{\text{BH}}/4 = \frac{\pi}{\kappa}(M - QU), \quad (18)$$

as asserted in [21]. If we were to calculate the action directly, as was done in [21], we would find that the sign difference of the QU term in the action is due to the fact that $F^2 = 2Q^2/r^4$ for the magnetic solution, but $F^2 = -2Q^2/r^4$ for the electric solution.

As we have said in the Introduction, the naive expectation that the rate of pair creation is simply approximated by the action ignores an important difference between the electric and magnetic cases. The partition function is

$$Z = \int d[g]d[A]e^{-I[g,A]}, \quad (19)$$

where the integral is over all metrics and potentials inside a boundary Σ^∞ at infinity, which agree with the given boundary data on Σ^∞ . Now for the Euclidean black holes, the appropriate boundary is $\Sigma^\infty = S_\infty^2 \times S^1$, and the boundary data are the three-metric h_{ij} and gauge potential A_i on the boundary at infinity. In the magnetic

case, one can evaluate the magnetic charge by taking the integral of F_{ij} over the (θ, ϕ) two-sphere lying in the boundary, so the magnetic charge is a boundary condition. That is, we are evaluating the partition function in a definite charge sector. In the electric case, however, A_i is constant on the boundary, so all we can construct is an integral of it over the boundary. This is the chemical potential $\omega = \int A_\tau d\tau$, where we define this integral to be in the direction of increasing τ . That is, we are evaluating the partition function in a sector of fixed ω . This can be written in a shorthand form as $Z(\beta, \omega)$. To obtain the partition function in a sector of definite charge, we have to introduce a charge projection operator in the path integral [6]. This gives²

$$Z(\beta, Q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega Q} Z(\beta, \omega). \quad (20)$$

We can think of ω as a canonical coordinate, in which case its canonically conjugate momentum is Q , and we can think of (20) as a Fourier transform.

Clearly, what we want to compare is the semiclassical approximation to the partition functions $Z(\beta, Q)$ in the magnetic and the electric case. For the magnetic case, the magnetic Reissner-Nordström solution provides the saddle-point contribution to the path integral, so

$$\ln Z(\beta, Q) = -I = -\beta M + \mathcal{A}_{\text{BH}}/4. \quad (21)$$

In the electric case, the Fourier transform (20) can also be calculated by a saddle point approximation. At the saddle point, $\omega = i\beta\Phi$, so

$$\begin{aligned} \ln Z(\beta, Q) &= -I + i\omega Q \\ &= -\beta(M - Q\Phi) + \mathcal{A}_{\text{BH}}/4 + i\omega Q \\ &= -\beta M + \mathcal{A}_{\text{BH}}/4. \end{aligned} \quad (22)$$

Thus we see that the semiclassical approximation to the partition function is the same for dual electric and magnetic black holes.

Alternatively, it is possible to construct a partition function $Z(\beta, Q)$ for the electric case directly; that is, we can write $Z(\beta, Q)$ in a path integral form for a suitable choice of action [16]. In the path integral, we want to use the action for which it is natural to fix the boundary data on Σ specified in the path integral (19). That is, we want to use an action whose variation gives the Euclidean equations of motion when the variation fixes these boundary data on Σ [18]. If we consider the action (6), we can see that its variation will be

$$\begin{aligned} \delta I &= (\text{terms giving the equations of motion}) \\ &+ (\text{gravitational boundary terms}) \\ &+ \frac{1}{4\pi} \int_{\Sigma} d^3x \sqrt{h} F^{\mu\nu} n_\mu \delta A_\nu, \end{aligned} \quad (23)$$

²There is a sign difference between this expression and the analogous expression in [6], but this is just due to a difference of conventions.

where n_μ is the normal to Σ and h_{ij} is the induced metric on Σ (see [18] for a more detailed discussion of the gravitational boundary terms). Thus, the variation of (6) will only give the equations of motion if the variation is at fixed gauge potential on the boundary, A_i .

For the magnetic Reissner-Nordström solutions, fixing the gauge potential fixes the charge on each of the black holes, as the magnetic charge is just given by the integral of F_{ij} over a two-sphere lying in the boundary. However, in the electric case, fixing the gauge potential A_i can be regarded as fixing ω . Holding the charge fixed in the electric case is equivalent to fixing $n_\mu F^{\mu i}$ on the boundary, as the electric charge is given by the integral of the dual of F over a two-sphere lying in the boundary. Therefore, the appropriate action is

$$I_{el} = I - \frac{1}{4\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu, \quad (24)$$

as its variation is

$$\begin{aligned} \delta I_{el} = & \text{(terms giving the equations of motion)} \\ & + \text{(gravitational boundary terms)} \\ & - \frac{1}{4\pi} \int_\Sigma d^3x \delta(\sqrt{h} F^{\mu\nu} n_\mu) A_\nu, \end{aligned} \quad (25)$$

and so it gives the equations of motion when $\sqrt{h} n_\mu F^{\mu i}$, and thus the electric charge, is held fixed. That is, if we use (24) in (19) in the electric case, the partition function we obtain is $Z(\beta, Q)$.

The observation that the magnetic charge must be imposed as a boundary condition in the path integral has another, more troubling consequence. In the derivation of the action for the asymptotically flat black holes above, we have assumed that periodically identified flat space is a suitable background, so we can take 2K_0 and E_0 in (7) to be the values of these quantities in flat space. However, a suitable background is one which agrees with the solution asymptotically; that is, it must satisfy the boundary conditions in the path integral (19). In the magnetic case, periodically identified flat space cannot satisfy these boundary conditions, as it has no magnetic charge. Flat space is not a suitable background to use in the evaluation of this action. The best we can do for single black holes is to compare the action of the non-extreme black holes with the action of the extreme black hole of the same charge, as this is a suitable background. It is natural to choose the actions of the extreme black holes so that the partition functions for fixed magnetic and electric charges are equal. Such problems will not arise in the case of pair creation in a cosmological background, as the instantons are compact, so there is no need for a suitable background solution to calculate the action.

III. REISSNER-NORDSTRÖM-de SITTER INSTANTONS

We will now describe the cosmological instanton, and calculate its action. The Reissner-Nordström-de Sitter

metric describes a pair of oppositely charged black holes at antipodal points in de Sitter space, as the Euclidean section has topology $S^2 \times S^2$. The spatial sections therefore have topology $S^2 \times S^1$, which may be thought of as a Wheeler wormhole, topology $S^2 \times R^1$, attached to a spatial slice of de Sitter space, topology S^3 . The metric is

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (26)$$

where

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2. \quad (27)$$

We restrict consideration to just purely magnetically or purely electrically charged solutions. The Maxwell field for the magnetically charged solution is (10), and the Maxwell field for the electrically charged solution is (11). In general, $V(r)$ has four roots, which we will label $r_1 < r_2 \leq r_3 \leq r_4$. The two roots r_2 and r_3 are the inner and outer black hole horizons, while r_4 is the cosmological horizon. The smallest root r_1 is negative, and thus has no physical significance.

We analytically continue $t \rightarrow i\tau$ to obtain a Euclidean solution. If the analytically continued metric is to be positive definite, r must lie between r_3 and r_4 , where $V(r)$ is positive. Then to have a regular solution, the surface gravities at r_3 and r_4 must be equal, so that the potential conical singularities at these two horizons can be eliminated with a single choice of the period of τ . This can be achieved in one of three ways: either $r_3 = r_4$, $|Q| = M$, or $r_2 = r_3$ [14–16]. Let us consider in detail the case where the roots r_3 and r_4 are coincident, which is analogous to the neutral black hole instanton studied in [11]. As in [11], the proper distance between $r = r_3$ and $r = r_4$ remains finite in the limit $r_3 \rightarrow r_4$, as we can see by making a similar change of coordinates. Let us set $r_3 = \rho - \epsilon$, $r_4 = \rho + \epsilon$. Then

$$V(r) = -\frac{\Lambda}{3r^2}(r - \rho - \epsilon)(r - \rho + \epsilon)(r - r_1)(r - r_2). \quad (28)$$

If we make a coordinate transformation

$$r = \rho + \epsilon \cos \chi, \quad \psi = A\epsilon\tau, \quad (29)$$

where

$$A = \frac{\Lambda}{3\rho^2}(\rho - r_1)(\rho - r_2), \quad (30)$$

then

$$V(r) \approx A\epsilon^2 \sin^2 \chi. \quad (31)$$

Thus, in the limit $\epsilon \rightarrow 0$, the metric becomes

$$ds^2 = \frac{1}{A}(d\chi^2 + \sin^2 \chi d\psi^2) + \frac{1}{B}(d\theta^2 + \sin^2 \theta d\phi^2), \quad (32)$$

where χ and θ both run from 0 to π , and ψ and ϕ both have period 2π . This metric has been previously men-

tioned in [15]. We assume that $B = 1/\rho^2 > A$ (this corresponds to real Q , as we see below). The cosmological constant is given by $\Lambda = (A + B)/2$, and the Maxwell field is

$$F = Q \sin \theta d\theta \wedge d\phi \quad (33)$$

in the magnetically charged case, and

$$F = -iQ \frac{B}{A} \sin \chi d\chi \wedge d\psi \quad (34)$$

in the electrically charged case, where $Q^2 = (B - A)/(2B^2)$. This metric is completely regular and, as the instanton is compact, it is extremely easy to compute its action:

$$I = -\frac{1}{16\pi} \int (R - 2\Lambda - F^2) = -\frac{\Lambda V^{(4)}}{8\pi} \pm \frac{Q^2 B^2 V^{(4)}}{8\pi}, \quad (35)$$

where $V^{(4)} = 16\pi^2/(AB)$ is the four-volume of the instanton. The action for the magnetic case is thus $I = -2\pi/B$, and for the electric case the action is $I = -2\pi/A$. Since the action for the instanton describing the creation of neutral black holes is $I = -2\pi/\Lambda$ [11], we have $I_{\text{magnetic}} > I_{\text{neutral}} > I_{\text{electric}}$. Further, $I_{\text{de Sitter}} > I_{\text{electric}}$ if $A < 2\Lambda/3$. Since the action is supposed to give the approximate rate for pair creation, this seems to say that de Sitter space should be disastrously unstable to the pair creation of large electrically charged black holes.

IV. CHARGE PROJECTION FOR REISSNER-NORDSTRÖM-de SITTER INSTANTONS

Clearly, there is an analogy between this problem and the difficulty with the Reissner-Nordström solution, and so what we need to do is to introduce a charge projection operator in the path integral in the electric case. However, as the instanton is compact, it looks like we do not have any boundary to specify boundary data on, and in particular no notion of a chemical potential.

However, we are again forgetting something. The pair creation of black holes in a de Sitter background is described, by the no-boundary proposal, by the propagation from nothing to a three-surface Σ with topology $S^2 \times S^1$. This process is described by a wave function

$$\Psi = \int d[g] d[A] e^{-I}, \quad (36)$$

where the integral is over all metrics and potentials on manifolds with boundary Σ , which agree with the given boundary data on Σ . This amplitude is dominated by a contribution from a Euclidean solution which has boundary Σ and satisfies the boundary conditions there. For pair creation of black holes, the instanton is in fact *half* of $S^2 \times S^2$. In the semiclassical approximation, $\Psi \approx e^{-I}$, where I is the action of this instanton.

In the usual approach reviewed in Sec. II, we take ad-

vantage of the fact that the instanton is exactly half of the bounce, so that the tunneling rate is $\Psi^2 = Z = e^{-I_b}$, where $I_b = 2I$ is the action of the bounce. This is helpful, as this latter action is easier to calculate, but in passing from Ψ to Z we have lost information about the boundary data on the surface on which the bounce is sliced in half. If there is a boundary at infinity, this is not very important,³ but in the cosmological case this information is crucial.

Consider the pair creation of charged black holes in a cosmological background. Then Σ has topology $S^2 \times S^1$, and the boundary data on Σ will be h_{ij} and A_i , the three-metric and gauge potential. In the magnetic case, we can again define the charge by the integral of F_{ij} over the S^2 factor (the charge in this case is the magnitude of the charge on each of the black holes), but in the electric case, we can fix only the potential

$$\omega = \int A, \quad (37)$$

where the integral is around the S^1 direction in Σ . This latter quantity is equal to the flux of the electric field across the disk. Let M_- be a Euclidean solution of the field equations which agrees with the given data on Σ , which is its only boundary. If M_- has topology $S^2 \times D^2$, which is the case we are interested in, the S^1 direction in Σ is the boundary of the two-disc D^2 .

For the boundary data which describes a pair of charged black holes, M_- will just be half the $S^2 \times S^2$ Euclidean section of the Reissner-Nordström de Sitter solution. Let us choose coordinates so that the boundary Σ corresponds to the surface $\psi = 0, \psi = \pi$ in the metric (32), and so that the integral in (37) is from the black hole horizon $\chi = \pi$ to the cosmological horizon $\chi = 0$ along $\psi = 0$, and back along $\psi = \pi$. The momentum canonically conjugate to ω is the electric charge Q . Now we are ready to make the Fourier transform

$$\Psi(Q, h_{ij}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega Q} \Psi(\omega, h_{ij}) \quad (38)$$

to obtain the wave function in a definite charge sector in the electric case.

We should make another Fourier transform, in both cases, as a natural requirement on the three-surface Σ is that its extrinsic curvature vanish. This guarantees that Σ bisects the bounce, and ensures that our manifold can be matched smoothly onto a Lorentzian extension. We should therefore perform a Fourier transform to trade h_{ij} for its conjugate momentum $\pi^{ij} = \sqrt{h}(K^{ij} - K h^{ij})$, where K^{ij} is the extrinsic curvature of Σ , and then set $\pi^{ij} = 0$. Thus

³It is easy to apply the methods we outline below to rederive the results of Sec. II using the instanton (half the bounce) to describe tunneling from a spatial slice of hot flat space to a spatial slice of an electrically charged Reissner-Nordström solution.

$$\Psi(Q, \pi^{ij}) = \frac{1}{2\pi} \int d[h_{ij}] e^{ih_{ij}\pi^{ij}} \Psi(Q, h_{ij}). \quad (39)$$

In the saddle-point approximation,

$$\Psi(Q, \pi^{ij} = 0) = \Psi(Q, h_{ij} = h_{ij}^0), \quad (40)$$

where h_{ij}^0 is the induced metric on the three-surface $\psi = 0, \psi = \pi$ in the Reissner–Nordström–de Sitter solution. That is, because we are setting $\pi^{ij} = 0$, there is no additional term in the semiclassical value which arises from this transformation.

For the electrically charged Reissner–Nordström–de Sitter instanton (32), the only vector potential which is regular everywhere on M_- is

$$A = iQ \frac{B}{A} \sin \chi \psi d\chi. \quad (41)$$

We have to insist that the gauge potential be regular on the instanton in the electric case to obtain gauge-independent results, as we can only determine the gauge potential on the boundary.⁴ Note that there is *no* electric vector potential regular everywhere on the Euclidean section of the electrically charged Reissner–Nordström–de Sitter solution, as (41) is not periodic in τ . Using (41), we see that in the semiclassical approximation, $\omega = 2\pi iQB/A$ and thus, in the electric case,

$$\ln \Psi(Q, \pi^{ij} = 0) = -I + i\omega Q = \frac{\pi}{A} - \frac{2\pi Q^2 B}{A} = \frac{\pi}{B}, \quad (42)$$

as the action of M_- is $-\pi/A$, half the action of the electric instanton. For the magnetic solution,

$$\ln \Psi(Q, \pi^{ij} = 0) = -I = \frac{\pi}{B}, \quad (43)$$

so the pair creation rate turns out to be identical in the two cases. As $\Psi^2 \leq e^{2\pi/\Lambda} < e^{3\pi/\Lambda}$, these processes are suppressed relative to both de Sitter space and the neutral black hole instanton of [11].

V. ELECTRIC ERNST INSTANTONS

Black holes may be pair created by a background electromagnetic field. An appropriate instanton which describes such pair creation is provided by the Ernst solution, which represents a pair of oppositely charged black holes undergoing uniform acceleration in a background electric or magnetic field. The magnetic case has been extensively discussed, notably in [8,9,3]. We now turn to the consideration of the electric case, to see if the pair creation rate is the same. An attempt was made to compare the electric case to a charged star instanton in [22].

However, the action for Ernst was not explicitly calculated. We find that the calculation of the pair creation rate in this case introduces several new features, but the pair creation rate given by $Z(\beta, Q)$ is identical to that obtained in the magnetic case.

We will review the electric Ernst and Melvin solutions in this section, and describe the calculation of the action in the following section. The solution describing the background electric field is the electric version of Melvin's solution [23],

$$ds^2 = \Lambda^2 (-dt^2 + dz^2 + d\rho^2) + \Lambda^{-2} \rho^2 d\varphi^2, \quad (44)$$

where

$$\Lambda = 1 + \frac{\hat{B}_M^2}{4} \rho^2, \quad (45)$$

and the gauge field is

$$A_t = \hat{B}_M z. \quad (46)$$

The Maxwell field is $F^2 = -2\hat{B}_M^2/\Lambda^4$, which is a maximum on the axis $\rho = 0$ and decreases to zero at infinity. The parameter $\hat{B}_M$ gives the value of the electric field on the axis.

The metric for the electric Ernst solution is

$$ds^2 = (x-y)^{-2} A^{-2} \Lambda^2 [G(y)dt^2 - G^{-1}(y)dy^2 + G^{-1}(x)dx^2] + (x-y)^{-2} A^{-2} \Lambda^{-2} G(x)d\varphi^2, \quad (47)$$

where

$$G(\xi) = (1 - \xi^2 - r_+ A \xi^3)(1 + r_- A \xi) \quad (48)$$

and

$$\Lambda = \left(1 + \frac{1}{2} B q x\right)^2 + \frac{B^2}{4A^2(x-y)^2} G(x), \quad (49)$$

while the gauge potential is [22]

$$A_t = -\frac{BG(y)}{2A^2(x-y)^2} \left[1 + \frac{1}{2} B q x + \frac{1}{2} B q (x-y)\right] - \frac{B}{2A^2} (1 + r_+ A y)(1 + r_- A y) \left(1 - \frac{1}{2} B q y\right) + qy + k, \quad (50)$$

where k is a constant, and $q^2 = r_+ r_-$.

If we label the roots of $G(\xi)$ by $\xi_1, \xi_2, \xi_3, \xi_4$ in increasing order, then x must be restricted to lie in $\xi_3 \leq x \leq \xi_4$ to obtain a metric of the right signature. Because of the conformal factor $(x-y)^{-2}$ in the metric, y must be restricted to $-\infty < y \leq x$. The axis $x = \xi_3$ points towards spatial infinity, and the axis $x = \xi_4$ points towards the other black hole. The surface $y = \xi_1$ is the inner black hole horizon, $y = \xi_2$ is the black hole event horizon, and $y = \xi_3$ the acceleration horizon. The black holes are non-extreme if $\xi_1 < \xi_2$, and extreme if $\xi_1 = \xi_2$. Note that it is *not* possible to choose k so that A_t vanishes at both $y = \xi_2$ and $y = \xi_3$. We choose k so that A_t vanishes at $y = \xi_3$.

⁴This can be clearly seen in the Reissner–Nordström case; we could set $A_t = 0$ at infinity if we did not insist that it be regular at the horizon.

As discussed in [8], to ensure that the metric is free of conical singularities at both poles, $x = \xi_3, \xi_4$, we must impose the condition

$$G'(\xi_3)\Lambda(\xi_4)^2 = -G'(\xi_4)\Lambda(\xi_3)^2, \quad (51)$$

where $\Lambda(\xi_i) \equiv \Lambda(x = \xi_i)$. For later convenience, we define $L \equiv \Lambda(x = \xi_3)$. We also define a physical electric field parameter $\hat{B}_E = BG'(\xi_3)/2L^{3/2}$. When (51) is satisfied, the spheres are regular as long as φ has period

$$\Delta\varphi = \frac{4\pi L^2}{G'(\xi_3)}. \quad (52)$$

As in the magnetic case [3], if we set $r_+ = r_- = 0$, the Ernst metric reduces to the Melvin metric in accelerated form:

$$ds^2 = \frac{\Lambda^2}{A^2(x-y)^2} \left[(1-y^2)dt^2 - \frac{dy^2}{1-y^2} + \frac{dx^2}{1-x^2} \right] + \frac{1-x^2}{\Lambda^2(x-y)^2 A^2} d\varphi^2, \quad (53)$$

where

$$\Lambda = 1 + \frac{\hat{B}_E^2}{4} \frac{1-x^2}{A^2(x-y)^2}. \quad (54)$$

The gauge field in this limit is

$$A_t = -\frac{\hat{B}_E(1-y^2)}{2A^2(x-y)^2}. \quad (55)$$

The acceleration parameter A is now a coordinate degree of freedom. Ernst also reduces to Melvin at large spatial distances, that is, as $x, y \rightarrow \xi_3$.

We Euclideanize (47) by setting $\tau = it$. In the non-extremal case, $\xi_1 < \xi_2$, the range of y is taken to be $\xi_2 \leq y \leq \xi_3$ to obtain a positive definite metric (we assume $\xi_2 \neq \xi_3$). To avoid conical singularities at the acceleration and black hole horizons, we take the period of τ to be

$$\beta = \Delta\tau = \frac{4\pi}{G'(\xi_3)} \quad (56)$$

and require

$$G'(\xi_2) = -G'(\xi_3), \quad (57)$$

which gives

$$\xi_2 - \xi_1 = \xi_4 - \xi_3. \quad (58)$$

The resulting Euclidean section has topology $S^2 \times S^2 - \{pt\}$, where the point removed is $x = y = \xi_3$. This instanton is interpreted as representing the pair creation of two oppositely charged black holes connected by a wormhole.

If the black holes are extremal, $\xi_1 = \xi_2$, the black hole event horizon lies at infinite spatial distance from the acceleration horizon, and gives no restriction on the period of τ . The range of y is then $\xi_2 < y \leq \xi_3$, and

the period of τ is taken to be (56). The topology of the Euclidean section is $R^2 \times S^2 - \{pt\}$, where the removed point is again $x = y = \xi_3$. This instanton is interpreted as representing the pair creation of two extremal black holes with infinitely long throats.

VI. ACTION IN ELECTRIC ERNST

Now, to calculate the pair creation rate, we need to calculate the action for the instanton. As in Sec. IV, an instanton describing the pair creation of black holes in a Melvin background is given by cutting the Euclidean section above in half. That is, the boundary Σ that we want the instanton to interpolate inside of consists of a three-boundary S_∞^3 “at infinity,” plus a boundary Σ_s which can be identified with the surface $\tau = 0, \tau = \beta/2$ in the Euclidean section.

Since we want to consider the pair creation rate at fixed electric charge, the appropriate action is (24). That is, in the instanton approximation, the partition function, and thus the pair creation rate, is approximately given by $Z(\beta, Q) \approx e^{-2I_{\text{Ernst}}}$, where I_{Ernst} is the action (24) of the instanton.

Because the Euclidean section is not compact, the physical action is only defined relative to a suitable background [17], which in this case is the electric Melvin solution. We need to ensure that we use the same boundary S_∞^3 in calculating the contributions to the action from the Ernst and Melvin metrics. This is achieved by insisting that the same boundary conditions are satisfied at the boundary in these two metrics [3]. That is, we insist that the Ernst and Melvin solutions induce the same fields on the boundary (up to contributions which vanish when we take the limit that the boundary tends to infinity).

Let us take the boundary S_∞^3 to lie at

$$x = \xi_3 + \epsilon_E \chi, \quad y = \xi_3 + \epsilon_E (\chi - 1), \quad (59)$$

in the Ernst solution, and define new coordinates by

$$\varphi = \frac{2L^2}{G'(\xi_3)} \varphi', \quad \tau = \frac{2}{G'(\xi_3)} \tau'. \quad (60)$$

We assume that S_∞^3 lies at

$$x = -1 + \epsilon_M \chi, \quad y = -1 + \epsilon_M (\chi - 1) \quad (61)$$

in the accelerated coordinate system in the Melvin solution. The metrics for the electric Ernst and Melvin solutions are the same as for the magnetic solutions, so we know from [3] that the induced metrics on the boundary can be matched by taking

$$\bar{A}^2 = -\frac{G'(\xi_3)^2}{2L^2 G''(\xi_3)} A^2 \quad (62)$$

and

$$\epsilon_M = -\frac{G''(\xi_3)}{G'(\xi_3)}\epsilon_E[1 + O(\epsilon_E^2)], \quad \hat{B}_M = \hat{B}_E[1 + O(\epsilon_E^2)]. \quad (63)$$

However, we cannot match the gauge potentials at the same time. We should work with a different gauge potential, as the gauge potential (50) is not regular at both the horizons in the spacetime. A suitable gauge potential, which is regular everywhere on the instanton, is

$$\begin{aligned} A &= -F_{x\tau}\tau dx - F_{y\tau}\tau dy \\ &= i\tau \left[\frac{B}{A^2(x-y)^3} G(y) \left(1 + \frac{1}{2} Bqx \right) \right] dx + i\tau \left[q \left(1 + \frac{1}{2} Bqx \right)^2 - \frac{B}{A^2(x-y)^3} G(x) \left(1 + \frac{1}{2} Bqx \right) \right. \\ &\quad \left. + \frac{B}{2A^2(x-y)^2} G'(x) \left(1 + \frac{1}{2} Bqx \right) - \frac{B^2 q}{4A^2(x-y)^2} G(x) \right] dy, \end{aligned} \quad (64)$$

and the induced gauge potential on S_∞^3 in the Ernst solution is

$$A_\chi = \frac{2iL^2\tau'\hat{B}_E}{A^2\epsilon_E G'(\xi_3)} \left[1 + \frac{G''(\xi_3)}{G'(\xi_3)}(\chi - 1)\epsilon_E + \frac{Bq\chi\epsilon_E}{L^{1/2}} \right], \quad (65)$$

while in the Melvin solution it is

$$A_\chi = \frac{i\tau\hat{B}_M}{A^2\epsilon_M} [1 + \epsilon_M(\chi - 1)], \quad (66)$$

so they are *not* matched by (62) and (63). [Note that, even if we worked in the gauge (50), the induced gauge potentials on the boundary still would not match.] This seemed for a long time to be an insuperable difficulty, but we have now realized that, in the electric case, we no longer want to match A_i . Instead, we should match $n_\mu F^{\mu i}$, and calculate the action (24), which will give the pair creation rate at fixed electric charge.

The induced value of $n_\mu F^{\mu i}$ on S_∞^3 in the Ernst solution is

$$n_\mu F^{\mu t} = \frac{A\epsilon_E^{1/2} G'(\xi_3)^{1/2} \hat{B}_E}{2L\lambda^3} \left[1 + \frac{G''(\xi_3)}{4G'(\xi_3)}\epsilon_E(2\chi + 1) \right], \quad (67)$$

where

$$\lambda = \frac{\hat{B}_E^2 L^2}{A^2 G'(\xi_3) \epsilon_E} \chi + \frac{\hat{B}_E^2 L^2 G''(\xi_3)}{2A^2 G'(\xi_3)^2} \chi^2 + 1, \quad (68)$$

while in the Melvin solution it is

$$n_\mu F^{\mu t} = \frac{\bar{A}\epsilon_M^{1/2} \hat{B}_M}{\sqrt{2}\Lambda^3} \left[1 - \frac{1}{4}\epsilon_M(2\chi + 1) \right], \quad (69)$$

where

$$\Lambda = \frac{\hat{B}_M^2}{2\bar{A}^2\epsilon_M} \chi - \frac{\hat{B}_M^2}{4\bar{A}^2} \chi^2 + 1. \quad (70)$$

We see that these two quantities are indeed matched by

(62) and (63).

The action (24) of the region of the Ernst solution inside Σ can be written as a surface term, as we can see by writing it in covariant form:

$$\begin{aligned} I_{el} &= \frac{1}{16\pi} \int d^4x \sqrt{g} (-R + F^2) - \frac{1}{8\pi} \int_\Sigma d^3x \sqrt{h} K \\ &\quad - \frac{1}{4\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu \\ &= -\frac{1}{8\pi} \int_\Sigma d^3x \sqrt{h} K - \frac{1}{8\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu, \end{aligned} \quad (71)$$

as the volume integral of R is zero by the field equations, and the volume integral of the Maxwell Lagrangian F^2 can be converted to a surface term by the field equations. The explicit surface term in (24) just reverses the sign of the electromagnetic surface term obtained from the F^2 volume integral; that is, it has the effect of reversing the sign of the electromagnetic contribution to the action.

Using the gauge choice (64), we see that the action is

$$\begin{aligned} I_{el} &= -\frac{1}{8\pi} \int_{S_\infty^3} d^3x \sqrt{h} K - \frac{1}{8\pi} \int_{\Sigma_s} d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu \\ &= -\frac{1}{8\pi} \int_{S_\infty^3} d^3x \sqrt{h} K - \frac{1}{16\pi} \frac{\beta}{2} \int dx dy d\varphi \sqrt{g} F^2. \end{aligned} \quad (72)$$

In the first line, we have used the fact that the extrinsic curvature of Σ_s vanishes, and that $n^\mu A^\nu F_{\mu\nu} = 0$ on S_∞^3 ; in the second line, we used (64). This is the same as the expression for the action in [3] (as the Maxwell term changes sign), and the matching conditions are the same, so we can use the calculation of the action in [3] to conclude that

$$I_{\text{Ernst}} = \frac{\pi L^2}{A^2 G'(\xi_3) (\xi_3 - \xi_1)}. \quad (73)$$

The pair creation rate is approximately $e^{-2I_{\text{Ernst}}}$, so it is thus identical to that for the magnetic case. Note that this applies to both extreme and nonextreme black holes. In particular, the pair creation of nonextreme black holes is enhanced over that of extreme black holes by a factor of $e^{\mathcal{A}_{\text{BH}}/4}$, as it was in the magnetic case [3].

VII. ENTROPY OF CHARGED BLACK HOLES

We turn now to a discussion of the thermodynamics of black holes. Consider first the asymptotically flat black holes. In the electric case, one can calculate the partition function for the grand canonical ensemble at temperature T and electrostatic chemical potential Φ . One does a path integral over all fields that have given period and potential at infinity. In the semiclassical approximation, the dominant contribution to the path integral comes from solutions of the field equations with the given boundary conditions. These are the electrically charged Reissner-Nordström solutions. The semiclassical approximation to the partition function is $Z(\beta, \omega) \approx e^{-I}$, where I is the action of the solution. But in the grand canonical ensemble, $\ln Z(\beta, \omega) = -\Omega/T$, where Ω is a thermodynamic potential [21],

$$\Omega = M - TS - Q\Phi. \quad (74)$$

Comparing this with the expression (18) for the action, one finds that the M and $Q\Phi$ terms cancel, leaving the entropy equal to a quarter of the area, $S = \mathcal{A}_{\text{BH}}/4$, as expected.

In the case of magnetic black holes, the entropy still comes out to be $\mathcal{A}_{\text{BH}}/4$, but the calculation is rather different. Since the magnetic charge is defined by the asymptotic form of the potential, or equivalently, by the choice of the electromagnetic fiber bundle, there is a separate canonical ensemble for each value of the magnetic charge, which is necessarily an integer, unlike the electric charge of an ensemble, which is a continuous variable.⁵ In the magnetic case, the charge is always quantized, even for an ensemble. There is thus no need for a chemical potential for magnetic charge. That is, the partition function Z depends on the charge, and should therefore be interpreted as the canonical partition function, so $\ln Z(\beta, Q) = -F/T$, where F is the free energy,

$$F = M - TS, \quad (75)$$

while the action is (15), and the entropy is therefore again $S = \mathcal{A}_{\text{BH}}/4$.

We should note that, if we make the Fourier transform (20) in the electric case, the partition function $Z(\beta, Q)$ is also interpreted as the canonical partition function. Therefore, this Fourier transform may be thought of as a Legendre transform giving the free energy in terms of the thermodynamic potential Ω [6]:

⁵We might add that the angular momentum of a black hole is a continuous variable, and is not quantized, because it is the expectation value in an ensemble, not a quantum number of a pure state. In the grand canonical ensemble, one therefore has to introduce angular velocity Ω as a chemical potential for angular momentum, like one introduces the electrostatic potential as a chemical potential for charge. The Hamiltonian gets an additional ΩJ surface term.

$$F(\beta, Q) = \Omega(\beta, \Phi) + Q\Phi. \quad (76)$$

The result we get for the entropy does not depend on whether we work with the partition function $Z(\beta, \omega)$ or $Z(\beta, Q)$ in the electric case.

The absence of the Maxwell surface term in the Hamiltonian for magnetic black holes means that they have higher action than their electric counterparts. For an extreme black hole, the region swept out by the surfaces of constant time covers the whole instanton, so $I = \beta H$ [17,3]. Now, as before, $H = M$ for a magnetic black hole, so the action of an extreme magnetic black hole is $I = \beta M$, where β is now an *arbitrary* period with which one can identify an extreme black hole. For the electric case, $H = M - Q\Phi$, but $Q = M$ implies $r_+ = M$, and thus $\Phi = 1$, so $I = 0$ for an extreme electric black hole. Both of these actions are proportional to β . This means that if you substitute the actions into the usual formula

$$S = - \left(\beta \frac{\partial}{\partial \beta} - 1 \right) \ln Z \quad (77)$$

for the entropy, where $Z \approx e^{-I}$, you find that both extreme electric and magnetic black holes have zero entropy, as previously announced [3].

Now for the cosmological black holes, we again need to work with the wave function $\Psi(Q, \pi^{ij} = 0)$ rather than the partition function Z . Because it does not depend on the temperature, Ψ^2 can be interpreted as the microcanonical partition function, or density of states [18]. In fact, it should be clear that Ψ represents a closed system, and the partition function should just be interpreted as counting the number of states, so the entropy should be $S = \ln Z$, or more accurately,

$$S = 2 \ln \Psi(Q, \pi^{ij} = 0). \quad (78)$$

Note that it is $\Psi(Q, \pi^{ij} = 0)$, and not $\Psi(\omega, \pi^{ij} = 0)$, which gives the microcanonical partition function. If we evaluate this entropy in the semiclassical approximation, where $\Psi(Q, \pi^{ij} = 0)$ is given by (42) and (43), we get

$$S = 2\pi/B = \mathcal{A}/4 \quad (79)$$

in both cases, as there are two horizons, which both have area $4\pi/B$, so the total area $\mathcal{A} = 8\pi/B$. That is, we find that the usual relation between entropy and area holds here too, despite the fact that Ψ has a very different interpretation in this case.

VIII. DISCUSSION

We have seen that the action of dual electric and magnetic solutions of the Einstein-Maxwell equations differ. This is presumably a general property of S duality, and initially led us to wonder whether S duality could be a symmetry of a quantum theory given by a path integral. However, we found that despite the difference in actions, the semiclassical approximation to the partition function in a definite charge sector was the same for dual electric and magnetic solutions. In particular, we found that the

rate at which electrically and magnetically charged black holes are created in a background electromagnetic field or in a cosmological background is the same.

The pair creation of both types of charged black holes in a cosmological background by the instanton studied here is suppressed relative to de Sitter space, as we might expect, and it is also more strongly suppressed than the creation of neutral black holes. The instantons describing pair creation of black holes in a cosmological background are studied in more detail in [16]. For all the instantons, the rate at which the pair creation occurs is suppressed relative to de Sitter space.

These calculations are all just semiclassical, but they do seem to offer some encouragement to the suggestion that electric-magnetic duality is more than just a sym-

metry of the equations of motion. The conclusion seems to be that duality is a symmetry of the quantum theory, but in a very nonobvious way. As Einstein said, "God is subtle, but He is not malicious."

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